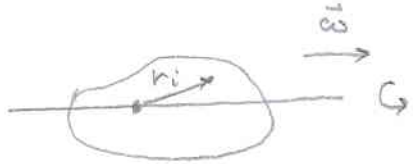
11/19/20

Review of dynamics of Rotation

- About a fixed axis



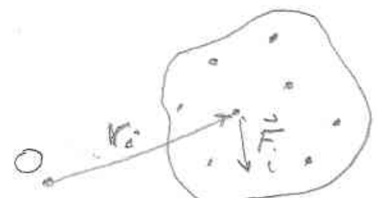
$$I = \sum_{i=1}^{N} \delta m_i (\hat{\omega} \times \vec{r}_i)^2 \qquad \frac{d\vec{r}_i}{dt} = \vec{\omega} \times \vec{r}_i$$

angular velocity

$$KE = \frac{1}{2} I \omega^2$$

moment of inertia

- Torque about an origin O:



$$\vec{\tau} = \sum_i \vec{r}_i \times \vec{F}_i$$

fixed origin

- Angular momentum about the origin O: $\quad \vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$

- Newton-like law for angular momentum

$$\frac{d\vec{L}}{dt} = \vec{\tau} = \underbrace{\vec{\tau}^{\,ext} = \sum_i \vec{r}_i \times \vec{F}_i^{ext}}_{\text{provided Newton's 3rd law is obeyed}}$$

For a general rigid body rotating with angular velocity $\vec{\omega}$, $\vec{L}$ is not simple:

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i \vec{r}_i \times (\delta m_i \overbrace{\vec{\omega} \times \vec{r}_i}^{\frac{d\vec{r}_i}{dt}})$$

$$= \sum_i \delta m_i \{ \vec{\omega}\, \vec{r}_i \cdot \vec{r}_i - \vec{r}_i\, (\vec{\omega} \cdot \vec{r}_i) \}$$

we can factor $\vec{\omega}$ out if we use components:

$$L_a = \sum_b \sum_i \delta m_i \{ \delta_{ab}\, \omega_b\, \vec{r}_i^{\,2} - (r_i)_a (r_i)_b\, \omega_b \}$$

$$= \sum_b \{ \underbrace{\sum_i \delta m_i (\delta_{ab} \vec{r}_i^{\,2} - (r_i)_a (r_i)_b)}_{} \} \omega_b$$

moment of inertia tensor $\longrightarrow I_{ab}$

$$L_a = \sum_b I_{ab}\, \omega_b \quad \longleftrightarrow \quad p^a = M v^a$$

$\vec{L}$ & $\vec{\omega}$ need not be parallel

However, $\hat{\omega} \cdot \vec{L}$ is simple

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$$= \sum_b \Big\{ \underbrace{\sum_i \delta m_i \left(\delta_{ab}\, \vec{r}_i^{\,2} - (r_i)_a (r_i)_b \right)} \Big\}\, \omega_b$$

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$\vec{L}$ & $\vec{\omega}$ need not be parallel

However, $\hat{\omega} \cdot \vec{L}$ is simple

$$\hat{\omega} \cdot \vec{L} = \hat{\omega} \cdot \sum_i \left(\vec{r}_i \times \delta m_i \frac{d\vec{r}_i}{dt} \right)$$

$$= \overset{A}{\hat{\omega}} \cdot \sum_i \{ \overset{B}{\vec{r}_i} \times \overset{C}{(\vec{\omega} \times \vec{r}_i)}\, \delta m_i \}$$

$A \cdot (B \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C}$

$$= \sum_i (\hat{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)\, \delta m_i$$

$$= \omega \sum_{i=1} \delta m_i\, |\hat{\omega} \times \vec{r}_i|^2 = \omega\, \underbrace{I_{\hat{\omega}}}$$

moment of inertia about axis // $\hat{\omega}$

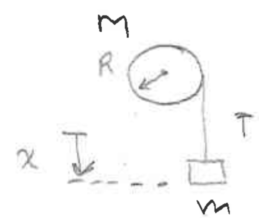
A special case

When $\hat{\omega}$ is fixed then

$$\hat{\omega} \cdot \left\{ \frac{d\vec{L}}{dt} = \vec{\tau}^{\,ext} \right\}$$

$$\frac{d}{dt} \big(\underbrace{\hat{\omega} \cdot \vec{L}}_{I_{\hat{\omega}}\, \omega} \big) = \hat{\omega} \cdot \vec{\tau}^{\,ext} \quad \text{or} \quad I_{\hat{\omega}} \frac{d\omega}{dt} = \tau^{ext}_{\hat{\omega}}$$

2. Use to solve earlier example



$I_\omega \dot{\omega} = RT$ $\qquad$ $m\ddot{x} = mg - T$

$\frac{1}{2} M R^{\cancel{2}} \left(\frac{\ddot{x}}{\cancel{R}} \right) = \cancel{R} T$ $\qquad$ add together

choose $\vec{\omega}$ into the paper

$$\left(\tfrac{1}{2} M + m \right) \ddot{x} = mg$$

$$\text{or} \quad \dot{x} = \frac{mgt}{\frac{1}{2} M + m} \quad ✓$$

$$\hat\omega\cdot\vec L = \hat\omega\cdot\sum_i\left(\vec r_i\times\delta m_i\frac{d\vec r_i}{dt}\right)$$

$$= \hat\omega\cdot\sum_i\left\{\overset{A}{\vec r_i}\times(\overset{B}{\vec\omega}\times\overset{C}{\vec r_i})\,\delta m_i\right\}\qquad A\cdot(B\times\vec C) = (\vec A\times\vec B)\cdot\vec C$$

$$= \sum_i(\hat\omega\times\vec r_i)\cdot(\vec\omega\times\vec r_i)\,\delta m_i$$

$$= \omega\sum_{i=1}\delta m_i\,|\hat\omega\times\vec r_i|^2 = \omega\,\underbrace{I_{\hat\omega}}$$

moment of inertia about axis // $\hat\omega$

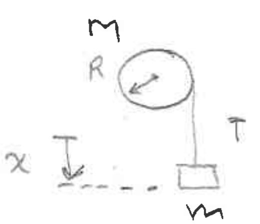
A special case

When $\hat\omega$ is fixed then

$$\hat\omega\cdot\left\{\frac{d\vec L}{dt} = \vec\tau^{\,ext}\right\}$$

$$\frac{d}{dt}(\underbrace{\hat\omega\cdot\vec L}_{I_{\hat\omega}\omega}) = \hat\omega\cdot\vec\tau^{\,ext}\quad\text{or}\quad I_{\hat\omega}\frac{d\omega}{dt} = \tau^{ext}_{\hat\omega}$$

2. Use to solve earlier example



$$I_\omega\dot\omega = RT \qquad m\ddot x = mg - T$$

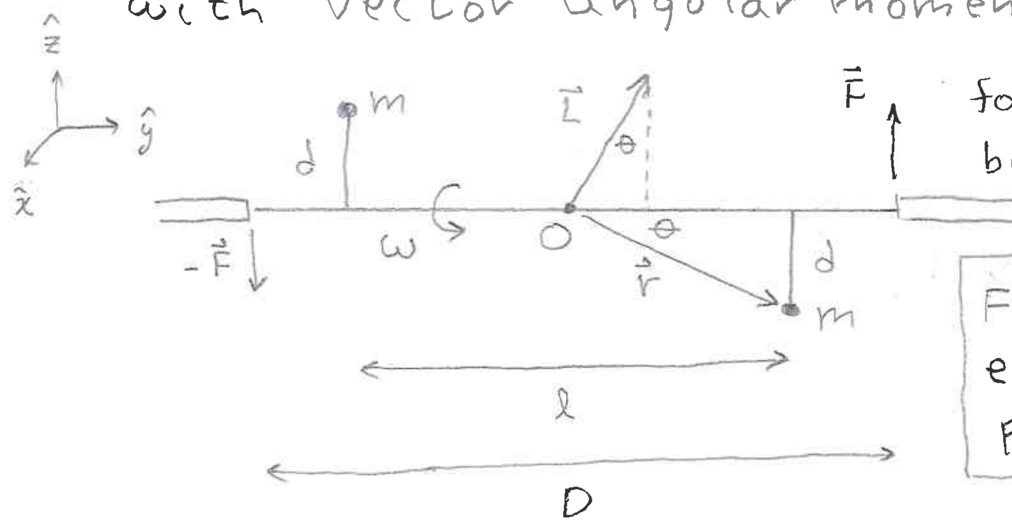
$$\frac12 M\cancel{R^2}\left(\frac{\ddot x}{\cancel{R}}\right) = \cancel{R}T$$ add together

choose $\vec\omega$ into the paper

$$\left(\tfrac12 M+m\right)\ddot x = mg$$

$$\text{or}\quad \dot x = \frac{mgt}{\frac12 M+m}\ \checkmark$$

3. More interesting problem with vector angular momentum



$\vec F$ for exerted by pivot

Find forces exerted by pivots

$\vec L$ in diagram is ⊥ to $\vec r$ & $\vec v$

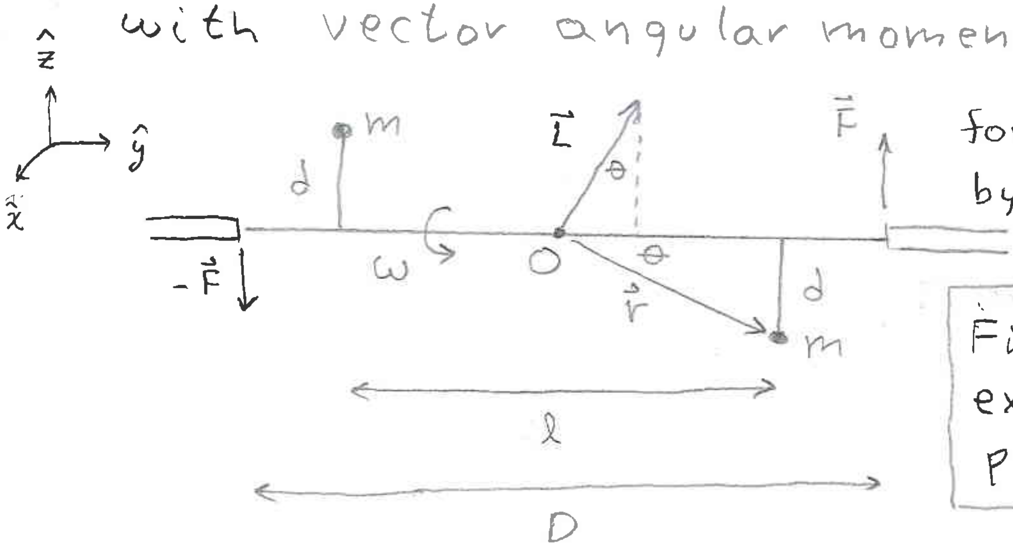
a) $$|\vec L| = 2m|\vec r\times(\vec\omega\times\vec r)| = 2mr(\omega d)$$
$$= 2m\omega d\sqrt{d^2+\left(\tfrac{l}{2}\right)^2}$$

b) $$\frac{d\vec L}{dt} = \vec\omega\times\vec L = \hat x\,\omega|L|\cos\theta \qquad \frac{l/2}{\sqrt{d^2+(l/2)^2}}$$
$$= \hat x\,m\omega^2 d\,l$$

$$= \vec\tau = 2\vec r\times\vec F = \hat x\,2\frac{D}{2}F$$

$$\vec F = \hat z\,\frac{m\omega^2 d\,l}{D}$$

3. More interesting problem with vector angular momentum



Find forces exerted by pivots

$\vec{L}$ in diagram is $\perp$ to $\vec{r}$ & $\vec{v}$

a) $|\vec{L}| = 2m|\vec{r} \times (\vec{\omega} \times \vec{r})| = 2mr(\omega d)$

$$= 2m\omega d\sqrt{d^2+\left(\tfrac{\ell}{2}\right)^2}$$

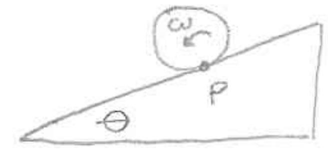
b) $\dfrac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L} = \hat{x}\,\omega|L|\cos\theta$ $\quad \dfrac{\ell/2}{\sqrt{d^2+(\ell/2)^2}}$

$$= \hat{x}\, m\omega^2 d\,\ell$$

$$= \vec{\tau} = 2\vec{r} \times \vec{F} = \hat{x}\, 2\,\frac{D}{2}\, F$$

$$\vec{F} = \hat{z}\,\frac{m\omega^2 d\ell}{D}$$

Important example

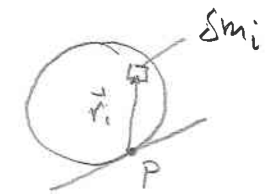


Find acceleration of cylinder of radius R and mass M rolling without slipping down the wedge.

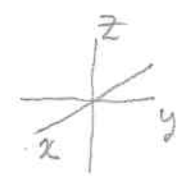
Method 1 $\quad$ Point P has zero velocity (no slipping) compute τ & L about P:

$$L = I_p\,\omega \qquad I_p = \tfrac{3}{2}MR^2$$

note we are using a fixed axis thru P $\perp$ to the page



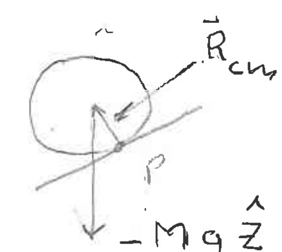
$$\vec{\tau} = \sum_i \vec{r}_i \times (-\delta m_i\, g\,\hat{z})$$



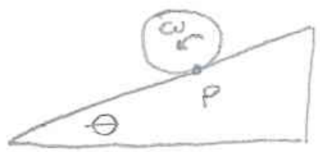
$$= -\underbrace{\sum_i \delta m_i\,\vec{r}_i}_{\vec{R}_{cm} M} \times \hat{z}\; g$$

$$= -Mg\,\vec{R}_{cm} \times \hat{z}$$

$$= MgR\,\hat{x}\sin\theta$$



Important example



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Method 1

Point P has zero velocity (no slipping) compute τ & L about P:

$$L = I_p \omega \qquad I_p = \frac{3}{2} M R^2$$

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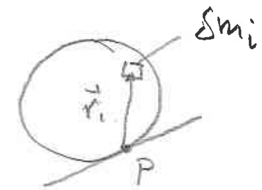
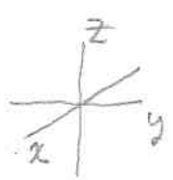


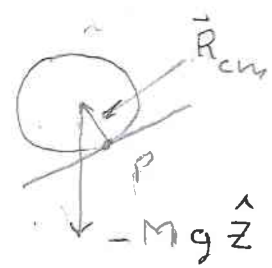
$$\vec{\tau} = \sum_i \vec{r}_i \times (-\delta m_i g \hat{z})$$

$$= - \underbrace{\sum_i \delta m_i \vec{r}_i}_{\vec{R}_{cm} M} \times \hat{z}\, g$$



$$= - M g \vec{R}_{cm} \times \hat{z}$$

$$= M g R \hat{x} \sin\theta$$



$$\frac{dL}{dt} = \tau \qquad \text{(equate } \hat{x} \text{ components)}$$

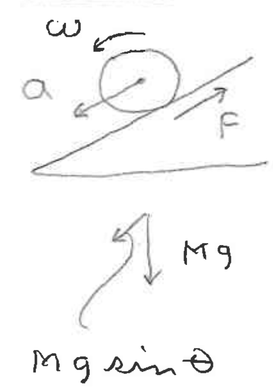
$$\frac{d\, I\omega}{dt} = M g R \sin\theta$$

$$\frac{3}{2} M R^2 \dot{\omega} = M g R \sin\theta$$

$$a_{cm} = R\dot{\omega} = \frac{2}{3} g \sin\theta$$

$< g \sin\theta$ result for block sliding on wedge

Method 2



Work with torques about cm:

$$I\dot{\omega} = FR$$

$$\frac{1}{2} M R^2 \frac{a}{R} = FR$$

$$Ma = Mg\sin\theta - F$$

$$= Mg\sin\theta - \frac{1}{2} Ma$$

$$\frac{3}{2} Ma = Mg\sin\theta$$

$$a = \frac{2}{3} g \sin\theta \checkmark$$