01/14/2021

UN2802 Accelerated Physics

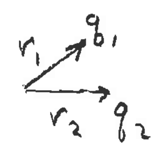
E & M followed by Quantum Mechanic

course website:

www.columbia.edu/~nhc1/UN2802

We have covered most of electro-statics

- Coulombs law:



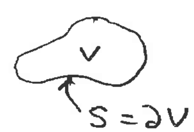
$$\vec{F}_{q_1 \text{ on } q_2} = q_1 q_2 \frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|^3}$$

- Electric field:

$$E(\vec{r}) = \sum_{i=1}^{N} q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3}$$

$$\vec{F}_{\substack{\text{on } \delta q \\ \text{at } \vec{r}}} = \delta q \, \vec{E}(\vec{r})$$

- Gauss' Law:



$$\int_{\partial V} \vec{E} \cdot \hat{n} \, ds = 4\pi \int_V \rho(\vec{r}) \, d^3 r$$

- Gauss' Theorem:

$$\int_{\partial V} \vec{E} \cdot \hat{n} \, ds = \int_V \vec{\nabla} \cdot \vec{E} \, d^3 r$$

$$\therefore \vec{\nabla} \cdot \vec{E} = 4\pi \rho$$

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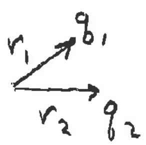

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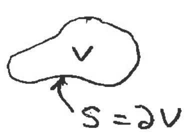

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$$\therefore \quad \vec{\nabla} \cdot \vec{E} = 4\pi \rho$$

- Scalar potential: $\vec{\nabla} \times \vec{E} = 0$ since the Coulomb force is radial. Define

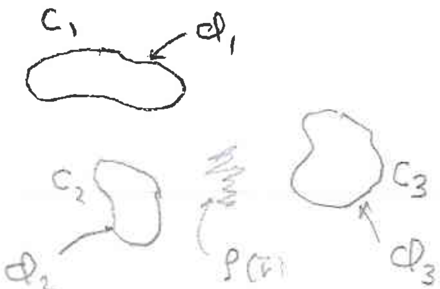
$$\varphi(\vec{r}) = -\oint_{r_0}^{\vec{r}} \vec{E}(r') \cdot d\vec{r}' \Rightarrow \vec{E}(\vec{r}) = -\vec{\nabla} \varphi$$

- Poisson's equation: $\vec{\nabla}^2 \varphi = -\vec{\nabla} \cdot \vec{E} = -4\pi \rho$

- Conductors: material within which charges can move so for electrostatics where there can be no flow of charge. $\vec{E} = 0$ inside and even parallel to the surface of a conductor

- Uniqueness theorem:

  Given conductors at specified potentials and a specified charge density $\rho(\vec{r})$



$$\varphi(r)\Big|_{r \in C_i} = \varphi_i$$

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$$\Rightarrow \varphi(\vec{r}) \text{ is unique}$$

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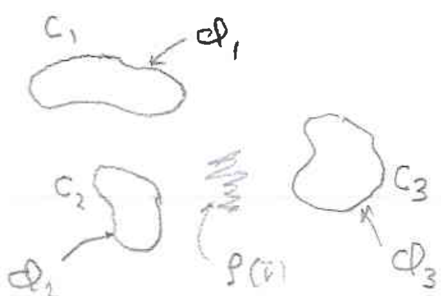

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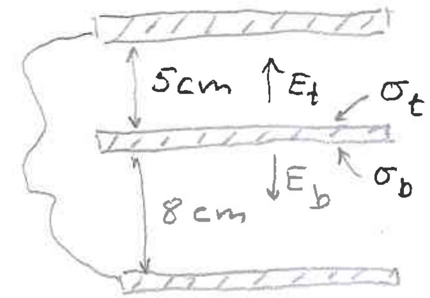
4. Complete problem #38 in Purcell



- large parallel conducting plate
- 10 esu/cm² placed on center plate
- top and bottom plate connect by wire
- find charge density on both surfaces of each plate

$\vec{E}$ is normal to plate

$$E\,\delta A = 4\pi\sigma\,\delta A \quad \text{or} \quad E = 4\pi\sigma$$

$$\sigma_t + \sigma_b = 10\ \text{esu/cm}^2$$

Top & bottom plate at same potential implies $5\text{cm}\,\vec{E}_t = 8\text{cm}\,\vec{E}_b$

$$5\sigma_t = 8\sigma_b \Rightarrow \sigma_b = \frac{5}{8}\sigma_t$$

$$\sigma_t = 10 - \sigma_b = 10 - \frac{5}{8}\sigma_t$$

$$\sigma_t\left(1+\frac{5}{8}\right) = 10 \qquad \sigma_t = \frac{8}{13}\,10\ \text{esu} \qquad \sigma_b = \frac{5}{13}\times 10\ \text{esu}$$

What about top & bottom plates

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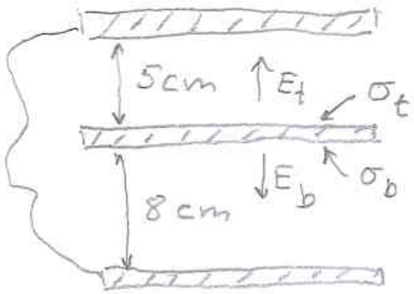

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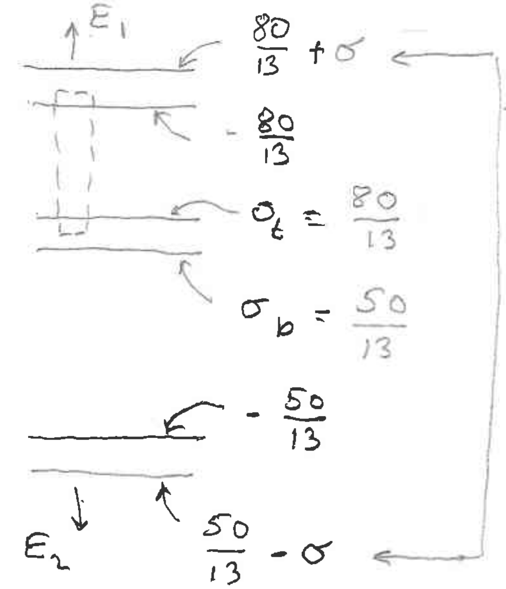


$+\sigma$ & $-\sigma$ require because together top and bottom plate carry no charge

Gauss' law requires

$E_1 + E_2 = 4\pi\cdot 10$

perhaps $E_1 = E_2 = 4\pi\cdot 5$ esu ?

$\frac{80}{13} + \sigma = \frac{50}{13} - \sigma \Rightarrow \sigma = \frac{1}{2}\left(\frac{50}{13} - \frac{80}{13}\right)$

$= -\frac{15}{13}$ esu

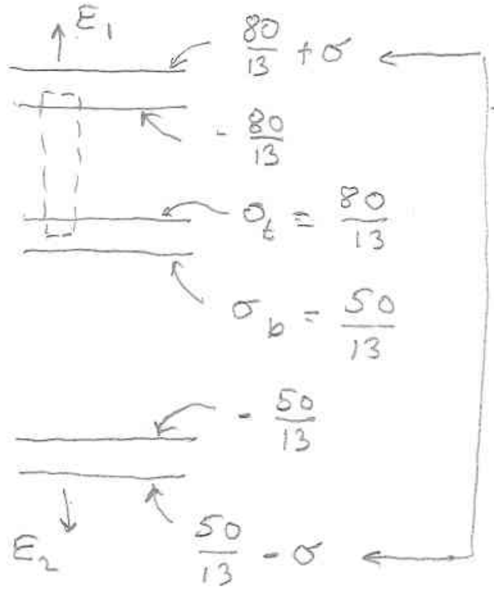
Proof energy density $= \frac{1}{8\pi}\vec{E}^2$

Energy $\propto (E_1 + \delta E)^2 + (-E_1 + \delta E)^2$

minimize: $\frac{\partial}{\partial\,\delta E}\left\{(E_1+\delta E)^2 + (-E_1+\delta E)^2\right\} = 0$

$= 2(E_1+\delta E) + (-E_1+\delta E) = 4\delta E = 0$

$\Rightarrow \delta E = 0$



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E Capacitance

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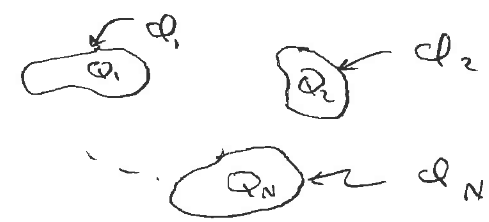

$\boxed{\text{Here choose } \Phi(r)\big|_{|\vec{r}|\to\infty} = 0}$

Solve N different problems:

k^{th} problem: $\Phi_i^{(k)} = \begin{cases} \Phi_k & i = k \\ 0 & i \neq k \end{cases}$

uniqueness theorem determines

$$\Phi^{(k)}(\vec{r}) \ \& \ Q_i^{(k)} = \frac{1}{4\pi}\int_{\partial C_i} [-\vec{\nabla}\cdot\Phi]\cdot\hat{n}\, ds$$

Since $\Phi^{(k)}(\vec{r})\big|_{\vec{r}\in C_k} = \Phi_k$

$$Q_i^{(k)} = C_{ik}\Phi_k \qquad 1 \le i \le k$$

All $Q_i^{(k)}$ are proportional to Φ_k

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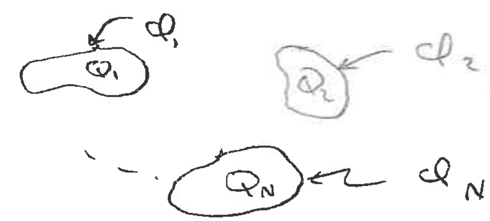

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Finally use the "principle of superposition" and add together these N solutions:

- $\Phi(\vec{r}) = \sum_{k=1}^{N} \Phi^k(\vec{r})$,
- $\Phi(\vec{r})\big|_{\vec{r}\in C_k} = \Phi_k$
- $Q_i = \sum_k Q_i^{(k)} = \sum_k C_{ik}\Phi_k$

coefficients of capacitance

2. Easiest example is a single conducting sphere of radius R

$$\Phi(r) = \begin{cases} \frac{Q_1}{r} & r > R \\ \frac{Q_1}{R} & r \le R \end{cases}$$

$$\Phi_1 = \Phi(\vec{r})\big|_{|\vec{r}|=R} = \frac{Q_1}{R}$$

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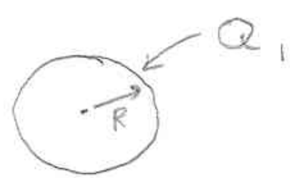

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3. For two conductors there are four C_{ij}:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \overbrace{\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}}^{C} \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix}$$

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = C^{-1}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \frac{1}{C_{11}C_{22} - C_{12}C_{11}} \begin{pmatrix} C_{22} & -C_{12} \\ -C_{21} & C_{11} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

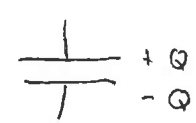
most important case is a neutral system with $Q_2 = -Q_1 = -Q$

$$\begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} = \frac{1}{C_{11}C_{22} - C_{12}C_{11}} \begin{pmatrix} C_{22}Q + C_{12}Q \\ -C_{21}Q - C_{11}Q \end{pmatrix}$$

So potential difference between C_1 & C_2

$$\Phi = \Phi_1 - \Phi_2 = \underbrace{\frac{C_{22} + C_{12} + C_{21} + C_{11}}{C_{11}C_{22} - C_{12}C_{11}}}_{1/C}\,Q$$

C is the capacitance of the two conductor system



$Q = CV$ where V is potential difference between two conductors

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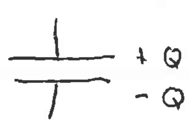

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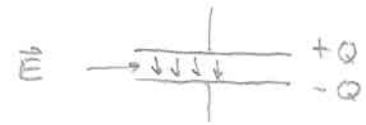
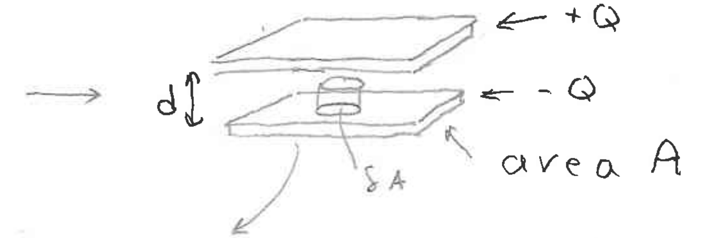
4. Parallel plate capacitor





$$\sigma = \frac{Q}{A}$$

$$E\,\delta A = -4\pi\sigma\,\delta A = -4\pi \frac{Q}{A}\,\delta A$$

$$\Phi_{top} - \Phi_{bot} \equiv V = -E \cdot d = \frac{4\pi Q}{A} d$$

or $V = \frac{1}{C} Q$ & $C = \frac{A}{4\pi d}$

5. Units

for esu $\quad C = \frac{Q}{V} \sim$ cm

for SI $\quad C = \frac{Q}{V} \sim \frac{\text{coulomb}}{\text{volt}} \equiv$ Farad

$$1 \text{ Farad} = \frac{1 \text{ Coulomb}}{1 \text{ Volt}} = \frac{3 \times 10^9 \text{ esu}}{\frac{1}{300} \text{ stat-volt}} = 9 \times 10^{11} \text{ cm}$$

$\uparrow$ $(2.99792458)^2$

a Farad is a large unit

a cm " " small unit