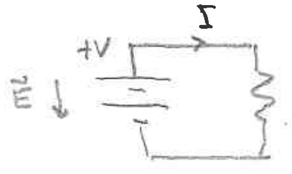
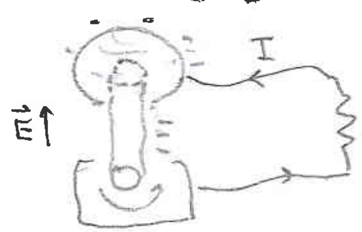
Actually there are two sources of EMF:

① $\oint_C \vec{E}\cdot d\vec{r} = 0$ and charge is forced by quantum mechanics to move against the $\vec{E}$ field



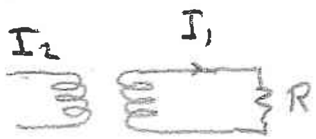
battery

or



Van de Graaff generator

② $\oint_C \vec{E}\cdot \vec{d} = -\frac{1}{c}\frac{d\Phi_B}{dt}$ Faraday's law



↑ large σ wire, $\vec{E}=0$

$IR = EMF = M_{12}\frac{dI_2}{dt}$

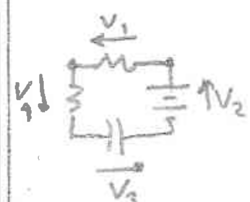
mutual inductance

We can now solve a multitude of problems with resistors, capacitors and batteries connected in interesting ways

Properties of electrostatics modified to allow current flow but forbidding net charge or rapidly changing currents;

Kirchoff's laws:

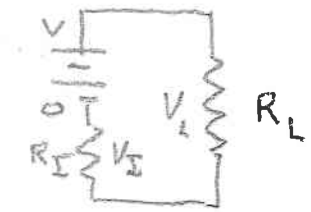
① Sum of all currents entering a node is zero



② Sum of all voltages around any closed circuit is zero.

Solve two problems:

$V = V_I + V_L$



a) Consider a battery with internal resistance R_I connected to a load resistance R_L. What value of R_L results in the largest power dissipation in R_L?

Power lost in R_L $= \frac{\Delta Q\, V_L}{\Delta t} = I V_L = I^2 R_L$

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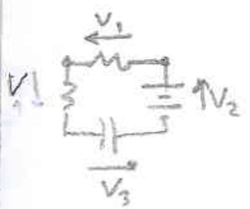

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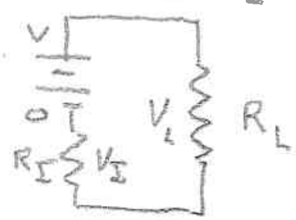

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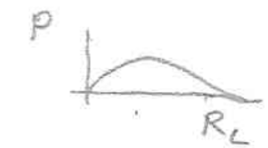
$V = I(R_I + R_L) \qquad P = \left[\dfrac{V}{R_L + R_I}\right]^2 R_L$

$P \to 0$ as $R_L \to 0$ or $R_L \to \infty$



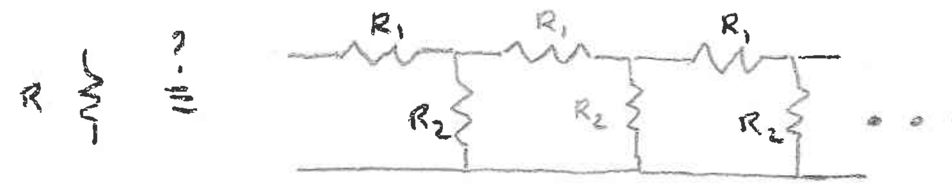
Find maximum:

$$0 = \frac{dP}{dR_L} = \frac{V^2}{(R_L + R_I)^2}\left[1 - \frac{2R_L}{R_L + R_I}\right]$$

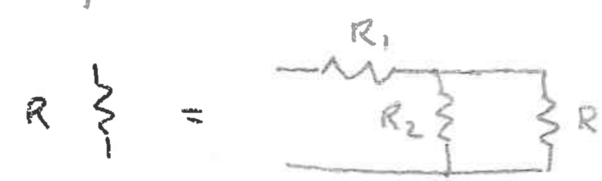
$$= \frac{V^2}{(R_L + R_I)^3}\left[R_I - R_L\right]$$

Maximum power dissapated in R_L when $R_L = R_I$: example of "impedance matching".

b) Find equivalent resistance to the infinite seriers of resistors



Recognize that R must obey



or $R = R_1 + \dfrac{1}{\frac{1}{R_2} + \frac{1}{R}}$

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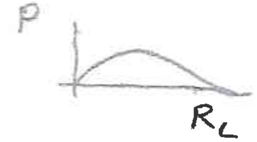

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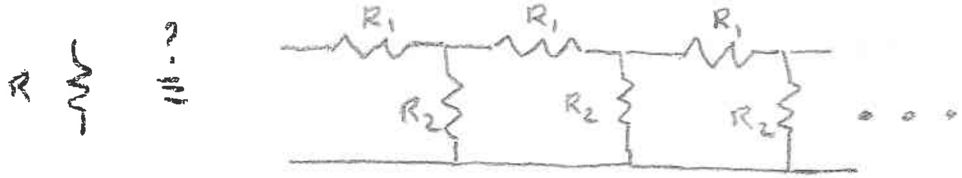

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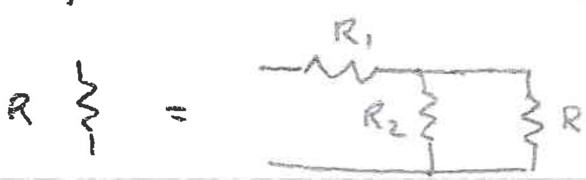

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or $R = R_1 + \dfrac{R_2 R}{R_2 + R}$

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or $R^2 - R R_1 - R_1 R_2 = 0$

or $R = \dfrac{R_1 \pm \sqrt{R_1^2 + 4R_1R_2}}{2}$

$= R_1\left[\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{R_2}{R_1}}\right]$

$R \to \begin{cases} \infty & \text{as } R_1 \to \infty \text{ or } R_2 \to \infty \\ R_1 & \text{as } R_2 \to 0 \end{cases}$

V Electrodynamics

A Motivation

1. Overview: we know

① $\vec\nabla \cdot \vec E = 4\pi\rho$ true in general

$\vec\nabla \times \vec E = 0$ true only when charge moves slowly

$\downarrow$

② $\vec\nabla \times \vec E + \frac{1}{c}\frac{\partial \vec B}{\partial t} = 0$ Faraday's law or $R = R_1 + \frac{R_2 R}{R_2 + R}$

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What about $\vec{E}$ & forces of charge?

$\frac{d\vec{p}}{dt} = q\vec{E}(\vec{r})$ for a charge at $\vec{r}$ with momentum $\vec{p}$

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$\frac{d\vec{p}}{dt} = q\left[\vec{E}(\vec{r}) + \frac{\vec{v}}{c} \times B(\vec{r})\right]$

consistent with special relativity if $\vec{p} = m\gamma_v \vec{v}$.

How is $\vec{B}(\vec{r})$ determined:

③ $\vec{\nabla} \cdot \vec{B} = 0$

④ $\vec{\nabla} \times \vec{B} \underbrace{- \frac{1}{c}\frac{\partial \vec{E}}{\partial t}}_{\text{Maxwell's displacement current}} = \underbrace{\frac{4\pi}{c}\vec{j}}_{\text{Ampere's law}}$

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Maxwell's equations were discovered from observations and experiment. They predict (require) relativity. We will go backwards: start with relativity and $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ and induce Maxwell's equations

2. Make $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ consistent with special relativity

$$\int_{\partial V} \vec{E} \cdot \hat{n} \, ds = 4\pi \int_V \rho(\vec{r})\, d^3r$$

is inconsistent with causality unless the only way $Q = \int_V \rho(\vec{r})\, d^3r$ can change is if current flows out through the boundary so that such flow at the boundary can directy affect $\vec{E}$ at the boundary!

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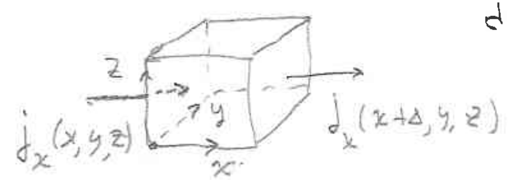
This is exactly what results if we relate the charge density $\rho(\vec{r})$ with the current density $\vec{J}(\vec{r})$ and require

$$\frac{d}{dt}\underbrace{\int_V \rho(\vec{r},t)\, d^3r}_{Q} = -\int_{\partial V} \vec{J}(\vec{r},t)\cdot\hat{n}\, ds :$$

The only way Q can change is for charge to flow through the surface of V: Charge cannot be created or destroyed - charge is <u>conserved</u>.

Consider the limit where V is very small:

$$\frac{d}{dt}\underbrace{\int_V \rho(\vec{r},t)\, d^3r}_{\sim \rho(\vec{r},t)\cdot V} = -\int_{\partial V} \vec{J}(\vec{r},t)\cdot\hat{n}\, ds$$



$$= -\int_V \vec{\nabla}\cdot\vec{J}(\vec{r},t)\, d^3r$$

$$\approx -\vec{\nabla}\cdot\vec{J}(\vec{r},t)\cdot V$$

Thus, $\dfrac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0$

the <u>continuity equation</u>

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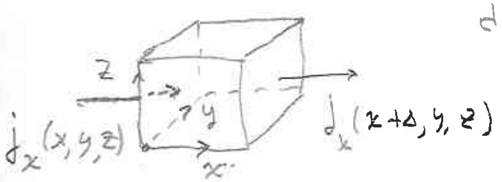

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the continuity equation

This continuity equation tells us to important things

① $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ is not obviously inconsistent with causality.

② $(j_x, j_y, j_z, c\rho)$ transform like the components of a four-vector.

Proof of ②: Since

$$0 = \vec{\nabla}\cdot\vec{j} + \frac{\partial}{\partial t}\rho = \frac{\partial}{\partial x}j_x + \frac{\partial}{\partial y}j_y + \frac{\partial}{\partial z}j_z - \left(-\frac{\partial}{\partial ct}\right)c\rho$$

must be zero in all reference systems and $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, -\frac{\partial}{\partial(ct)}\right)$ transforms as a four-vector, our equation must be the Lorentz-invariant product of two four-vectors $\Rightarrow$ $(\vec{j}, c\rho)$ is a 4-vector

(x, y, z) & $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ are 3-vectors

Why the minus sign in $\left(\vec{\nabla}, -\frac{1}{c}\frac{\partial}{\partial t}\right)$?

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Proof of 2: Since

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Why the minus sign in $\left(\vec{\nabla}, -\frac{1}{c}\frac{\partial}{\partial t}\right)$?

Comes from minus sign in the Lorentz-invariant length. Assume $f(x)$ is a Lorentz invariant function of x, then

$$f(x+\delta x) = f(x) \underbrace{+\left(\frac{\partial}{\partial x_1} f(x)\right)\delta x_1 + \left(\frac{\partial}{\partial x_2} f(x)\right)\delta x_2 + \left(\frac{\partial}{\partial x_3} f(x)\right)\delta x_3 + \left(\frac{\partial}{\partial ct} f(x)\right) c\,\delta t}_{\text{must be Lorentz invariant}} + O(\delta x^2)$$

use $x_4 = ct$

$\sum_{\mu=1}^{4} \frac{\partial f}{\partial x_\mu}\delta x_\mu$ can be Lorentz invariant only if is the dot product between two four vectors

$$= \sum_{\mu=1}^{3} \frac{\partial f}{\partial x_i}\delta x_i - \left(-\frac{\partial f}{\partial (ct)}\right)\delta(ct)$$

the needed minus

Useful language:

x_μ is a contravariant vector

$\frac{\partial}{\partial x_\mu}$ is a covariant vector