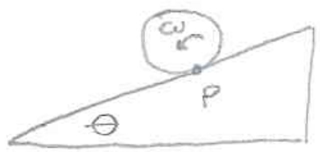
Important example

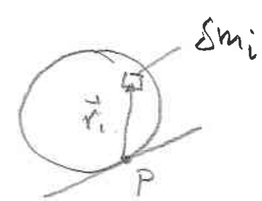



Find acceleration of cylinder of radius $R$ and mass $M$ rolling without slipping down the wedge.

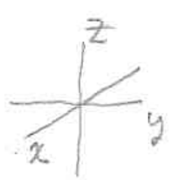
Method 1  Point $P$ has zero velocity (no slipping) compute $\tau$ & $L$ about $P$:

$$L = I_p \omega \qquad I_p = \frac{3}{2} M R^2$$

note we are using a fixed axis thru $P$ $\perp$ to the page



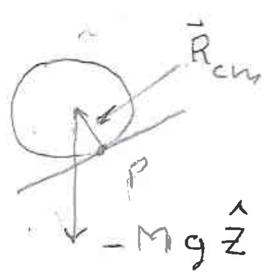

$$\vec{\tau} = \sum_i \vec{r}_i \times (-\delta m_i g \hat{z})$$




$$= - \underbrace{\sum_i \delta m_i \vec{r}_i}_{\vec{R}_{cm} M} \times \hat{z}\, g$$

$$= - M g \vec{R}_{cm} \times \hat{z}$$

$$= M g R \hat{x} \sin\theta$$






$$\frac{dL}{dt} = \tau \qquad \text{(equate } \hat{x} \text{ components}$$

Conservation of L

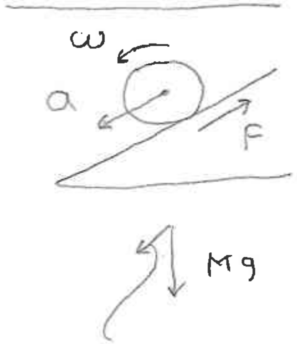
$$\frac{d\, I\omega}{dt} = M g R \sin\theta$$

$$\frac{3}{2} M R^2 \dot{\omega} = M g R \sin\theta$$

$$a_{cm} = R\dot{\omega} = \frac{2}{3} g \sin\theta$$

$< g \sin\theta$ result for block sliding on wedge

Method 2




Mg sin θ

Work with torques about cm:

$$I\dot{\omega} = F R$$

$$\frac{1}{2} M R^2 \frac{a}{R} = F R$$

$$M a = M g \sin\theta - F$$

$$= M g \sin\theta - \frac{1}{2} M a$$

$$\frac{3}{2} M a = M g \sin\theta$$

$$a = \frac{2}{3} g \sin\theta \; \checkmark$$

$$\frac{dL}{dt} = \tau \qquad \text{(equate } \hat{x} \text{ components}$$

$$\frac{d\, I\omega}{dt} = MgR\sin\theta$$

$$\frac{3}{2}MR^2\dot{\omega} = MgR\sin\theta$$

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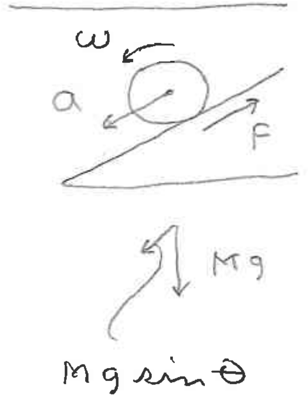
## Method 2




Work with torques about cm:

$$I\dot{\omega} = FR$$

$$\frac{1}{2}MR^2\frac{a}{R} = FR$$

$$Ma = Mg\sin\theta - F$$
$$= Mg\sin\theta - \frac{1}{2}Ma$$

$$\frac{3}{2}Ma = Mg\sin\theta$$

$$a = \frac{2}{3}g\sin\theta \ \checkmark$$



This method is also subtle

$I\dot{\omega} = FR$ equate $\frac{dL}{dt}$ & $\tau$

computed about cm which is an accelerating, non-inertial system !!

This is OK since fictitious force due to acceleration exerts no torque about cm.

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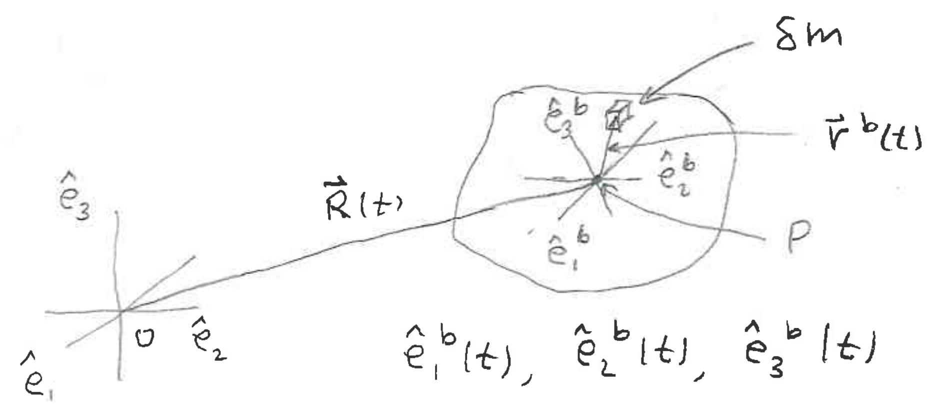
## C General motion of a rigid body

1. Angular velocity done better




$\hat{e}_1^b(t)$, $\hat{e}_2^b(t)$, $\hat{e}_3^b(t)$ are orthonormal axes fixed in body

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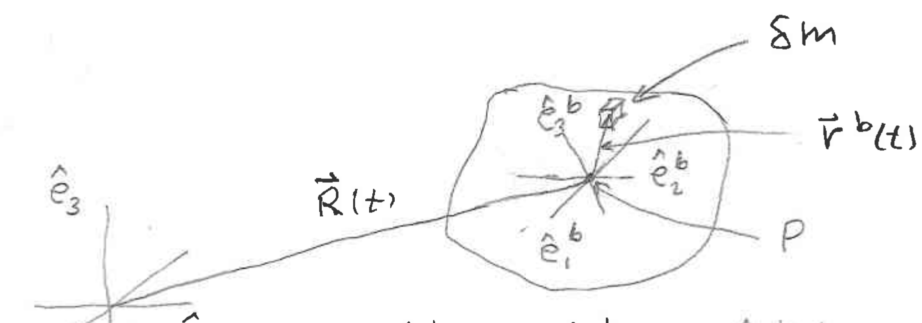


$\hat{e}_1^b(t)$, $\hat{e}_2^b(t)$, $\hat{e}_3^b(t)$

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The rotating axes fixed in the body are related to the fixed axes by a time dependent rotation matrix

Recall $\hat{e}_i^b(t) = \sum_{j=1}^{3} M_{ij}(t)\,\hat{e}_j$

& $\hat{e}_j(t) = \sum_{i=1}^{3} [M(t)]^t_{ji}\,\hat{e}_i^b(t)$

$$\vec{r}_i(t) = \vec{R}(t) + \vec{r}_i^{\,b}(t)$$

$$\vec{v}(t) = \frac{d\vec{R}(t)}{dt} + \frac{d}{dt}\vec{r}_i(t)$$

$$\vec{r}^{\,b}(t) = \sum_{i=1}^{3} \underbrace{(r^b)_i}_{\text{constant}}\,\underbrace{\hat{e}_j^b(t)}_{\text{rotating}}$$

$$\frac{d\vec{r}^{\,b}(t)}{dt} = \sum_{i=1}^{3}(r^b)_i\,\frac{d\hat{e}_i^b}{dt}(t)$$

$\sum_k (M^t)_{jk}\,\hat{e}_k^b$

$$= \sum_{i=1}^{3}(r_i^b)\left[\sum_j \frac{dM_{ij}}{dt}\,\hat{e}_j\right]$$

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$$= \sum_{i=1}^{3}\sum_{k=1}^{3}(r^b)_i\underbrace{\left[\sum_{j=1}^{3}\frac{dM_{ij}}{dt}M^t_{jk}\right]}_{\left(\frac{dM}{dt}M^t\right)_{ik}}\hat{e}^b_k$$

But $M(t)\,M(t)^t = I$

$$\therefore\ \frac{dM}{dt}M^t + M\frac{dM^t}{dt} = 0$$

Or $\frac{dM}{dt}M^t = -M\frac{dM^t}{dt} = -\left(\frac{dM}{dt}M^t\right)^t$

$$\Rightarrow\ \frac{dM}{dt}M^t = \begin{pmatrix} 0 & +\omega_3 & -\omega_2 \\ -\omega_3 & 0 & +\omega_1 \\ +\omega_2 & -\omega_1 & 0 \end{pmatrix}$$

& $$\frac{d\vec{r}^{\,b}}{dt} = \begin{pmatrix} r^b_1 & r^b_2 & r^b_3 \end{pmatrix}\begin{pmatrix} 0 & +\omega_3 & -\omega_2 \\ -\omega_3 & 0 & +\omega_1 \\ +\omega_2 & -\omega_1 & 0 \end{pmatrix}\begin{pmatrix} \hat{e}^b_1 \\ \hat{e}^b_2 \\ \hat{e}^b_3 \end{pmatrix}$$

$$= (-\omega_3 r^b_2 + \omega_2 r^b_3)\,\hat{e}^b_1 + (\omega_3 r_1 - \omega_1 r_3)\,\hat{e}^b_2 + (-\omega_2 r_1 + \omega_1 r_2)\,\hat{e}^b_3$$

$$= \vec{\omega}\times\vec{r}^{\,b}$$

$$= \sum_{i=1}^{3} (r^b)_i \left[ \sum_{j=1}^{3} \frac{dM_{ij}}{dt} \left( \sum_{k=1}^{3} M^t_{jk}\, \hat{e}^b_k \right) \right]$$

$$= \sum_{i=1}^{3} \sum_{k=1}^{3} (r^b)_i \underbrace{\left[ \sum_{j=1}^{3} \frac{dM}{dt}_{ij} M^t_{jk} \right]}_{\left( \frac{dM}{dt} M^t \right)_{ik}} \hat{e}^b_k$$

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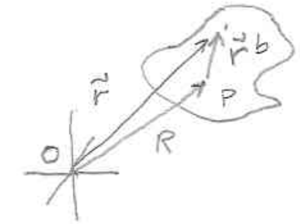
$$= \vec{\omega} \times \vec{r}^b$$



Thus motion of a rigid body can be described as translation of a point P fixed in the body and rotation about P




$$\vec{r}(t) = \vec{R}(t) + \vec{r}^b(t), \quad \frac{d\vec{r}(t)}{dt} = \frac{d\vec{R}(t)}{dt} + \vec{\omega}(t) \times \vec{r}^b(t)$$

Use this resut to determine angular momentum about O:

$$\vec{L} = \sum_n \left[ \vec{R}(t) + \vec{r}^b_n(t) \right] \times \left[ \frac{d\vec{R}}{dt}(t) + \vec{\omega}(t) \times \vec{r}^b_n(t) \right] \delta m_n$$

This is complicated unless we choose $P =$ center-of-mass so $\sum_n \vec{r}^b_n(t)\, \delta m_n = 0$

Then $\quad \vec{L} = \underbrace{\vec{R}(t) \times \frac{d\vec{R}}{dt} M}_{\vec{L} \text{ of com about O}} + \underbrace{\sum_n \vec{r}^b_n \times (\vec{\omega} \times \vec{r}^b_n)\, \delta m_n}_{\vec{L} \text{ of body about the center of mass}}$

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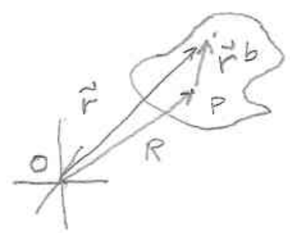


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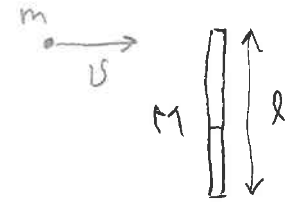
2. Work out two examples

(a)




mass m moving with velocity $v \perp$ to rod of length $l$ & mass M (at rest) strikes and sticks to the end of the rod. Describe the resulting motion.

* CM motion moves with $(m+M)\vec{V}_{cm} = m\vec{v}$

or $\vec{V}_{cm} = \frac{m}{m+M}\vec{v}$

Note location of cm must be d above center where $(m+M)d = m\frac{l}{2} + M\,0 \Rightarrow d = \frac{m}{m+M}\frac{l}{2}$

* Rotation about CM

$$L_{int} = m v\left(\frac{l}{2} - d\right) = \frac{mMv}{m+M}\frac{l}{2}$$

$$L_{final} = I_{cm}\,\omega$$

$$I_{cm} = \underbrace{m\left(\frac{l}{2} - d\right)^2}_{I\text{ for }m} + \underbrace{\frac{1}{12}Ml^2 + Md^2}_{\text{parallel axis theorem for rod about new CM}}$$

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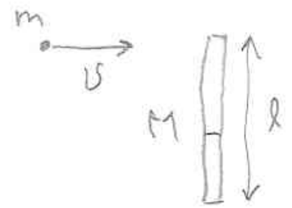
(a) 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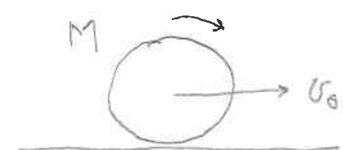
$$I_{cm} = \left\{ \frac{mM^2}{(m+M)^2} + \frac{1}{3}M + \frac{Mm^2}{(m+M)^2} \right\} \frac{\ell^2}{4}$$

$$= \frac{3Mm + M(m+M)}{m+M}\frac{\ell^2}{12} = \frac{(4m+M)M}{m+M}\frac{\ell^2}{12}$$

$$\& \ \omega = \frac{L_{int}}{I_{cm}} = \frac{m+M}{(4m+M)M}\frac{12}{\ell^2} \times \frac{mMv}{m+M}\frac{\ell}{2}$$

$$= \frac{6m}{4m+M}\frac{v}{\ell}$$

(b) 


A bowling ball of mass $M$ and radius $R$ is released moving at velocity $v_0$ and begins to slide with coefficient of friction $\mu$. What is its final velocity

$$\cancel{M}\frac{dv}{dt} = -\mu\cancel{M}g \qquad v(t) = v_0 - \mu g t$$

$$I\dot{\omega} = \tau = \mu M g R$$

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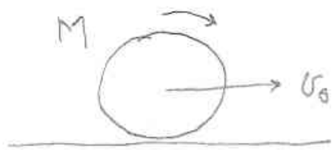


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$$I\dot{\omega} = \tau = \mu M g R$$



$$\dot{\omega} = \frac{\mu M g R}{I} = \frac{\mu M g R}{\frac{2}{5}MR^2} = \frac{5\mu g}{2R}$$

$$\& \quad \omega(t) = \frac{5\mu g}{2R}\,t$$

stops sliding when

$$R\omega(t) = v(t)$$

$$\frac{5}{2}\mu g t = v_0 - \mu g t$$

$$\frac{7}{2}\mu g t = v_0 \,, \quad t = \frac{2}{7}\frac{v_0}{\mu g}$$

$$v = \omega R = \frac{5\mu g}{2}\,\frac{2}{7}\frac{v_0}{\mu g} = \frac{5}{7}v_0$$