Physics UN2802 Professor Christ
due 04/01/2021

Assignment #20

Don’t miss the problems on the second and third pages!

(Typo in 174.(a) corrected.)

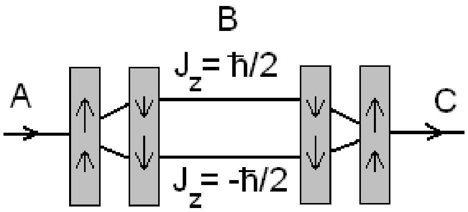
Reading:

March 30: French and Taylor 4-1 and 4-2

Problems:

170. Purcell 9.8 (Note, the center of the metal box lies on the line $x = \pi/2k$, $y = \pi/2k$.)

171. French and Taylor 2-5



172. The apparatus shown on the right uses a series of regions of inhomogenous magnetic field to separate an incoming beam of spin-1/2 neutral atoms into their $J_z = \pm\hbar/2$ components and then to recombine them back into a single beam. The magnetic fields are arranged so that the only effect on the quantum state when passing from region A to region B and from region B to region C is the spatial separation shown in the figure. Assume that the atoms in the incoming beam are each in the quantum state with $J_y = +\hbar/2$.

    (a) If an additional magnetic field in the $+z$-direction is imposed in region B on the atoms passing on the upper path ($J_z = +\hbar/2$) which rotates the phase of the states passing on the upper path through an angle ϕ, what will be the resulting average value of $\vec{J}$ found for the atoms in region C?

    (b) Assume that the atoms have $\vec{\mu} = \gamma\vec{J}$ with $\gamma = e/mc$ where e and m are the electron’s charge and mass. If the atoms are traveling at a speed of $3 \cdot 10^5$ cm/sec and this additional magnetic field is present for 10 cm of the upper path in region B, use the time translation operator $\exp(+i\vec{\mu} \cdot \vec{B}t/\hbar)$ to determine how strong an additional magnetic field (measured in Gauss) is needed in this region to produce a change in phase of $\phi = +\pi/2$ radians for the states passing through the upper path?

173. A spin-1/2 particle with magnetic moment $\vec{\mu} = \gamma \vec{J}$ is initially in the state

$$|+\frac{1}{2}\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Use the basis of states with $J_z = \pm\frac{\hbar}{2}$ and $J^i = \frac{\hbar}{2}\sigma^i$, $1 \leq i \leq 3$ where σ^i are the three conventional sigma matrices. A constant magnetic field $\vec{B} = B\hat{y}$ is imposed in the y direction.

(a) Find the resulting state $|\psi(t)\rangle$ as a function of the time t.

(b) Find the average values of the three components of the angular momentum of the particle at the time t: $\langle\psi(t)|J_i|\psi(t)\rangle$, $1 \leq i \leq 3$.

174. Consider a quantum system composed of **two** spin-1/2 particles, call them D and E. A general quantum state ψ can be written:

$$\psi = a_{11}|1\rangle_D \otimes |1\rangle_E + a_{12}|1\rangle_D \otimes |2\rangle_E + a_{21}|2\rangle_D \otimes |1\rangle_E + a_{22}|2\rangle_D \otimes |2\rangle_E \qquad (1)$$

where the individual states $|1\rangle_D$, $|2\rangle_D$, $|1\rangle_E$, $|2\rangle_E$ represent particle D with $J_z = +\hbar/2$, particle D with $J_z = -\hbar/2$, particle E with $J_z = +\hbar/2$, particle E with $J_z = -\hbar/2$ respectively.

Next, define the two angular momentum operators $\vec{J}^D$ and $\vec{J}^E$ which affect only the states $|d_i\rangle$ and $|e_i\rangle$ respectively and can be written in terms of Pauli matrices in the usual way. Thus,

$$\begin{aligned} J_i^D\left(|a\rangle_D \otimes |b\rangle_E\right) &= \frac{\hbar}{2}\sum_{k=1,2}\sigma^i_{k,a}(|k\rangle_D \otimes |b\rangle_E) & (2) \\ J_i^E\left(|a\rangle_D \otimes |b\rangle_E\right) &= \frac{\hbar}{2}\sum_{k=1,2}\sigma^i_{k,b}(|a\rangle_D \otimes |k\rangle_E), & (3) \end{aligned}$$

so that J_i^D affects only the $|a\rangle_D$ vectors and J_i^E affects only the $|b\rangle_E$ vectors. For example:

$$J_y^D\left(|2\rangle_D \otimes |1\rangle_E\right) = \frac{-i\hbar}{2}|1\rangle_D \otimes |1\rangle_E. \qquad (4)$$

Finally define the total angular momentum operator $\vec{J}^{\text{tot}} = \vec{J}^D + \vec{J}^E$.

(a) Show that the state $|1\rangle_D \otimes |1\rangle_E$ is an eigenstate of J_z^{tot} with eigenvalue $J_z = \hbar$.

(b) Show by direct evaluation that this state is also an eigenvector of the operator $(\vec{J}^{\text{tot}})^2 = (J_x^{\text{tot}})^2 + (J_y^{\text{tot}})^2 + (J_z^{\text{tot}})^2$ with eigenvalue $2\hbar^2$.

(c) Show by direct evaluation that the state $|1\rangle_D \otimes |2\rangle_E - |2\rangle_D \otimes |1\rangle_E$ is also an eigenstate of the operators J_z^{tot} and $(\vec{J}^{\text{tot}})^2$ with eigenvalue 0 for each.

175. If O is a hermitian operator, show that for any state $|\psi\rangle$ the matrix element $\langle\psi|O^2|\psi\rangle$ must be non-negative.

176. Consider a hermitian operator O acting on a complex vector space of finite dimension N. Consider a "sphere" of states $|\psi\rangle$ with unit norm, $|| \, |\psi\rangle \, ||^2 = 1$. Make the reasonable assumption that there should exist a vector $|\psi_0\rangle$ which lies on this sphere for which the expression,

$$\Big(|\psi_0\rangle, O|\psi_0\rangle\Big) = \langle\psi_0|O|\psi_0\rangle \tag{5}$$

takes a value less than or equal to the value of $\langle\psi|O|\psi\rangle$ found for any state $|\psi\rangle$ on this sphere. (Prove the truth of this assumption if you wish.) Consider a second, unit vector $|\delta\psi\rangle$ which lies in the $N-1$ dimensional space of vectors perpendicular to $|\psi_0\rangle$:

$$\Big(|\delta\psi\rangle, |\psi_0\rangle\Big) = \langle\delta\psi|\psi_0\rangle = 0. \tag{6}$$

(a) Define the vector

$$|\psi_\epsilon\rangle = \frac{1}{\sqrt{1+|\epsilon|^2}}\Big(|\psi_0\rangle + \epsilon|\delta\psi\rangle\Big) \tag{7}$$

where ϵ is a complex number. Show that the state $|\psi_\epsilon\rangle$ also has unit length.

(b) Since $|\psi_0\rangle$ is a state of unit length which minimizes the expectation value of O among those states of unit length we can conclude that

$$\Big(|\psi_\epsilon\rangle, O|\psi_\epsilon\rangle\Big) \geq \Big(|\psi_0\rangle, O|\psi_0\rangle\Big). \tag{8}$$

Show that if evaluated to first order in the small complex number ϵ this inequality implies that for all states $|\delta\psi\rangle$

$$\Big(|\delta\psi\rangle, O|\psi_0\rangle\Big) = 0. \tag{9}$$

(c) Explain how this result implies that $O|\psi_0\rangle$ must be proportional to $|\psi_0\rangle$ or

$$O|\psi_0\rangle = \lambda_0|\psi_0\rangle, \tag{10}$$

if the proportionality constant is labeled λ_0, proving O has at least one eigenvector.

(d) Consider the $N-1$ dimensional space of vectors that are orthogonal to $|\psi_0\rangle$ and repeat the above argument to find a second eigenvector $|\psi_1\rangle$ of O.

(e) Repeating the above procedure, deduce that any hermitian O acting on a finite dimensional vector space of dimension N can be diagonalized.