Comes from minus sign in the Lorentz-invariant length. Assume $f(x)$ is a Lorentz invariant function of x, then

$$f(x+\delta x) = f(x) \overbrace{+ \left(\frac{\partial}{\partial x_1} f(x)\right)\delta x_1 + \left(\frac{\partial}{\partial x_2} f(x)\right)\delta x_2 + \left(\frac{\partial}{\partial x_3} f(x)\right)\delta x_3 + \left(\frac{\partial}{\partial ct} f(x)\right) c\,\delta t}^{\text{must be Lorentz invariant}} + O(\delta x^2)$$

use $x_4 = ct$

$$\sum_{\mu=1}^{4} \frac{\partial f}{\partial x_\mu}\,\delta x_\mu$$ can be Lorentz invariant only if is the dot product between two four vectors

$$= \sum_{\mu=1}^{3} \frac{\partial f}{\partial x_i}\,\delta x_i - \left(-\frac{\partial f}{\partial (ct)}\right)\delta(ct)$$

the needed minus

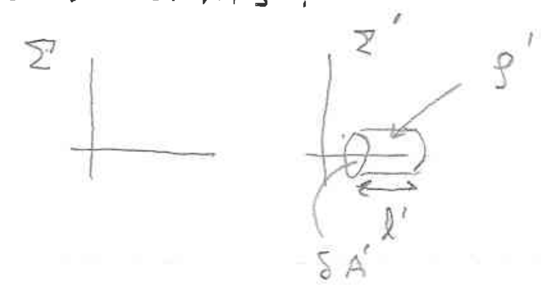
Useful language:

x_μ is a contravariant vector

$\frac{\partial}{\partial x_\mu}$ is a covariant vector

01/26/2021

Knowing how $(\vec{J}, c\rho)$ transform we can determine how charge transforms:



in Σ'

$$Q' = \delta A' \, l' \rho'$$

$$\vec{J}' = 0$$

Σ calculates the charge Q in the moving volume

$$Q = l\,\delta A\,\rho = \frac{l'}{\gamma}\,\delta A'\rho$$

where $c\rho = \gamma(c\rho' + \frac{v}{c}\underset{0}{\underset{\parallel}{J'}}) = c\gamma\rho'$

$$\therefore\ Q = \frac{l'}{\gamma}\,\delta A'\,\gamma\rho' = Q'$$

and charge is Lorentz invariant!

We began with the idea that charge must be conserved $\sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} J_\mu = 0$ and concluded that charge was a Lorentz scalar — this is too strong!

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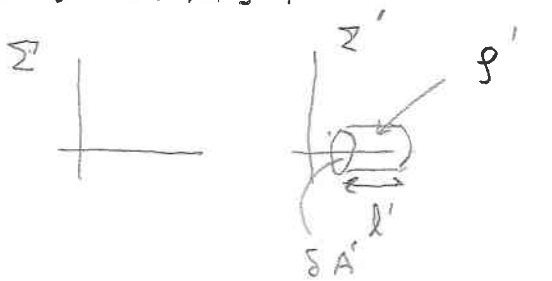

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$\sum_{\mu=1}^{4}\frac{\partial}{\partial x_\mu} j_\mu = 0 \Rightarrow j_\mu$ is a 4-vector is the simpest possibility. There could be an extra index:

$$\sum_{\mu=1}^{4}\frac{\partial}{\partial x_\mu} T_{\mu\nu} = 0 \qquad \nu = 1, 2, 3, 4$$

This would require 4 types of "charges" not one - inconsist with Nature.

However, if we are not discussing charge but momentum and energy there ARE four quantities and

$T_{\mu,4}(\vec{r})$ is the vector of currents $(\mu = 1,2,3)$ and density of energy/c

an $T_{\mu,i}(\vec{r})$ is the vector of currents $(\mu = 1,2,3)$ and density of momentum in the i^{th} direction

$T_{\mu\nu}$ is the energy-momentum tensor

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How can we view $\vec{\nabla}\cdot\vec{E} = \frac{4\pi}{c} j_4$ as an equation connecting 4-vector?

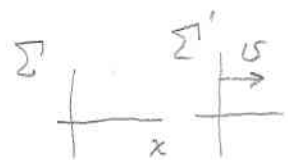
We could guess that

$$\vec{\nabla}\cdot\vec{E} = \sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} F_{\mu 4} = \underbrace{\sum_{i=1}^{3} \frac{\partial F_{i4}}{\partial x_i}}_{\vec{\nabla}\cdot\vec{E}} + \underbrace{\frac{\partial F_{44}}{\partial (ct)}}_{?}$$

What should we do about extra term ?

Lorentz transform $F_{\mu\nu}$ from Σ to Σ':

Recall $x'_\mu = \sum_{\nu=1}^{4} L_{\mu\nu} x_\nu$ ↖ 4x4 matrix



$$L_{\mu\nu} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\frac{v}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{v}{c}\gamma & 0 & 0 & \gamma \end{pmatrix}$$

Then $F'_{\mu\nu} = \sum_{\alpha,\beta} L_{\mu\alpha} L_{\nu\beta} F_{\alpha\beta}$

If $F_{\alpha\beta} = \pm F_{\beta\alpha}$, then $F'_{\mu\nu} = \pm F_{\nu\mu}$

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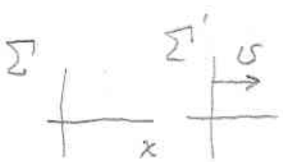

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Then $F_{44} = -F_{44} \Rightarrow F_{44} = 0$.

Assume

$$F_{\mu\nu} = \begin{pmatrix} 0 & -B_3 & B_2 & E_1 \\ B_3 & 0 & -B_1 & E_2 \\ -B_2 & B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix}$$

and $\vec{\nabla}\cdot\vec{E} = 4\pi\rho = \frac{4\pi}{c} j_4$ becomes

four equation $\sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} F_{\mu\nu} = \frac{4\pi}{c} j_\nu$ $\nu = 1, 2, 3, 4$

What equation comes from $\nu = 3$

$$\underbrace{\frac{\partial}{\partial x_1} B_2 - \frac{\partial}{\partial x_2} B_1}_{(\vec{\nabla}\times\vec{B})_3} - \frac{\partial}{\partial ct} E_3 = \frac{4\pi}{c} j_3$$

or $\vec{\nabla}\times\vec{B} - \underbrace{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}_{} = \frac{4\pi}{3}\vec{j}$ & $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$

magnetic field

Maxwell's displacement current

Ampere's law

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How do $\vec{E}$ & $\vec{B}$ transform?

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$$= \begin{pmatrix} 0 & -\gamma B_3' - \beta\gamma E_2' & \gamma B_2' - \beta\gamma E_3' & \gamma^2(1-\beta^2)E_1' \\ \gamma B_3' + \beta\gamma E_2' & 0 & -B_1' & \gamma E_2' + \gamma\beta B_3' \\ -\gamma B_2' + \beta\gamma E_3' & B_1' & 0 & \gamma E_3' - \gamma\beta B_2' \\ \gamma^2(\beta^2-1)E_1' & -\beta\gamma B_3' - \gamma E_2 & \beta\gamma B_2' - \gamma E_3' & 0 \end{pmatrix}$$

$B_3 = \gamma B_3' + \gamma\beta E_2'$ $\qquad$ $E_3 = \gamma E_3' - \gamma\beta B_2'$

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$B_1 = B_1'$ $\qquad$ $E_1 = E_1'$

use $\parallel$ & $\perp$ for components $\parallel$ & $\perp$ to $\vec{v}$

$B_\parallel = B_\parallel'$ $\qquad$ $E_\parallel = E_\parallel'$

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This is an interesting alternative to the transformation of $(\vec{x}, ct)$:

$$x_\parallel = \gamma\left(x_\parallel' + \frac{v}{c}\cdot ct'\right) \qquad ct = \gamma\left(ct' + \frac{v}{c}x_\parallel\right)$$
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$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} F_{\mu\nu} = \frac{4\pi}{c} j_\nu \Rightarrow \begin{matrix} \vec{\nabla}\cdot\vec{E} = \frac{4\pi}{c}\rho \\ \vec{\nabla}\times\vec{B} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \end{matrix}$$

tells use $\vec{\nabla}\cdot\vec{E}$ & $\vec{\nabla}\times\vec{B}$. What about $\vec{\nabla}\times\vec{E}$ & $\vec{\nabla}\cdot\vec{B}$?

Observe replacing $\vec{E}$ by $\vec{B}$ and $\vec{B}$ by $-\vec{E}$ does not change transformation laws:

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The same!

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The same!

Thus

$$F_{\mu\nu} = \begin{pmatrix} 0 & -B_3 & B_2 & E_1 \\ B_3 & 0 & -B_1 & E_2 \\ -B_2 & B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix} \; \& \; \tilde F_{\mu\nu} = \begin{pmatrix} 0 & E_3 & -E_2 & B_1 \\ -E_3 & 0 & E_1 & B_2 \\ E_2 & -E_1 & 0 & B_3 \\ -B_1 & -B_2 & -B_3 & 0 \end{pmatrix}$$

Both transform like 2nd rank tensors

Missing equations are $\displaystyle\sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu}\tilde F_{\mu\nu} = \frac{4\pi}{c} j^*_\nu$

j^*_ν will be a new current of magnetic charge which has never been seen: $j^*_\nu = 0$

$$\sum_{\mu=1}^{4} \frac{\partial \tilde F_{\mu\nu}}{\partial x_\mu} = 0 \quad \Rightarrow \quad \begin{aligned} &\vec\nabla\cdot\vec B = 0 \\ &\underbrace{\vec\nabla\times\vec E - \frac{1}{c}\frac{\partial \vec B}{\partial t} = 0}_{\text{Faraday's law}} \end{aligned}$$

One final equation is need before all of E & M is complete:

How does $\vec B$ affect a charge?