Physics UN2802 Professor Christ
due 04/15/2021

Assignment #22

Don't miss the problems on page two

Problem 186 has been clarified.

Reading:

April 12 French & Taylor Chapter 8.1-8,6, Optional: Nielsen & Chuang 1.1-1.3
April 15 French & Taylor Chapter 8.7-8.12 Optional: Nielsen & Chuang 1.4

Problems:

183. Consider a wave function for a quantum state defined in three dimensions which depends on the three variable x, y and z: $\psi(x, y, z)$. Define the three "orbital" angular momentum operators by:

$$\begin{aligned} L_x &= y(-i\hbar\frac{\partial}{\partial z}) - z(-i\hbar\frac{\partial}{\partial y}) \\ L_y &= z(-i\hbar\frac{\partial}{\partial x}) - x(-i\hbar\frac{\partial}{\partial z}) \\ L_z &= x(-i\hbar\frac{\partial}{\partial y}) - y(-i\hbar\frac{\partial}{\partial x}) \end{aligned}$$

(a) Show that $[L_x, L_y] = i\hbar L_z$

(b) Show that the quantum state $\psi(x, y, z) = N \exp\{-(x^2 + y^2 + z^2)/2D^2\}$ is a zero eigenvector of L_i for each i:

$$L_i\psi(x, y, z) = 0.$$

(c) Find the eigenvalues for the operator L_z for each of the states below:

$$\begin{aligned} \psi_A(x, y, z) &= N_A(x + i\,y)e^{-(x^2+y^2+z^2)/2D^2} \\ \psi_B(x, y, z) &= N_B(z)e^{-(x^2+y^2+z^2)/2D^2} \\ \psi_C(x, y, z) &= N_C(x - i\,y)e^{-(x^2+y^2+z^2)/2D^2}. \end{aligned}$$

184. Consider a particle of charge q and mass m whose one-dimensional motion is determined by the simple harmonic oscillator Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2.$$

(a) What is the normalized wave function for the ground state and what are the allowed energies for this system?

(b) An electric field E parallel to the one-dimensional motion is suddenly applied. What is the new Hamiltonian, the new ground state wave function and the new allowed energies?

(c) Assume that the system was initially in its ground state and that the electric field was turned on so suddenly that the system remains in that initial state. What is the probability that a measurement of the energy of the system after the electric field has been applied returns the new ground state value? (Hint: this can be determined from the wave functions or by relating the lowering operators for the initial and final systems and using the coherent-state analysis presented in class.)

185. Consider the tensor product of two complex vector spaces D_M and E_N with dimensions M and N respectively. Let $|d_i\rangle$, $1 \le i \le M$ and $|e_j\rangle$, $1 \le j \le N$ be orthonormal sets of basis vectors for each of these vector spaces. Thus, vectors u and v in D_M and E_N can be written:

$$u = \sum_{i=1}^{M} u_i |d_i\rangle$$
$$v = \sum_{j=1}^{N} v_j |e_j\rangle.$$

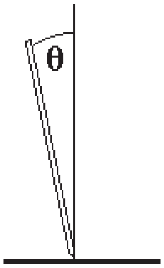
Define the $M \times N$ combinations $|d_i\rangle \otimes |e_j\rangle$, $1 \le i \le M$, $1 \le j \le N$ as an orthonormal basis in the tensor product $D_M \otimes E_N$ of the vector spaces D_M and E_N. If the product $u \otimes v$ is defined as:

$$u \otimes v = \sum_{i=1}^{M} \sum_{j=1}^{N} u_i v_j |d_i\rangle \otimes |e_j\rangle, \tag{1}$$

show that if w and x are elements of D_N and E_N respectively then the inner products of u, v, w and x obey:

$$(w \otimes x, u \otimes v) = (w, u) \cdot (x, v). \tag{2}$$

186. Combine uncertainty principle constraints on the initial conditions with a classical description of the resulting time-dependent motion to estimate the maximum length of time a pencil can be balanced on its point. Simplify the problem by treating the pencil's motion as restricted to a two-dimensional plane. Locate the pencil's position by the angle θ that it makes with the vertical direction.



(a) Make a rough numerical estimate in cgs units of the moment of inertia, I, of the pencil, rotating about its point in the 2-dimensional plane.

(b) Given an initial value of $\theta = \theta_0$ and the initial angular momentum L_0, find the resulting angular position, $\theta(t)$ at a later time t predicted by classical mechanics in the approximation that $\theta(t) \ll 1$. (Hint: this is similar to simple harmonic motion but now calculated about a point of unstable equilibrium, with sinh and cosh solutions.)

(c) Determine initial values θ_0 and L_0 which are consistent with the uncertainty relation $\Delta\theta\ \Delta L \ge \hbar$ and yet allow the pencil to remain vertical for as long a time as possible.

(d) Roughly, how long can the pencil remain with $\theta < 0.1$?