Physics UN2802 | Professor Christ
due 04/8/2021

Assignment #21

Don't miss the problems on the later pages 2 and 3

Problem 182. clarified

Reading:

April 6	French and Taylor 4-3
April 8	French and Taylor 4-4, 4-5

Problems:

177. French and Taylor 3-5

178. A physical "observable" $\mathcal{O}$ is represented in quantum mechanics by an operator O which acts on the vectors of a complex vector space. The set of vectors $|\lambda_n\rangle_{0\le n\le\infty}$ forms a basis for this vector space. Each of these vectors corresponds to a definite value λ_n of the observable $\mathcal{O}$ and obeys:

$$O|\lambda_n\rangle = \lambda_n|\lambda_n\rangle.$$

These states are normalized to unit length and are mutually orthogonal:

$$\begin{aligned}\langle\lambda_n|\lambda_m\rangle &= 1 \quad \text{for} \quad n=m \\ \langle\lambda_n|\lambda_m\rangle &= 0 \quad \text{for} \quad n\ne m.\end{aligned}$$

(a) Consider the quantum state

$$|\psi\rangle = \frac{1}{2}|\lambda_1\rangle + \frac{\sqrt{3}}{2}|\lambda_2\rangle.$$

What is the probability that a measurement of $\mathcal{O}$ will return the value λ_1? The value λ_2? What is the average result that will be obtained if O is measured many times on identical copies of the above state $|\psi\rangle$?

(b) For the quantum state

$$|\psi\rangle = N\sum_{n=0}^{\infty}\frac{a^n}{\sqrt{n!}}|\lambda_n\rangle$$

find the value of the constant N which will insure that this new state $|\psi\rangle$ has unit length: $\langle\psi|\psi\rangle = 1$. Here a is a fixed complex number.
(Hint: Use the Taylor series for the exponential $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$).

179. Consider a spin-1 particle, again using the basis of eigenstates of J_z now with eigenvalues $+\hbar$, 0 and $-\hbar$. Use the three, 3×3 angular momentum matrices given in problem 168. Assume the particle has magnetic moment $\vec{\mu} = \gamma \vec{J}$ and that a constant magnetic field B is applied in the $+z$ direction.

   (a) Initially the particle is in the state with $J_y = +\hbar$. Find that state in the prescribed basis. Assume this is the state of the system at $t = 0$.

   (b) Find the resulting state $|\psi(t)\rangle$ as a function of the time t.

   (c) Find the average values of the three components of the angular momentum of the particle at the time t: $\langle\psi(t)|J_i|\psi(t)\rangle$, $1 \le i \le 3$.

180. If the quantum state of a particle is given by the wave function

$$\psi(x) = N \begin{cases} 0 & x \le -d \\ d + x & -d \le x \le 0 \\ d - x & 0 \le x \le d \\ 0 & d \le x \end{cases}$$

   (a) Determine the constant N so the wave function $\psi(x)$ will obey the normalization condition

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

   (b) What is the probability that the particle will be found in the interval $-d/2 \le x \le +d/2$?

181. Show that if the wave function $\psi(x)$ is real then its Fourier transform, $\widetilde{\psi}(p)$ obeys the condition $\widetilde{\psi}(p)^* = \widetilde{\psi}(-p)$. Recall that

$$\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ipx/\hbar}\, \psi(x) dx.$$

182. Consider a particle moving in a periodic box of length 20 and unit mass in an appropriate set of units in which $\hbar = 1$. Assume that initially the particle is in a localized state described by the wave function

$$\psi(x) = N \begin{cases} 0 & 0 \le x \le x_1 \\ 1 - \cos\left(2\pi \frac{x - x_1}{x_2 - x_1}\right) & x_1 \le x \le x_2 \\ 0 & x_2 \le x \le 20 \end{cases},$$

where $x_1 = 2$ and $x_2 = 4$. Use Python to answer the following questions. It may be convenient to build upon the two Fourier series examples linked to the web site: PythonPage.

(a) Determine the normalization constant N accurate to four decimal places. [Here using the Python routine numpy.quad, introduced in the second Fourier series Python example, to numerically evaluate an integral that you could compute analytically may be the easiest approach.]

(b) Express the wave function $\psi(x)$ as a Fourier series using the quantized momenta $p_m = 2\pi m/L$ where $-100 \leq m \leq 100$. Submit as your answer to this part of the problem a graph of the difference between the exact function $\psi(x)$ and the function obtained from this truncated Fourier series approximation. [As in part (a), the recommended approach is to use numpy.quad to simplify the evaluation of the Fourier coefficients – let the computer do the work for you.]

(c) Introduce the energy operator $H = p^2/2$ and use the time development

$$|\psi(t)\rangle = e^{-i\frac{p_{\text{op}}^2}{2}t}|\psi(t=0\rangle$$

implied by the Schrodinger equation:

$$i\frac{\partial}{\partial t}\psi(x,t) = \frac{p_{\text{op}}^2}{2}\,\psi(x,t)$$

to determine $\psi(x,t)$ at the later times $t = 0.2n$ where the integer n varies between 0 and 6. (Recall that the action of p_{op} that appears in the energy operator H is very simple when acting on each term in the Fourier series expansion worked out part (b).) Submit a composite graph plotting $\psi(x)$ for each of these times as your answer to this part of this problem.

(d) Extend the length of the periodic box to 80 and repeat your answer to part (c) but now use the initial wave function

$$\psi_{\text{new}}(x) = e^{ip_0x}\psi(x),$$

where the constant momentum $p_0 = 60$. [Here we are using the operator $\exp(ip_0x_{\text{op}}/\hbar)$ to "translate" the average momentum of the state $|\psi\rangle$ from zero to p_0.]