Use new math tools to express energy in terms of $\vec{E}(\vec{r})$

$$U = \frac{1}{2}\int_V d^3r \int d^3r' \, \rho(\vec{r}) \frac{1}{|\vec{r}-\vec{r}\,'|} \rho(\vec{r}\,')$$

$$= \frac{1}{2}\int_V d^3r \underbrace{\rho(\vec{r})}_{+\frac{1}{4\pi}\vec{\nabla}\cdot\vec{E}} \left\{ \underbrace{\int d^3r' \frac{1}{|\vec{r}-\vec{r}\,'|}\rho(r')}_{\Phi(\vec{r})} \right\}$$

$$= \frac{1}{8\pi}\int_V d^3r \left\{ \vec{\nabla}\cdot\left[\vec{E}(\vec{r})\,\Phi(\vec{r})\right] - \vec{E}(\vec{r})\cdot\vec{\nabla}\Phi \right\}$$

$$= \underbrace{\frac{1}{8\pi}\int_{\partial V} \hat{n}\cdot\left[\underbrace{\vec{E}(\vec{r})\,\Phi(\vec{r})}_{\sim\frac{1}{r^3}}\right] \underbrace{dS}_{r^2}}_{\text{vanish as } V \text{ grows}} + \frac{1}{8\pi}\int d^3r \, \vec{E}(\vec{r})^2$$

$$U = \int d^3r \left(\frac{\vec{E}(r)^2}{8\pi}\right)$$

energy density in the electric field

December 10, 2020

E Conductors

1. Electrical properties of materials

Recall all materials are made of atoms with a heavy compact positive charge nucleus surrounded by a much larger cloud of negatively charged electrons.

Two comon behaviors when $\vec{E}$ is applied:

insulator
a) All electrons a bound to a nucleus or cluster of nuclei and external $\vec{E}(\vec{r}) \neq 0$ does not make them move

conductor
b) One or more electron associated with some types of atoms present are accelerated by external $\vec{E} \neq 0$ and can flow from atom to atom in the direction of $-\vec{E}$

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$$U = \int d^3r \left(\frac{\vec E(r)^2}{8\pi}\right)$$

energy density in the electric field

Theory of electron motion

$$\vec a_e = -\frac{\vec E e}{m_e}, \qquad \vec v_e = -\frac{\vec E e}{m_e} t$$

negative of electron's charge

but $\vec v_e$ does not increase without bound as t grows: the electron will collide with something in the material. Thus,

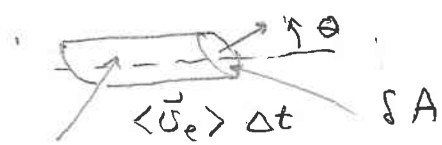
$$\langle \vec v_e \rangle = -\frac{\vec E e}{m_e}\overbrace{\langle t \rangle}^{\tau_{col}}$$

The average time between collisions

If there are n_e conduction electrons per unit volume the result is a flux or current of charge $\vec j(\vec r) = -e\, n_e \langle \vec v_e \rangle$

$$= \underbrace{\frac{e^2 \tau_{col}\, n_e}{m_e}}_{=\sigma} \vec E(\vec r)$$

σ conductivity of material



charge thru δA per time

$$= \frac{-e n_e \langle \vec v_e \rangle \Delta t \, \delta A \cos\theta}{\Delta t} = \vec j\cdot\hat n\,\delta A$$

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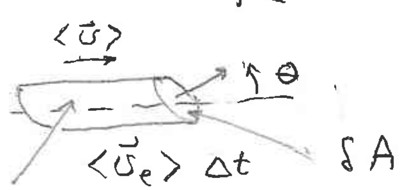

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charge thru δA per time

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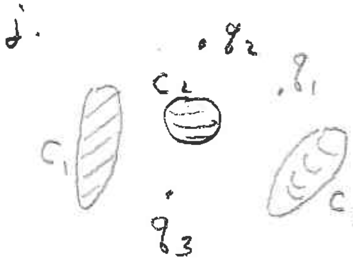
$\vec{j}(\vec{r}) = \sigma \vec{E}(\vec{r})$ is Ohm's law

Typically the thermal velocity of the conduction electrons is much greater than the slow drift caused by $\vec{E}(\vec{r})$.

For electrostatics nothing should be moving so $\vec{j}(\vec{r}) = 0$ in a conductor which implies $\vec{E}(\vec{r}) = 0$.

Consider a system of fixed charges q_i at $\vec{r}_i$ & conductors C_j.



The charge that is free to move in the conductors must move to make $\vec{E} = 0$ everywhere inside each conductor

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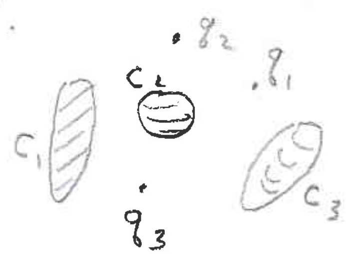

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Locating the charges on the conductors poses a new type of problem. Can be solved by:

a) solving Poisson's with the conductors providing boundary conditions

b) Simple tricks using symmetry, image charges and Gaussian surfaces.

2. Uniqueness theorem

Given K conductors at known potentials Φ_i, $1 \le i \le K$ and the charge density $\rho(\vec{r})$ outside the conductors there is a unique potential $\Phi(\vec{r})$ which obeys:

- $\lim_{|\vec{r}| \to \infty} \Phi(\vec{r}) = 0$
- $\vec{\nabla}^2 \Phi(r) = 0$ outside of conductors
- $\Phi(r) = \Phi_i$ on surface of ith conductor

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Proof: Assume that both $\varphi(\vec{r})$ and $\varphi'(\vec{r})$ obey these conditions, then their difference $\delta\varphi(\vec{r}) = \varphi(\vec{r}) - \varphi'(\vec{r})$ obeys:

$\nabla^2 \delta\varphi = 0$ outside the conductors

$\delta\varphi(\vec{r}) = 0$ at the surface of the conductors

We need one important property of solutions to Laplace's equation:

For a sphere centered at $\vec{r}_0$ inside a region where $\vec{\nabla}^2 \delta\varphi(\vec{r}) = 0$

$$\underbrace{\frac{1}{4\pi R^2} \int_S \varphi(\vec{r})\, ds}_{\text{average of } \varphi(r) \text{ over the sphere}} = \varphi(r_0)$$

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This property implies that $\delta\varphi(\vec{r})$ cannot reach a local maximum or minimum in the region between the conductors!

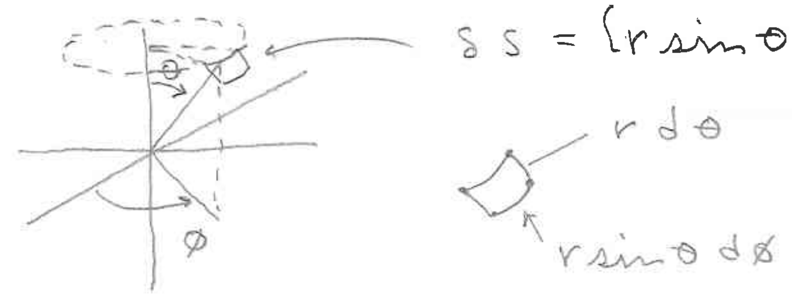
Thus the largest and smallest values of $\delta\varphi$ must occur on the surface of a conductor or as $|\vec{r}| \to \infty$ - all places where $\delta\varphi(r)$ is zero.

To show $\frac{1}{4\pi R^2}\int_{S_R} \varphi(\vec{r})\, dS = \varphi(r_0)$

define $A(\vec{r}) = \frac{1}{4\pi r^2}\int_{S_r} \varphi(\vec{r})\, dS$

Introduce polar coordinates

$\delta S = [r \sin\theta\, d\phi][r\, d\theta]$



This property implies that $\delta\Phi(\vec{r})$ cannot reach a local maximum or minimum in the region between the conductors!

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To show $\frac{1}{4\pi R^2}\int_{S_R} \Phi(\vec{r})\, dS = \Phi(r_0)$

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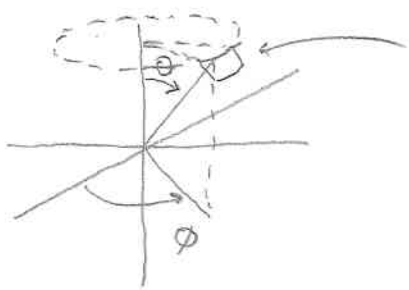

$\delta S = [r \sin\theta\, d\phi][r\, d\theta]$

Thus

$$A(r) = \frac{1}{4\pi \cancel{r^2}}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, \sin\theta\, \cancel{r^2}\, \Phi(\theta,\phi,r)$$

$$\frac{d}{dr}A(r) = \frac{1}{4\pi}\int_0^{\pi} d\theta \int_0^{2\pi} d\phi\, \sin\theta\, \underbrace{\frac{d}{dr}\Phi(\theta,\phi,r)}_{-\vec{E}\cdot\hat{r}}$$

$$= -\frac{1}{4\pi R^2}\int \vec{E}(\vec{r})\cdot\hat{n}\, dS = Q_{incl} = 0$$

and $A(R) = \lim_{r\to 0} A(\vec{r}) = \Phi(r_0)$ as claimed

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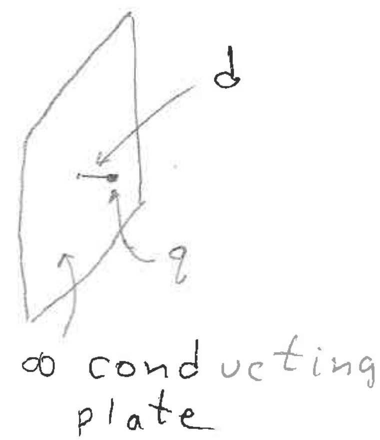
3. Famous problem: find $\vec{E}(\vec{r})$ and surface charge when a charge q is located a distance d from the surface of a large conducting plate



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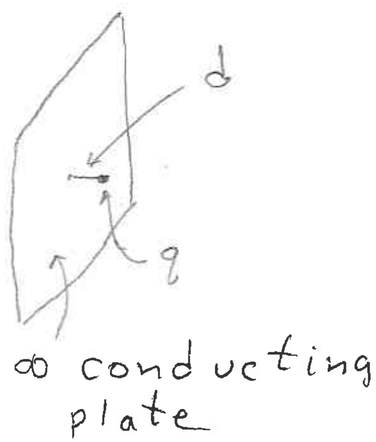
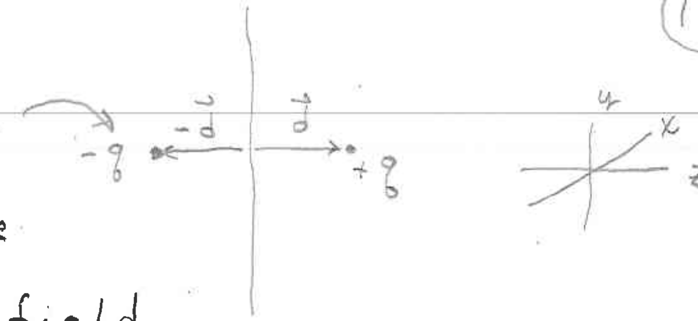


Edge view:

add fictitious "image" charge to construct field for which conductor is a surface of constant potential. Since the solution is unique this must be it!

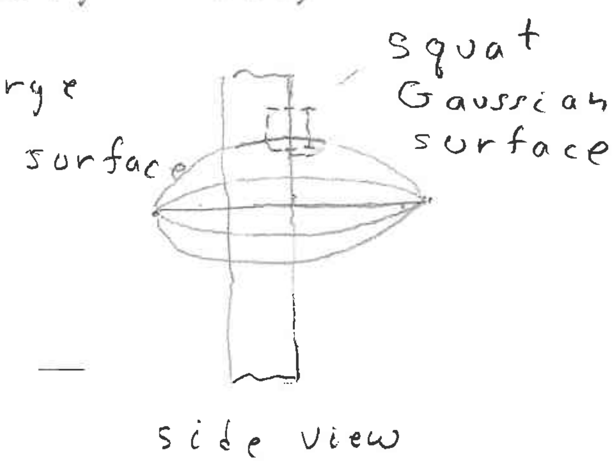


$$\Phi(\vec{r}) = \frac{-q}{|\vec{r}+\vec{d}|} + \frac{q}{|\vec{r}-\vec{d}|}$$

when $z=0$

$$\Phi(x,y,0) = \frac{-q}{\sqrt{x^2+y^2}} + \frac{q}{\sqrt{x^2+y^2}} = 0$$

Find the charge density on the surface of the plate



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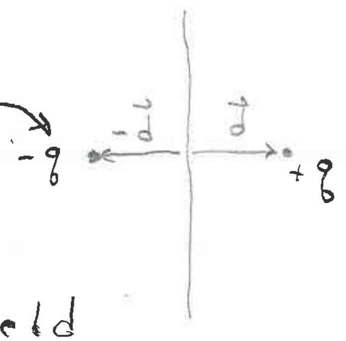

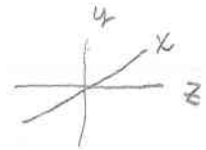

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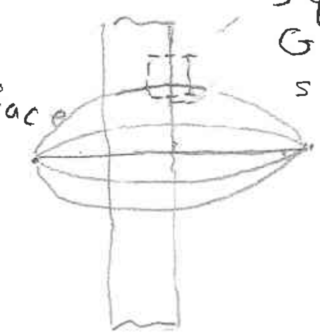
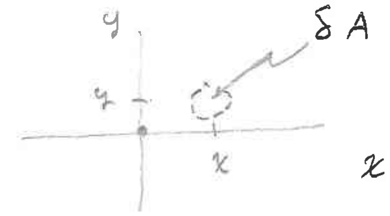




$$\oint \vec{E}\cdot\hat{n}\, ds = 4\pi Q$$

$$\delta A\ \vec{E}(x,y,0)\cdot\hat{n} = 4\pi\sigma\ \delta A$$

charge density on surface

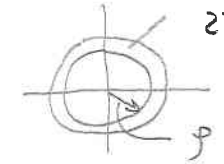
$$E_z(x,y,z) = -\frac{\partial}{\partial z}\left\{ -\frac{q}{\sqrt{x^2+y^2+(z+d)^2}} + \frac{q}{\sqrt{(x^2+y^2+(z-d)^2}} \right\}$$

$$= -\frac{q(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} + \frac{q(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}}$$

$$\therefore\ \sigma(x,y) = \frac{1}{4\pi} E_z(x,y,0)$$

$$= -\frac{1}{4\pi}\frac{2qd}{\sqrt{x^2+y^2+d^2}}$$

area = $2\pi\rho\, d\rho$

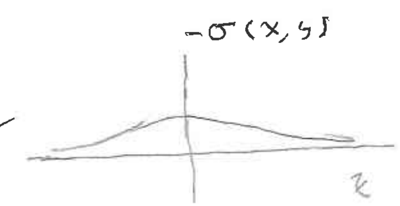


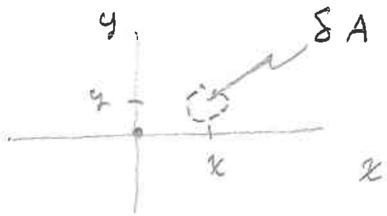
Note charge on plate Q

$$Q = \int_0^\infty \sigma(x,y)\, 2\pi\rho\, d\rho = \int_0^\infty \left(-\frac{1}{2\pi}\frac{dq}{(\rho^2+d^2)^{3/2}}\right) 2\pi\rho\, d\rho$$

$$= -\frac{1}{2} q d \int_0^\infty d\rho^2 \frac{1}{(\rho^2+d^2)^{3/2}}$$

$$= -qd \left.\frac{1}{(\rho^2+d^2)^{1/2}}\right|_0^\infty = -q \checkmark$$





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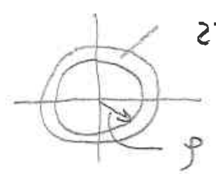

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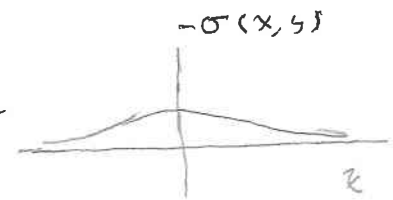
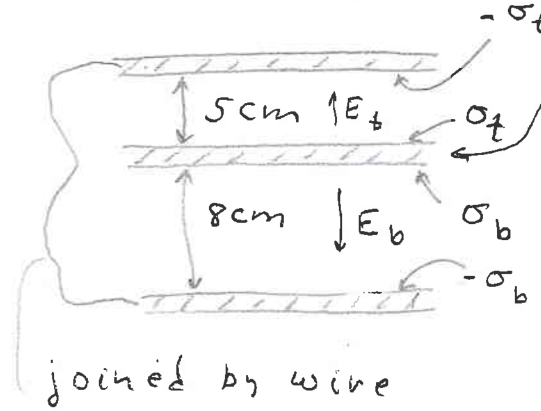
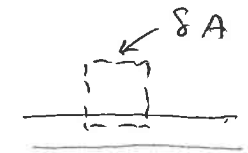


4. Solve another problem from Purcell (#38)



charge 10 esa/cm²

How is charge distributed on each surface?



$E_t = 4\pi\sigma_t$

$E_b = 4\pi\sigma_b$

$E\,\cancel{\delta A} = 4\pi\sigma\,\cancel{\delta A}$

$\varphi_t = \varphi_b \Rightarrow 5\text{ cm } E_t = 8\text{ cm } E_b$

$5\sigma_t = 8\sigma_b \qquad \sigma_t + \sigma_b = 10$

$5\sigma_t = 8(10-\sigma_t) \Rightarrow \sigma_t = \frac{80}{13}$

$\sigma_b = 10 - \sigma_t = \frac{130-80}{13} = \frac{50}{13}$

Expect field above top plate to be $2\pi \times 10\ \frac{\text{esu}}{\text{cm}^2}$ & field below bottom to be $-2\pi\cdot 10\ \frac{\text{esu}}{\text{cm}^2}$

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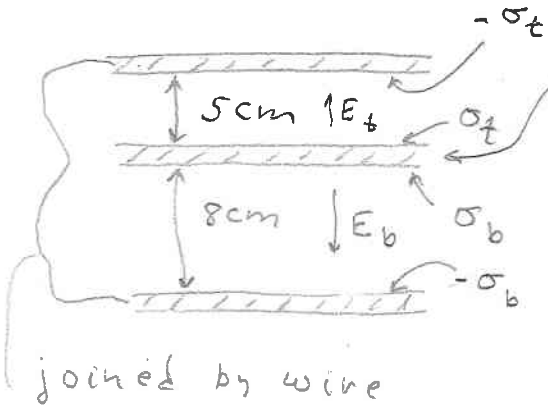

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$+5$
$-80/13$ $\Big\}$ total $= \frac{65-80}{13} = -\frac{15}{13}$

$80/13$
$50/13$

$-50/13$
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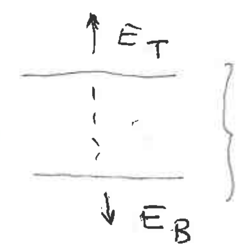

σ_{total}

$$E_T + E_B = 4\pi\sigma_{tot}$$

assume $E_B = -E_T$

(no extra field applied)

top plus bottom energy density

$$\frac{\partial}{\partial E_0}\frac{1}{8\pi}\left\{(E_0+E_T)^2 + (E_0-E_B)^2\right\}$$

$$= \frac{1}{4\pi}\left\{(E_0+E_T) + (E_0-E_B)\right\} = \frac{1}{2\pi}E_0$$

smallest when $E_0 = 0$