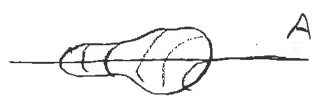
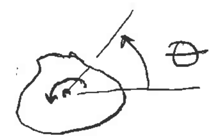
1. Rotation about a fixed axis A



side view

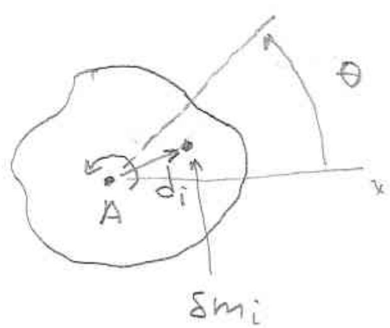


end view

Treat the solid as a sum of small masses δm_i

$$M = \sum_{i=1}^{N} \delta m_i$$

Each δm_i is located a distance d_i from the axis and will move in a circle of radius d_i as the body rotates about the axis A.



for $\dot{\theta} = \omega$
δm_i has velocity
$v_i = d_i \omega$ and
the total kinetic

$$\text{energy} = \sum_i \frac{1}{2} \delta m_i v_i^2 = \frac{1}{2} \underbrace{\sum_i \delta m_i d_i^2}_{I} \omega^2$$

Thus, $K.E. = \frac{1}{2} I_A \omega^2$ where

$I_A = \sum_i \delta m_i d_i^2$ is the moment of inertia of the body about the axis A.

Note I can be viewed as a "moment" of the mass distribution: a sum over each bit of mass δm_i weighted by d_i^2

mass density at $\vec{r}$

$$I_A = \sum_i d_i^2 \delta m_i = \int d^3r \, \rho(\vec{r}) \, d(\vec{r})$$

a "2nd moment" of $\rho(\vec{r})$

$$M = \sum_i \delta m_i = \int d^3r \, \rho(r)$$

the "0th moment" of $\rho(\vec{r})$

Thus, $K.E. = \frac{1}{2} I_A \omega^2$ where

$I_A = \sum_i \delta m_i d_i^2$ is the moment of inertia of the body about the axis A.

Note I can be viewed as a "moment" of the mass distribution: a sum over each bit of mass δm_i weighted by d_i^2

mass density at $\vec{r}$

$$I_A = \sum_i d_i^2 \, \delta m_i = \int d^3r \, \rho(\vec{r}) \, d(\vec{r})$$

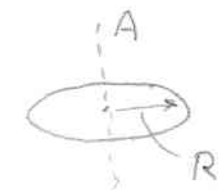
a "2nd moment" of $\rho(\vec{r})$

$$M = \sum_i \delta m_i = \int d^3 r \, \rho(r)$$

the "0th moment" of $\rho(\vec{r})$

2. Examples

a) Find I_A for a ring of radius R and mass M for A ⊥ to ring passing thru center



$$I = \sum_i \delta m_i d_i^2 = \sum_i \delta m_i R^2 = M R^2$$

b) Same ring but A thru center in plane of ring

$$I = 2 \int_0^{\pi} d\theta \cdot \underbrace{\frac{M}{2\pi}} \cdot \underbrace{[R \sin\theta]^2}_{d^2(\theta)}$$

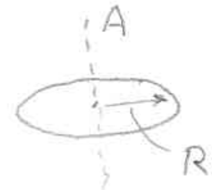
mass density per radian

$$= M \cdot 2 \, \frac{1}{2\pi} \underbrace{\int_0^{\pi} d\theta \, \sin^2\theta}_{\pi/2} \, R^2$$

$$= \frac{1}{2} M R^2$$

2. Examples

a) Find I_A for a ring of radius R and mass M for $A \perp$ to ring passing thru center



$$I = \sum_i \delta m_i \, d_i^2 = \sum_i \delta m_i R^2 = M R^2$$

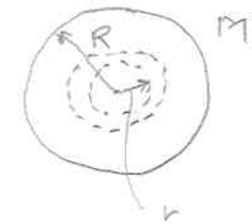
b) Same ring but A thru center in plane of ring

$$I = 2 \int_0^{\pi} d\theta \cdot \underbrace{\frac{M}{2\pi}}_{\substack{\text{mass density}\\ \text{per radian}}} \cdot \underbrace{[R \sin\theta]^2}_{d^2(\theta)}$$

$$= M \cdot 2 \, \frac{1}{2\pi} \underbrace{\int_0^{\pi} d\theta \sin^2\theta}_{\pi/2} \, R^2$$

$$= \frac{1}{2} M R^2$$

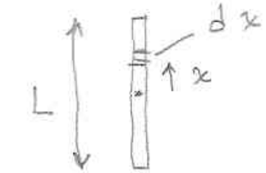
c) Uniform disk of mass M and radius R about perpendicular axis thru center



$$I = \int_0^R \underbrace{dr \, 2\pi r}_{\text{area of ring}} \underbrace{\frac{M}{\pi R^2}}_{\substack{\text{mass}\\ \text{per unit}\\ \text{area}}} r^2$$

$$= 2 \frac{M}{R^2} \int_0^R dr \, r^3 = 2 \frac{M}{R^2} \frac{1}{4} R^4 = \frac{1}{2} M R^2$$

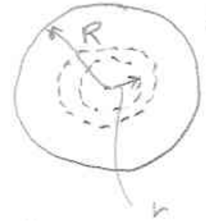
d) Narrow rod of length L and mass M about perpendicular axis thru its center.



$$I = 2 \int_0^{L/2} dx \, \frac{M}{L} x^2 = 2 \, \frac{1}{3} \left(\frac{L}{2}\right)^3 \times \frac{M}{L}$$

$$= \frac{1}{12} M L^2$$

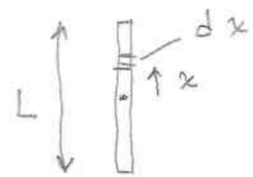
c) Uniform disk of mass M and radius R about perpendicular axis thru center



$$I = \int_0^R \underbrace{dr\, 2\pi r}_{\text{area of ring}}\ \underbrace{\frac{M}{\pi R^2}}_{\substack{\text{mass}\\\text{per unit}\\\text{area}}}\ r^2$$

$$= 2\frac{M}{R^2}\int_0^R dr\, r^3 = 2\frac{M}{R^2}\frac{1}{4}R^4 = \frac{1}{2}MR^2$$

d) Narrow rod of length L and mass M about perpendicular axis thru its center.

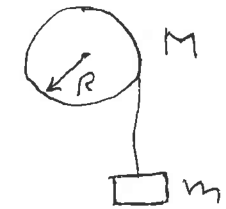


$$I = 2\int_0^{L/2} dx\, \frac{M}{L}\, x^2 = 2\,\frac{1}{3}\left(\frac{L}{2}\right)^3 \times \frac{M}{L}$$

$$= \frac{1}{12}ML^2$$

3. Problem:

Cylinder of radius R and mass M wound with rope attached to falling mass m rotates without friction about its axis of symmetry. If m starts from rest and falls a distance x, find its velocity.



Use energy conservation

$$\underbrace{mgx}_{\substack{\text{lost potential}\\\text{energy}}} = \underbrace{\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2}_{\substack{\text{gained kinetic}\\\text{energy}}}$$

$I = \frac{1}{2}MR^2$

$v = \omega R$

$$mgx = \frac{1}{2}mv^2 + \frac{1}{4}MR^2 \times \left(\frac{v}{R}\right)^2 \quad \text{or}$$

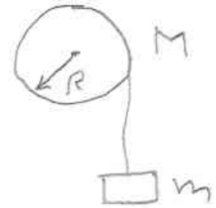
$$v(x) = \sqrt{\frac{4mgx}{2m + M}}$$

---

4. Add a little math: angular velocity vector and cross product

3. Problem:

Cylinder of radius R and mass M wound with rope attached to falling mass m rotates without friction about its axis of symmetry. If m starts from rest and falls a distance x, find its velocity.



Use energy conservation

$$\underbrace{mgx}_{\text{lost potential energy}} = \underbrace{\tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2}_{\text{gained kinetic energy}}$$

$$I = \tfrac{1}{2}MR^2$$

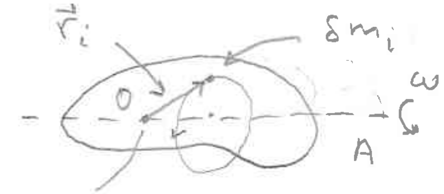
$$v = \omega R$$

$$mgx = \tfrac{1}{2}mv^2 + \tfrac{1}{4}MR^2 \times \left(\frac{v}{R}\right)^2 \quad \text{or}$$

$$v(x) = \sqrt{\frac{4mgx}{2m+M}}$$

---

4. Add a little math: angular velocity vector and cross product



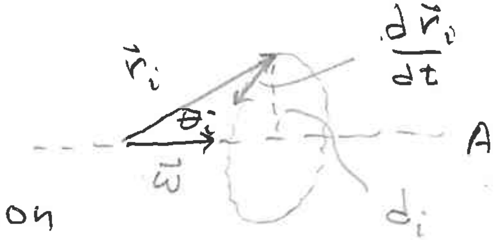
origin O on axis A

Introduce unit vector $\hat{n}$ parallel to A with right hand rule giving diretion of rotation

Define $\vec{\omega} = \hat{n}\omega$.

Then velocity of $\vec{r}_i$ locating δm_i is

$$\frac{d\vec{r}_i}{dt} = \vec{\omega} \times \vec{r}_i$$

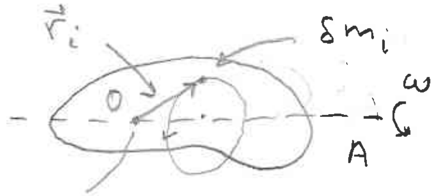


- $\vec{\omega} \times \vec{r}_i$ points in correct direction
- $|\vec{\omega} \times \vec{r}_i| = |\vec{\omega}||\vec{r}_i| \sin\theta_i = |\vec{\omega}| d_i$

Now $I_A = \sum_i \delta m_i d_i^2$

$$= \sum_i \delta m_i (\hat{n} \times \vec{r}_i)^2$$

a fundamentally simpler expression



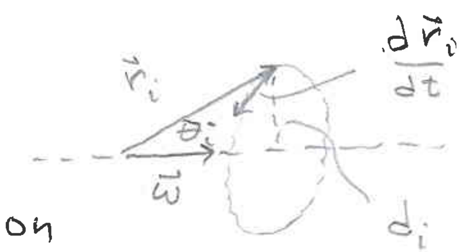
origin O
on axis A

Introduce unit vector $\hat{n}$ parallel to A with right hand rule giving diretion of rotation

Define $\vec{\omega} = \hat{n}\,\omega$.

Then velocity of $\vec{r}_i$ locating δm_i is

$$\frac{d\vec{r}_i}{dt} = \vec{\omega} \times \vec{r}_i$$



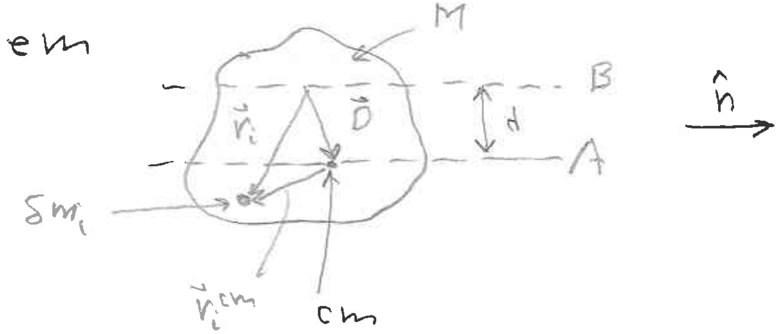
- $\vec{\omega} \times \vec{r}_i$ points in correct direction
- $|\vec{\omega} \times \vec{r}_i| = |\vec{\omega}||\vec{r}_i| \sin\theta_i = |\vec{\omega}|\, d_i$

Now $I_A = \sum_i \delta m_i d_i^2$

$$= \sum_i \delta m_i (\hat{n} \times \vec{r}_i)^2$$

a fundamentally simpler expression

5. Use to prove parallel axis theorem



$$I_B = I_A + Md^2$$

$$I_B = \sum_{i=1}^{N} \delta m_i |\hat{n} \times \vec{r}_i|^2$$

$$= \sum_{i=1}^{N} \delta m_i \left[\hat{n} \times (\vec{r}_i^{\,cm} + \vec{D})\right] \cdot \left[\hat{n} \times (\vec{r}_i^{\,cm} + \vec{D})\right]$$

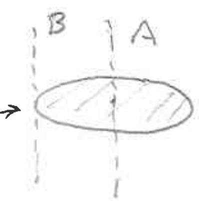
$$= \sum_{i=1}^{N} \delta m_i \Big\{ (\hat{n} \times \vec{r}_i^{\,cm})^2 + \overbrace{(\hat{n} \times \vec{D})^2}^{d^2}$$

$$+ \cancel{2[\hat{n} \times \vec{r}_i^{\,cm}] \cdot [\hat{n} \times \vec{D}]} \quad 0$$

since $\sum_i \delta m_i \vec{r}_i^{\,cm} = \vec{0}$

$$= I_A + Md^2$$

Example
dish of radius R & mass M

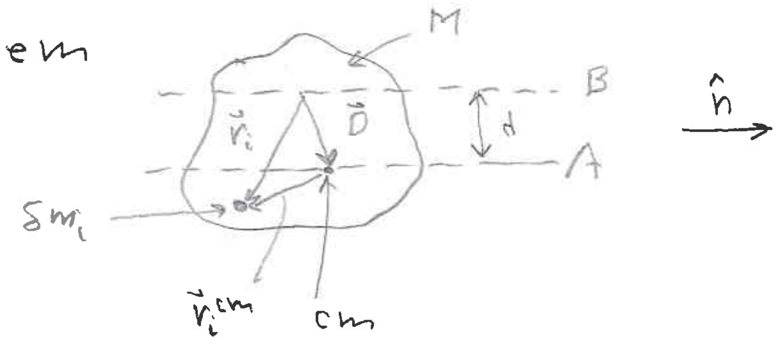


$$I_B = I_A + MR^2$$

$$= \tfrac{1}{2}MR^2 + MR^2$$

$$= \tfrac{3}{2}MR^2$$

5. Use to prove parallel axis theorem



$$I_B = I_A + M d^2$$

$$I_B = \sum_{i=1}^{N} \delta m_i |\hat n \times \vec r_i|^2$$

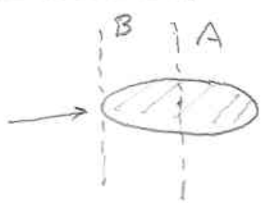
$$= \sum_{i=1}^{N} \delta m_i \left[\hat n \times (\vec r_i^{\,cm} + \vec D)\right]\cdot\left[\hat n \times (\vec r_i^{\,cm} + \vec D)\right]$$

$$= \sum_{i=1}^{N} \delta m_i \Big\{ (\hat n \times \vec r_i^{\,cm})^2 + \overbrace{(\hat n \times \vec D)^2}^{d^2} + \cancelto{0}{2[\hat n \times \vec r_i^{\,cm}]\cdot[\hat n \times \vec D]}$$

since $\sum_i \delta m_i \vec r_i^{\,cm} = \vec 0$

$$= I_A + M d^2$$

Example
dish of
radius R
& mass M



$$I_B = I_A + M R^2$$
$$= \frac{1}{2} M R^2 + M R^2$$
$$= \frac{3}{2} M R^2$$

B Angular momentum (30)

recall in the linear case:

$$\frac{d}{dt}\left[\frac{1}{2} m \vec v^{\,2}\right] = \Big(\underbrace{m \frac{d^2 \vec r}{dt^2}}_{F}\Big)\cdot \vec v$$

1. for rotational case:

$$\frac{d}{dt}\left[\frac{1}{2}\sum_{i=1}^{N} \delta m_i \vec v_i^{\,2}\right] = \sum_{i=1}^{N} \overbrace{\Big(\delta m_i \frac{d\vec v_i}{dt}\Big)}^{F_i}\cdot(\vec \omega \times \vec r_i)$$

$$= \underbrace{\sum_{i=1}^{N} (\vec r_i \times \vec F_i)}_{\vec \tau \,\equiv\, \text{torque}} \cdot \vec \omega$$

the rotational analogue of force.

Examin torque:

$$\tau = \sum_{i=1}^{N} \vec r_i \times \vec F_i = \sum_{i=1}^{N} \vec r_i \times \delta m_i \frac{d^2 \vec r_i}{dt^2} \qquad \vec A \times \vec A = 0$$

$$= \underbrace{\frac{d}{dt}\Big\{\sum_{i=1}^{N} \vec r_i \times \overbrace{\delta m_i \frac{d\vec r_i}{dt}}^{\vec p_i}\Big\}}_{\vec L \,=\, \text{angular momentum}} - \cancelto{0}{\sum_{i=1}^{N} \frac{d\vec r_i}{dt} \times \delta m_i \frac{d\vec r_i}{dt}}$$

$$\vec \tau = \frac{d\vec L}{dt} \quad \Big(\text{analogue of } \vec F = \frac{d\vec P}{dt}\Big)$$

B Angular momentum (30) (150)

recall in the linear case:

$$\frac{d}{dt}\left[\frac{1}{2} m \vec{v}^2\right] = \underbrace{\left(m \frac{d^2\vec{r}}{dt^2}\right)}_{F}\cdot \vec{v}$$

1. for rotational case:

$$\frac{d}{dt}\left[\frac{1}{2} \sum_{i=1}^{N} \delta m_i \vec{v}_i^2\right] = \sum_{i=1}^{N} \overbrace{\left(\delta m_i \frac{d\vec{v}_i}{dt}\right)}^{F_i}\cdot(\vec{\omega}\times\vec{r}_i)$$

$$= \underbrace{\sum_{i=1}^{N} (\vec{r}_i\times\vec{F}_i)}_{\vec{\tau} \,\equiv\, \text{torque}}\cdot\vec{\omega}$$

the rotational analogue of force.

Examin torque:

$$\tau = \sum_{i=1}^{N} \vec{r}_i\times\vec{F}_i = \sum_{i=1}^{N} \vec{r}_i\times\delta m_i \frac{d^2 \vec{r}_i}{dt^2} \qquad \vec{A}\times\vec{A} = 0$$

$$= \underbrace{\frac{d}{dt}\left\{\sum_{i=1}^{N} \vec{r}_i \times \overbrace{\delta m_i \frac{d\vec{r}_i}{dt}}^{\vec{p}_i}\right\}}_{\vec{L} \,=\, \text{angular momentum}} - \underbrace{\cancel{\sum_{i=1}^{N} \frac{d\vec{r}_i}{dt}\times\delta m_i\frac{d\vec{r}_i}{dt}}}_{0}$$

$$\vec{\tau} = \frac{d\vec{L}}{dt} \qquad \left(\text{analogue of } \vec{F} = \frac{d\vec{P}}{dt}\right)$$

(151)

$$\vec{L} \equiv \sum_{i=1}^{N} \vec{r}_i\times\vec{p}_i \qquad \vec{P} = \sum_{i=1}^{N} p_i$$

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \sum_i \vec{r}_i\times\vec{F}_i \qquad \frac{d\vec{P}}{dt} = \sum_i \vec{F}_i$$

If $\vec{F}_i = \vec{F}_i^{ext} + \sum_{j\neq i}\vec{F}_{j\,on\,i}$

$$\frac{d\vec{L}}{dt} \overset{?}{=} \sum_i \vec{r}_i\times F_i^{ext} = \vec{\tau}^{ext} \qquad \frac{d\vec{P}}{dt} = \sum_i \vec{F}_i^{ext} = \vec{F}^{ext}$$

Proof $\quad \vec{\tau} = \sum_i \vec{r}_i\times\left[\vec{F}_i^{ext} + \sum_{j\neq i}\vec{F}_{j\,on\,i}\right]$

Look at second term: second term with i,j pair

$$\vec{r}_i\times\vec{F}_{j\,on\,i} + \vec{r}_j\times\underbrace{\vec{F}_{i\,on\,j}}_{-\vec{F}_{j\,on\,i}} \qquad \text{(Newton's 3rd law)}$$

$$= (\vec{r}_i-\vec{r}_j)\cdot\vec{F}_{j\,on\,i} = 0$$

since $\vec{F}_{j\,on\,i} \parallel \vec{r}_i - \vec{r}_j$

$$\therefore \quad \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i\times\vec{F}_i^{ext} \equiv \vec{\tau}^{ext}$$

$$\vec{L} \equiv \sum_{i=1}^{N} \vec{r}_i \times \vec{p}_i \qquad \vec{P} = \sum_{i=1}^{N} p_i$$

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i \qquad \frac{d\vec{P}}{dt} = \sum_i \vec{F}_i$$

If $\vec{F}_i = \vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{j\,on\,i}$

$$\frac{d\vec{L}}{dt} \stackrel{?}{=} \sum_i \vec{r}_i \times F_i^{ext} = \vec{\tau}^{\,ext} \qquad \frac{d\vec{P}}{dt} = \sum_i \vec{F}_i^{ext} = \vec{F}^{ext}$$

proof $\quad \vec{\tau} = \sum_i \vec{r}_i \times \left[\vec{F}_i^{ext} + \sum_{j \neq i} \vec{F}_{j\,on\,i} \right]$

Look at second term: $\quad$ second term with i,j pair

$$\vec{r}_i \times \vec{F}_{j\,on\,i} + \vec{r}_j \times \underbrace{\vec{F}_{i\,on\,j}}_{-\vec{F}_{j\,on\,i}}$$

(Newton's 3rd law)

$$= (\vec{r}_i - \vec{r}_j) \cdot \vec{F}_{j\,on\,i} = 0$$

since $\vec{F}_{j\,on\,i} \parallel \vec{r}_i - \vec{r}_j$

$$\therefore \quad \frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i^{ext} \equiv \vec{\tau}^{\,ext}$$

For a general rigid body rotating with angular velocity $\vec{\omega}$, $\vec{L}$ is not simple:

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times (\delta m_i \overbrace{\vec{\omega} \times \vec{r}_i}^{\frac{d\vec{r}_i}{dt}})$$

$$= \sum_i \delta m_i \left\{ \vec{\omega}\, \vec{r}_i \cdot \vec{r}_i - \vec{r}_i\, (\vec{\omega} \cdot \vec{r}_i) \right\}$$

we can factor $\vec{\omega}$ out if we use components:

$$L_a = \sum_b \sum_i \delta m_i \left\{ \delta_{ab}\, \omega_b\, \vec{r}_i^{\,2} - (r_i)_a (r_i)_b\, \omega_b \right\}$$

$$= \sum_b \Big\{ \underbrace{\sum_i \delta m_i \left(\delta_{ab}\, \vec{r}_i^{\,2} - (r_i)_a (r_i)_b \right)}_{} \Big\}\, \omega_b$$

moment of inertia tensor $\longrightarrow I_{ab}$

$$L_a = \sum_b I_{ab}\, \omega_b \longleftrightarrow p^a = M v^a$$

$\vec{L}$ & $\vec{\omega}$ need not be parallel

However, $\hat{\omega} \cdot \vec{L}$ is simple