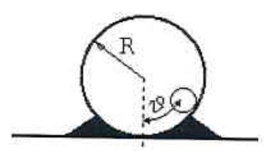
12/15/20 ①

Review Session for Final

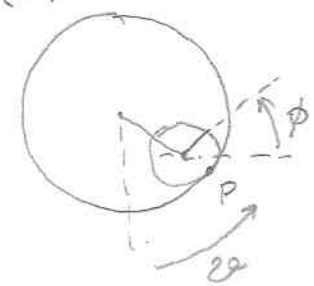
1. A small cylinder of radius r rolls without slipping inside a larger, fixed cylinder of radius R.



(a) Find the acceleration of the smaller cylinder (*e.g.* $d^2\vartheta/dt^2$) as a function of ϑ. [8 points]

(b) If the small cylinder is released from rest at an angle $\vartheta_0 \ll 1$, how long does it take to reach the bottom? [12 points]

1(a)



Point P is stationary. Compute velocity of cm of rolling cylinder two ways

$$(R-r)\dot{\vartheta} = -r\dot{\phi}$$

where ϕ is rotation angle for cylinder.

Thus, $\ddot{\phi} = \left(\frac{R-r}{r}\right)\ddot{\vartheta}$. For rotation about P: $I_P\ddot{\phi} = mgr\sin\vartheta$

$$-\left(1+\tfrac{1}{2}\right)\cancel{m}r^{\cancel{2}}\left(\frac{R-r}{\cancel{r}}\right)\ddot{\vartheta} = \cancel{m}g\cancel{r}\sin\vartheta$$

$\tau = mgr\sin\vartheta$

$$\ddot{\vartheta} = -\frac{2}{3}\left(\frac{g}{R-r}\right)\sin\vartheta$$

$$\approx -\frac{2}{3}\left(\frac{g}{R-r}\right)\vartheta$$

②

1(b) Oscillation frequency

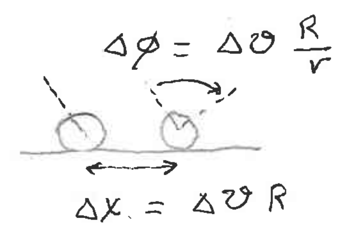
$$\omega_{osc} = \left[\frac{2}{3}\,\frac{g}{R-r}\right]^{1/2}, \quad \tau\,\omega_{osc} = 2\pi$$

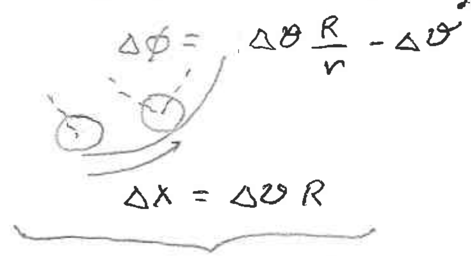
Time to roll from ϑ_0 to bottom

$$= \frac{\tau}{4} = \frac{2\pi}{4\omega_{osc}} = \frac{\pi}{2}\left[\frac{3(R-r)}{2g}\right]^{1/2}$$

Note: motion of cylinder is tricky

$v_P = \dot{\vartheta}R$





$$\Delta\phi = \Delta\vartheta\left(\frac{R-r}{r}\right)$$

1 (b) Oscillation frequency

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Note: motion of cylinder is tricky

$v_p = \dot{\vartheta} R$

$\Delta\phi = \Delta\vartheta\,\frac{R}{r}$

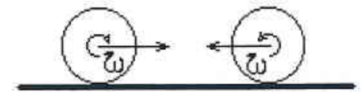
$\Delta x = \Delta\vartheta R$

$\Delta\phi = \Delta\vartheta\,\frac{R}{r} - \Delta\vartheta$

$\Delta x = \Delta\vartheta R$

$$\Delta\phi = \Delta\vartheta\left(\frac{R-r}{r}\right)$$

2. Two identical billiard balls of radius R and mass M rolling with velocities $\pm\vec{v}$ collide elastically, head-on. Assume that after the collision they have both reversed motion and are still rolling.



(a) Find the impulse which the surface of the table must exert on each ball during its reversal of motion. [10 points]

(b) What impulse is exerted by one ball on the other? [10 points]

Recall that a time varying force $F(t)$ exerts an impulse $\mathbf{I} = \int F(t)dt$.

(a) Look at left ball

$$\Delta P = I_B - I_T \qquad \Delta L_{cm} = I_T R$$

$$\Delta P = \int \frac{dP}{dt}\,dt = \underbrace{\int F\,dt}_{I_{tot}} \qquad \Delta L = \int \frac{dL}{dt}\,dt = \int F_T R\,dt = R I_T$$

$$I_T = \frac{1}{R}\,\Delta L_{cm} = \frac{1}{R}\cdot I_{cm}\cdot 2\omega = \frac{1}{R}\left(\frac{2}{5}MR^2\right)2\omega = \frac{4}{5}M\omega R$$

(b) $I_B = \Delta P + I_T = 2M\overset{\omega R}{v} + \frac{4}{5}M\omega R = \frac{14}{5}M\omega R$

(3)

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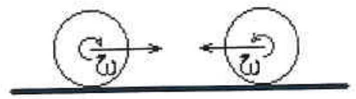

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$= R I_T$

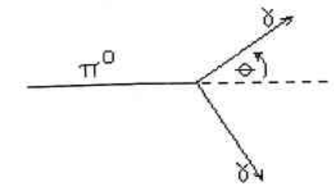
$I_T = \frac{1}{R}\Delta L_{cm} = \frac{1}{R}\cdot I_{cm}\cdot 2\omega = \frac{1}{R}\left(\frac{2}{5}MR^2\right)2\omega$

$= \frac{4}{5}M\omega R$

(b) $I_B = \Delta P + I_T = 2M\overset{\omega R}{v} + \frac{4}{5}M\omega R = \frac{14}{5}M\omega R$

(4)

3. A π^0 particle of mass $m_\pi = 140MeV/c^2$ is moving with velocity v. It decays into two massless photons. Find the energy of one of the photons as a function of the angle θ between the original direction of the π and the momentum of the decay photon. [20 points]



Start with the four-vector equation for energy-momentum conservation: $p_\pi = k_1 + k_2$

Recall $p_\pi^2 = -m_\pi^2 c^2$, $k_1^2 = k_2^2 = 0$

Use $\hat{x} \parallel \vec{p}_\pi$: $p_\pi = (m_\pi \gamma_v v,\ 0,\ 0,\ m_\pi \gamma_v c)$

$k_1 = \left(\frac{E_1}{c}\cos\theta,\ \frac{E_1}{c}\sin\theta,\ 0,\ \frac{E_1}{c}\right)$

Isolate unwanted k_2 on RHS

$p_\pi - k_1 = k_2 \Rightarrow (p_\pi - k_1)^2 = k_2^2 = 0$

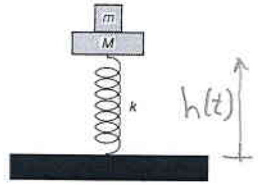
$0 = p_\pi^2 - 2p_\pi\cdot k_1 + k_1^2 = -m_\pi^{\cancel{2}} c^2 - 2\cancel{m_\pi}\gamma_v E_1 \frac{v}{c}\cos\theta$

$-2\cancel{m_\pi}\gamma_v E_1$

$= -m_\pi c^2 + 2\gamma_v E_1\left(1 - \frac{v}{c}\cos\theta\right) \qquad E_1 = \frac{\sqrt{1 - v^2/c^2}\ m_\pi c^2}{2\left(1 - \frac{v}{c}\cos\theta\right)}$

(5)

4. A platform of mass M is supported by a massless spring with spring constant k and equilibrium length l. The spring and platform are constrained to move in the vertical direction. A block of mass m rests on this platform.



(a) Find the equilibrium height of the block-platform combination. [2 points]
(b) Find the frequency of small oscillation about that equilibrium. [8 points]

The block-platform combination is pushed downward below this equilibrium position an additional distance D and then released from rest at $t = 0$.

(c) What is the largest value, $D_{\max}$, of D for which the block will never lose contact with the platform? [5 points]
(d) If the mass-platform combination is released from rest after being pushed downward by an amount $3D_{\max}$ what is the maximum height reached by the block? [5 points]

(a) Let $h(t)$ be heigt of blocks as above & h_0 be equilibrium height. Balance upward forces:

$$0 = -(M+m)g - (h_0 - l)k$$

or $h_0 = l - \frac{(M+m)g}{k}$

(b) Let $h(t) = x(t) + h_0$. Then

$$(M+m)\ddot{x} = -kx(t) \Rightarrow \ddot{x} = -\underbrace{\frac{k}{M+m}}_{\omega^2} x$$

$$\omega = \sqrt{\frac{k}{M+m}}$$

(6)

(c) When do the blocks separate?

Let F be upward force of M on m

$$m\ddot{h} = -mg + F$$
$$M\ddot{h} = -Mg - F - k(h-l)$$

multiply top equation by M and bottom equation by m & subtract to eliminate $\ddot{h}$

$$0 = MF + mF + k(h-l)m$$

$F = -k(h-l)\frac{m}{M+m}$ is positive for $h<l$

Thus, top block separates when $h = l$.

For $x(0) = -D$ & $\dot{x}(0) = 0$

$$h(t) = l - \frac{(M+m)g}{k} - D\cos\omega t$$

and the maximum value of $h(t)$ will be l (so masses separate)

if $D_{max} = \frac{M+m}{k}g$

(c) When do the blocks separate?

Let F be upward force of M on m

$$m\ddot{h} = -mg + F$$

$$M\ddot{h} = -Mg - F - k(h-\ell)$$

multiply top equation by M and bottom equation by m & subtract to eliminate $\ddot{h}$

$$0 = MF + mF + k(h-\ell)m$$

$F = -k(h-\ell)\dfrac{m}{M+m}$ is positive for $h<\ell$

Thus, top block separates when $h=\ell$.

For $x(0) = -D$ & $\dot{x}(0) = 0$

$$h(t) = \ell - \frac{(M+m)g}{k} - D\cos\omega t$$

and the maximum value of $h(t)$ will be ℓ (so masses separate)

if $D_{max} = \dfrac{M+m}{k}g$

(d) If we start with $D = 3D_{max}$

then $h(t) = \ell - D_{max} - 3D_{max}\cos\omega t$

they separate at t_0 when

$$-D_{max} - 3D_{max}\cos\omega t_0 = 0$$

or $\cos\omega t_0 = -\frac{1}{3}$

$v(t_0) = +3D_{max}\,\omega\sin\omega t_0$

so m will travel upward Δh where

$$\Delta h\, mg = \frac{1}{2}m\,v(t_0)^2$$

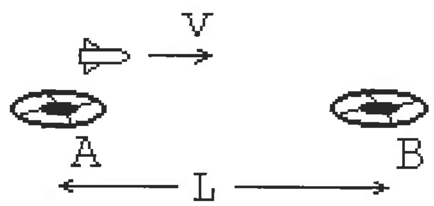
$$\Delta h \cancel{m} g = \frac{\cancel{m}}{2}9\left(\frac{(M+m)g}{k}\right)^2\frac{k}{M+m}\,\overbrace{\sin^2\omega t_0}^{1-\cos^2\omega t_0}$$

$$\Delta h = \frac{9}{2g}\cdot\frac{M+m}{k}\cdot g^2\,\underbrace{\left(1-\frac{1}{9}\right)}_{\frac{8}{9}}$$

Thus, m reaches the height

$$\ell + \Delta h = \ell + 4\frac{(M+m)g}{k}$$

5. Two space stations, A and B, are separated by a distance L and are at rest with respect to each other. The two stations lie in the same time zone and have accurately synchronized clocks. A rocket moving at a constant speed v travels from station A to station B. When the rocket pilot passes the first station A, she sets the ship's clock to agree with that of station A. For simplicity assume that they both read zero when the rocket passes station A.



(a) What is the time t_B shown on the clock of station B when the rocket goes by B? [5 points]

(b) What time is displayed on the ship's clock, t'_B when the rocket passes station B? Explain your result from the perspective of an observer in station B. [4 points]

(c) Explain the result found in (b) from the perspective of the rocket pilot. [3 points]

(d) Explain the result found in (a) from the perspective of the rocket pilot. [8 points]

(a) $t_B = L/v$

(b) $t'_B = \frac{1}{\gamma}\frac{L}{v}$ station B sees a moving clock running slow

(c) The rocket pilot sees the distance L between the stations as a moving length which is shortened

$t'_B = (L/\gamma) \times \frac{1}{v}$

(d) The rocket pilot assigns the coordinates $x' = 0$, $t' = L/\gamma v$ to the event of reaching B.

She performs a Lorentz transformation to predict

$$x_B = \gamma\,(x'_B + v t'_B) = \gamma\,(0 + v t'_B)$$
$$= \gamma v \left(\frac{L}{\gamma v}\right) = L \checkmark$$

$$t_B = \gamma\left(t'_B + \frac{v}{c^2} x'_B\right) = \gamma\left(t'_B + \frac{v}{c^2} 0\right)$$
$$= \gamma\left(\frac{L}{\gamma v}\right) = \frac{L}{v} \checkmark$$

She could also say that B's clock is running slow but reads an later time than A

$$t_B = \underbrace{\frac{1}{\gamma}\left(\frac{L}{\gamma v}\right)}_{\text{slow}} + \underbrace{\frac{v}{c^2} L}_{\text{out of sync}} = \frac{L}{v}\left(1 - \frac{v^2}{c^2} + \frac{v^2}{c^2}\right) = \frac{L}{v} \checkmark$$