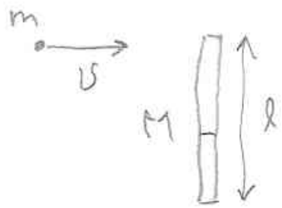
2. Work out two examples

(a)



mass m moving with velocity $v \perp$ to rod of length ℓ & mass M (at rest) strikes and sticks to the end of the rod. Describe the resulting motion.

* CM motion moves with $(m+M)\vec{V}_{cm} = m\vec{v}$

or $\vec{V}_{cm} = \frac{m}{m+M}\vec{v}$

Note location of cm must be d above center where $(m+M)d = m\frac{\ell}{2} + M\,0 \Rightarrow d = \frac{m}{m+M}\frac{\ell}{2}$

* Rotation about CM

$$L_{int} = mv\left(\frac{\ell}{2} - d\right) = \frac{mMv}{m+M}\frac{\ell}{2}$$

$$L_{final} = I_{cm}\,\omega$$

$$I_{cm} = \underbrace{m\left(\frac{\ell}{2} - d\right)^2}_{I \text{ for } m} + \underbrace{\frac{1}{12}M\ell^2 + Md^2}_{\text{parallel axis theorem for rod about new CM}}$$

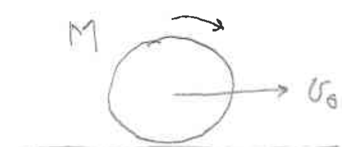
$$I_{cm} = \left\{\frac{mM^2}{(m+M)^2} + \frac{1}{3}M + \frac{Mm^2}{(m+M)^2}\right\}\frac{\ell^2}{4}$$

$$= \frac{3Mm + M(m+M)}{m+M}\frac{\ell^2}{12} = \frac{(4m+M)M}{m+M}\frac{\ell^2}{12}$$

$$\& \ \omega = \frac{L_{int}}{I_{cm}} = \frac{m+M}{(4m+M)M}\frac{12}{\ell^2} \times \frac{mMv}{m+M}\frac{\ell}{2}$$

$$= \frac{6m}{4m+M}\frac{v}{\ell}$$

(b)



A bowling ball of mass M and radius R is released moving at velocity v_0 and begins to slide with coefficient of friction μ. What is its final velocity

$$\cancel{M}\frac{dv}{dt} = -\mu\cancel{M}g \qquad v(t) = v_0 - \mu g t$$

$$I\dot{\omega} = \tau = \mu M g R$$

$$I_{cm} = \left\{ \frac{mM^2}{(m+M)^2} + \frac{1}{3}M + \frac{Mm^2}{(m+M)^2} \right\} \frac{\ell^2}{4}$$

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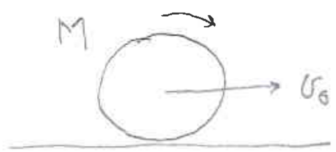

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$$I \dot{\omega} = \tau = \mu M g R$$

$$\dot{\omega} = \frac{\mu M g R}{I} = \frac{\mu M g R}{\frac{2}{5} M R^2} = \frac{5\mu g}{2R}$$

$$\& \ \omega(t) = \frac{5\mu g}{2R} t$$

stops sliding when

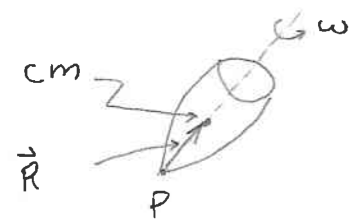
$$R\omega(t) = v(t)$$

$$\frac{5}{2}\mu g t = v_0 - \mu g t$$

$$\frac{7}{2}\mu g t = v_0, \quad t = \frac{2}{7}\frac{v_0}{\mu g}$$

$$v = \omega R = \frac{5\mu g}{2} \frac{2}{7} \frac{v_0}{\mu g} = \frac{5}{7} v_0$$

3. Motion of rapidly spinning top



$\nearrow \vec{\omega} \qquad \vec{L} = I\vec{\omega}$

mass M

Find the effect of the torque exerted by gravity about P. Assume $\vec{\omega}$ is so large that any small additional rotation of the top can be neglected.

$$\dot{\omega} = \frac{\mu M g R}{I} = \frac{\mu M g R}{\frac{2}{5} M R^2} = \frac{5\mu g}{2R}$$

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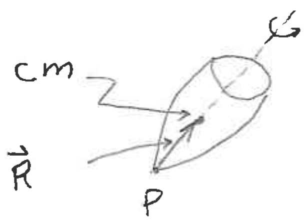

$\nearrow \vec{\omega}$ $\quad \vec{L} = I\vec{\omega}$

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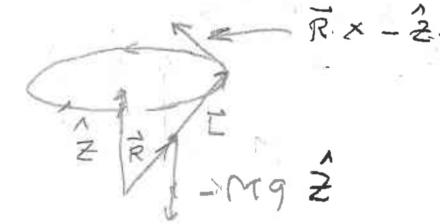
$$\frac{d\vec{L}}{dt} = \vec{\tau} = \vec{R} \times (-mg\hat{z})$$

Use $\vec{R} = I\vec{\omega}\,\frac{R}{I\omega} = \vec{L}\,\frac{R}{I\omega}$

$$\frac{d\vec{L}}{dt} = \frac{R}{I\omega}\vec{L} \times (-Mg\hat{z}) = \underbrace{\frac{MgR}{I\omega}}_{\omega_{prec}} \hat{z} \times \vec{L}$$

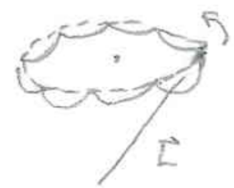
$\Rightarrow$ $\vec{L}$ precesses with angular velocity

$$\vec{\omega}_{prec} = \frac{MgR}{I\omega}\hat{z}$$



nutation

Note if rotation axis is held fixed and then released $\vec{L}$ will begin to rotate but dip down:



Nutation - added angular momentum associate with precession, $\Delta\vec{L} \propto +\hat{z}$ comes from decrease in L_z from spin on original axis.

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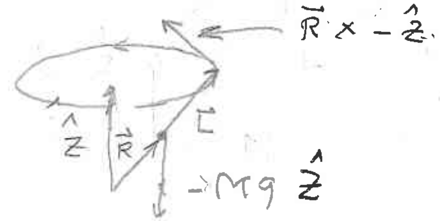

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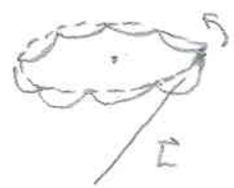

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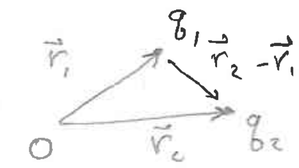
IV Electrostatics

A. Coulomb's law

1. Introduction:



Inverse square law force between to charges q_1 and q_2

$$\vec{F}_{1\,on\,2} = k\,\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|}\,\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}$$

Two common sets of units

a) esu $\quad k = 1 \quad 1\ \underline{esu} = \sqrt{dyne}\ cm$

b) SI: $\quad k = 8.9875 \times 10^9\ \frac{Newton \cdot m^2}{(Coulomb)^2}$

esu & Coulomb are unit of charge

Note force between two 1 Coulomb charges 1 cm apart is

$$\approx \left[9 \times 10^9\ Newton \cdot m^2\right] \times \frac{1}{(0.01\,m)^2} \times \frac{10^5\,dyne}{Newton}$$

$$= 9 \times 10^{18}\ dyne$$

$$1\ Coulomb \equiv 3 \times 10^9\ esu$$

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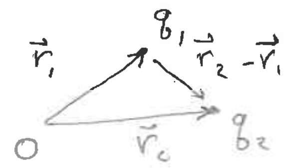

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$$\vec{F}_{1 \text{ on } 2} = k \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}$$

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$$1 \text{ Coulomb} \equiv 3 \times 10^9 \text{ esu}$$

In contrast to cgs and mks units in mechanics Coulomb and esu unit must be used in different equations, while $\vec{F} = m \dfrac{d^2\vec{r}}{dt^2}$ is valid for both cgs and mks units

Like gravity, the Coulomb force is a fundamental force that comes from forces between elementary particles at the atomic level. Because, like gravity it is long range, $F \propto \frac{1}{r^2}$, it is easily recognized at a macroscopic scale

- the strong nuclear force $F_{\text{strong}} \sim \dfrac{e^{-\frac{r}{10^{-13}\text{cm}}}}{r^2}$
- the weak force $F_{\text{weak}} \sim \dfrac{e^{-\frac{r}{10^{-16}\text{cm}}}}{r^2}$

can't be seen at long distances.

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2. Principle of superposition:

Just as with gravity, electrostatic forces add $\vec{F}_{1+2 \text{ on } 3} = \vec{F}_{1 \text{ on } 3} + \vec{F}_{2 \text{ on } 3}$

Given N charges $q_1, q_2, \dots q_N$ at positions $\vec{r}_1, \vec{r}_2, \dots \vec{r}_N$ their force on an $N+1^{st}$ charge q at $\vec{r}$ is given by

$$\vec{F}_{\text{on } q} = \sum_{i=1}^{N} \frac{\vec{r}_q - \vec{r}_i}{|\vec{r}_q - \vec{r}_i|^3} q q_i$$

[This simple linear superposition fails for extremely strong fields near point-like atomic charges.]

3. Define the electric field $\vec{E}(\vec{r})$:

Since $\vec{F}_{\text{on } q}$ is proportional to q we can factor out q and define

$$\vec{E}(\vec{r}) = \lim_{\delta q \to 0} \frac{1}{\delta q} \vec{F}_{\text{on } \delta q} = \sum_{i=1}^{N} \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3} q_i$$

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The limit $\delta q \to 0$ allows for the possibility that δq might shift the position of the other charges. $\vec{E}(\vec{r})$ does not depend on the small test charge used to measure it.

While $\vec{E}(\vec{r})$ looks like a foolish rescaling of $\vec{F}_{\text{on } q}$ its introduction is profound!

- The potential energy of the assembled N charges

$$\mathcal{E} = \frac{1}{2} \sum_{i \neq j} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \int d^3 r \underbrace{\frac{\vec{E}(r)^2}{8\pi}}_{\text{energy density!}}$$

- Introduction of $\vec{E}(\vec{r})$ can make electrostatic become causal. modify relation between $\vec{r}_i$ & $\vec{E}$ to allow changes in $\vec{r}_i$ to propagate to give changes in $\vec{E}(\vec{r})$

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4. Recall result for potential energy for gravity

$$U(\vec{r}_1, \vec{r}_2) = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$$

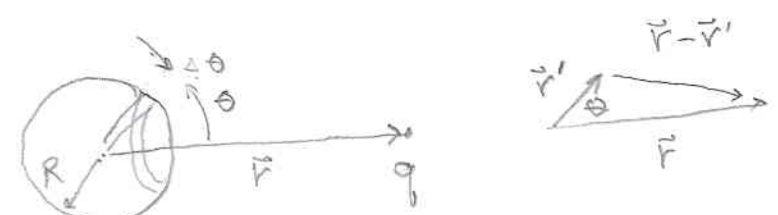
$$\vec{F}_{1\ on\ 2} = -\vec{\nabla}_{r_2}\left[-\frac{G m_1 m_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}\right]$$

$$= -\frac{G m_1 m_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_1 - \vec{r}_2|^3}$$

for two charges $U(r_1, r_2) = +\dfrac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$

for N charges $U(\vec{r}_1, \ldots \vec{r}_N) = \dfrac{1}{2} \sum\limits_{i \neq j} \dfrac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}$

5. Find the potential energy of a charge q displaced by $\vec{r}$ from the center of a sphere (not a ball) of charge Q with radius R



$|r - r'|^2 = r^2 + R^2 - 2rR\cos\theta$

4. Recall result for potential energy

for gravity $U(\vec{r}_1, \vec{r}_2) = -\dfrac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|}$

$$\vec{F}_{1 \text{ on } 2} = -\vec{\nabla}_{r_2}\left[-\frac{G m_1 m_2}{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2+(z_1-z_2)^2}}\right]$$

$$= -\frac{G m_1 m_2 (\vec{r}_2 - \vec{r}_1)}{|\vec{r}_1 - \vec{r}_2|^3} \checkmark$$

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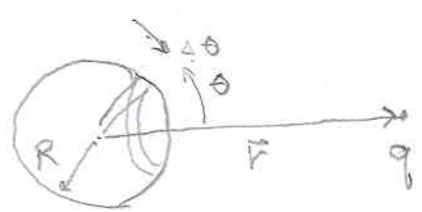

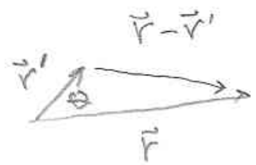

$$|r - r'|^2 = r^2 + R^2 - 2rR\cos\theta$$

Add up the potential energy of each ring of charge, equidistant from q:

$$U(r) = \int_0^{\pi} \underbrace{d\theta\, R\, 2\pi R \sin\theta}_{\text{area of ring}} \underbrace{\frac{Q}{4\pi R^2}}_{\text{surface density of charge on sphere}} \frac{q}{\sqrt{R^2 + r^2 - 2Rr\cos\theta}}$$

let $z = \cos\theta$

$$= \frac{Qq}{2}\int_{-1}^{1} dz \frac{1}{\sqrt{R^2+r^2-2Rrz}}$$

$$= \frac{Qq}{2}\left(-\frac{1}{2Rz}\right) 2\Big\{\underbrace{\underbrace{\sqrt{R^2+r^2-2Rr}}_{|R-r|} - \underbrace{\sqrt{R^2+r^2+2Rr}}_{|R+r|}}_{-2\times\text{smaller}(R,r)}\Big\}$$

$$= Qr \frac{1}{\text{larger}(R,r)}$$