Thus

$$F_{\mu\nu} = \begin{pmatrix} 0 & -B_3 & B_2 & E_1 \\ B_3 & 0 & -B_1 & E_2 \\ -B_2 & B_1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{pmatrix} \quad \& \quad \tilde{F}_{\mu\nu} \begin{pmatrix} 0 & E_3 & -E_2 & B_1 \\ -E_3 & 0 & E_1 & B_2 \\ E_2 & -E_1 & 0 & B_3 \\ -B_1 & -B_2 & -B_3 & 0 \end{pmatrix}$$

Both transform like 2^{nd} rank tensors

Missing equations are $\sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} \tilde{F}_{\mu\nu} = \frac{4\pi}{c} j^*_\nu$

j^*_ν will be a new current of magnetic charge which has never been seen: $j^*_\nu = 0$

$$\sum_{\mu=1}^{4} \frac{\partial \tilde{F}_{\mu\nu}}{\partial x_\mu} = 0 \quad \Rightarrow \quad \underbrace{\begin{aligned} &\vec{\nabla} \cdot \vec{B} = 0 \\ &\vec{\nabla} \times \vec{E} - \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0 \end{aligned}}_{\text{Faraday's law}}$$

One final equation is need before all of E & M is complete:

How does $\vec{B}$ affect a charge?

Lorentz force law

Start in Σ' where charge q is at rest at $t'=0$, $\vec{p}'=0$ at $t=0$

$\Delta t'$ is the time elapsed in Σ' read by a clock at rest in Σ' close to the charge q.

$$\Delta \vec{p}' = q\vec{E}'\Delta t' + \cancel{q\vec{B}'\Delta t'}$$

What does Σ see?

Δp_μ is a four vector:

$$\Delta \vec{p}_\perp = \Delta p'_\perp \qquad \Delta p_\parallel = \gamma\left(\Delta p'_\parallel + \frac{v}{c}\frac{\Delta E'}{c}\right)$$

$$\Delta E = \frac{\Delta p'^2}{2m} \to 0 \text{ as } \Delta p' \to 0$$

$$\frac{\Delta \vec{p}_\perp}{\Delta t} = \frac{\Delta \vec{p}'_\perp}{\gamma \Delta t'} = \frac{1}{\gamma} q \vec{E}'_\perp = \frac{1}{\gamma} q \gamma \left[\vec{E}_\perp + \frac{\vec{v}}{c} \times \vec{B}\right]$$

$$\frac{\Delta p_\parallel}{\Delta t} = \frac{\gamma \Delta p'_\parallel}{\gamma \Delta t'} = q E'_\parallel = q E_\parallel$$

Thus, $\frac{d\vec{p}}{dt} = q\left[\vec{E} + \frac{\vec{v}}{c} \times \vec{B}\right]$

a relativistic version of Newton's law

01/28/21

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Thus, $$\frac{d\vec{p}}{dt} = q\left[\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right]$$

a relativistic version of Newton's law

B Magnetostatics:

large currents that are slowly changing: $\vec{\nabla}\cdot\vec{B} = 0 \quad \vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{j}$

(similar to electrostatics with $\vec{\nabla}\cdot\vec{E} = 4\pi\rho$ & $\vec{\nabla}\times\vec{E} = 0$)

Before we work out a general solution, look at simple problems that can be solved using symmetry & Stokes theorem (like symmetry and Gauss' law for electrostatics).

a) Find $\vec{B}$ produced by a long thin wire carrying a current I:

side view end view

In what direction does $\vec{B}$ point?

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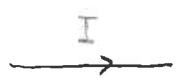


side view

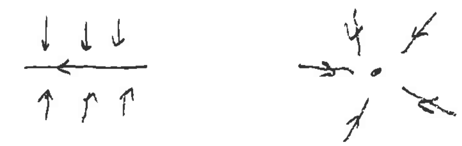
end view

In what direction does $\vec{B}$ point?

a) can $\vec{B}$ be radially outward?

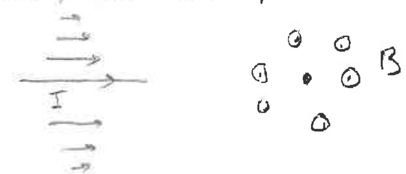
but $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$

implied $\vec{B}$ should change direction when I changes direction



Clear not consist since nothing has really changed!

b) Can $\vec{B}$ be parallel to I?



No, since the sign of $\vec{B}$ depends on whether we use left- or righ-handed coordinates but $\vec{B} \parallel$ to I does not

$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$ RH $\qquad \hat{e}_1 \times \hat{e}_2 = -\hat{e}_3$ LH

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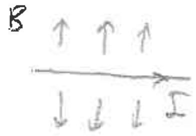

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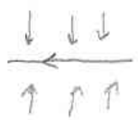

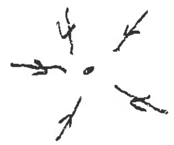

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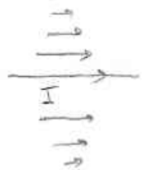

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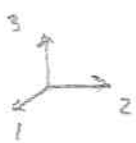


$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3$

RH

$\hat{e}_1 \times \hat{e}_2 = -\hat{e}_3$

LH

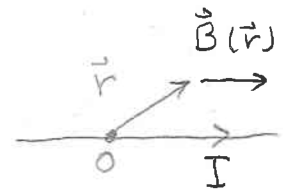
This argument in more detail:

$$\vec{\nabla}_r \times \vec{B}(\vec{r}) = \frac{4\pi}{c} \vec{j}(\vec{r})$$

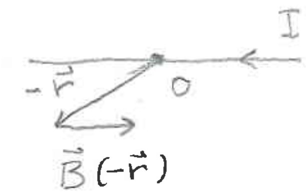
$$\Rightarrow \quad -\vec{\nabla}_r \times \vec{B}(-\vec{r}) = \frac{4\pi}{c} \vec{j}(-\vec{r})$$

$$\frac{\partial f}{\partial x}(x)\Big|_{-x} = -\frac{\partial f(-x)}{\partial x}\Big|_{x=x}$$

$\Rightarrow$ $\vec{B}(-\vec{r})$ solves problem with new current $\vec{j}^{\,new}(\vec{r}) = -\vec{j}(-\vec{r})$



$\Rightarrow$

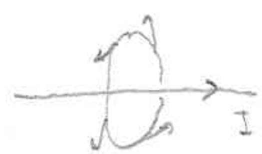


$\vec{B} \parallel I$

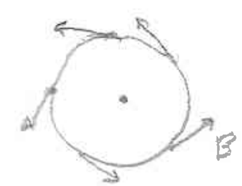
$\vec{B}$ antiparallel to I

a contradiction!

c) can $\vec{B}$ be circular? Yes!



perspective



end view

direction given by right-hand role

This argument in more detail:

$$\vec\nabla_r \times \vec B(\vec r) = \frac{4\pi}{c}\vec j(\vec r)$$

$$\left.\frac{\partial f}{\partial x}(x)\right|_{-x} = -\left.\frac{\partial f(-x)}{\partial x}\right|_{x=x}$$

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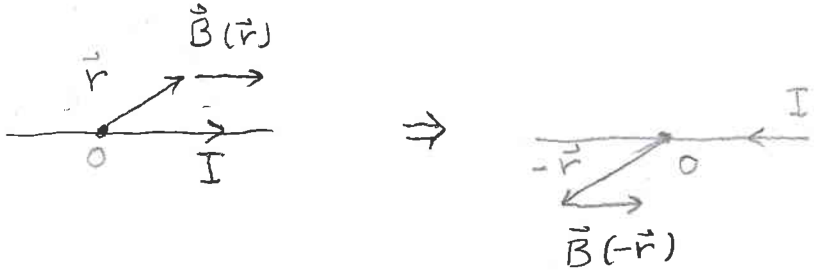

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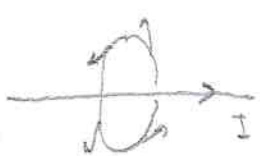

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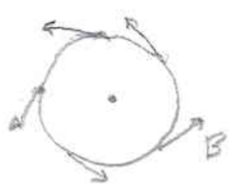
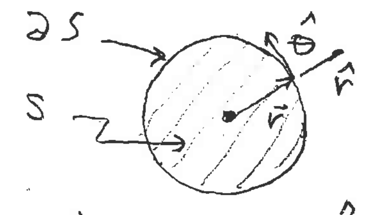


end view

direction given by right-hand rule

Having determined the direction of $\vec B(\vec r)$ from symmetry we can use Stokes theorem to find its magnitude.

$$\vec\nabla \times \vec B - \cancel{\frac{1}{c}\frac{\partial \vec E}{\partial t}} = \frac{4\pi}{c}\vec j$$



$\vec B(\vec r) = B(r)\hat\theta$

$$\underbrace{\oint_{\partial S} \vec B\cdot d\vec l}_{2\pi r\, B(r)} = \int (\vec\nabla\times\vec B)\cdot\hat n\, dS$$

$$= \frac{4\pi}{c}\underbrace{\int \vec j\cdot\hat n\, dS}_{I}$$

$$B(r) = \frac{2I}{cr}$$

Note: actual direction of $\vec B$ given by right-hand rule

esu units for $\vec B$:

$$\frac{esu/sec}{cm/sec\cdot cm} = \frac{esu}{cm^2} \quad \text{same as } \vec E$$

for magnetic field $\frac{esu}{cm} \doteq$ Gauss

In S.I units B is measured in Tesla

1 Tesla $= \frac{1\,amp}{m}$; $\vec\nabla\times\vec B = \mu_0 \vec j$

$\mu_0 = 4\pi\times10^{-7}$

Having determined the direction of $\vec{B}(\vec{r})$ from symmetry we can use Stokes theorem to find its magnitude.

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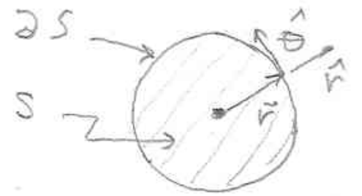

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In SI units B is measured in Tesla

$$1\ \text{Tesla} = \frac{1\,\text{amp}}{\text{m}} \quad \text{\&} \quad \vec{\nabla}\times\vec{B} = \mu_0\vec{J}$$

$$\mu_0 = 4\pi\times10^{-7}$$

A Gauss is a small unit, the size of the Earth's B field near the surface. Compare Gauss & Tesla

consider $B(r)$ 1 meter from wire carrying 1 Amp:

esu

$$B = \frac{2I}{rc} = \frac{2\ \overbrace{\text{Coulomb}}^{3\times10^9\ \text{esu}}/\text{sec}}{100\ \text{cm}\ \ 3\times10^{10}\,\text{cm/sec}}$$

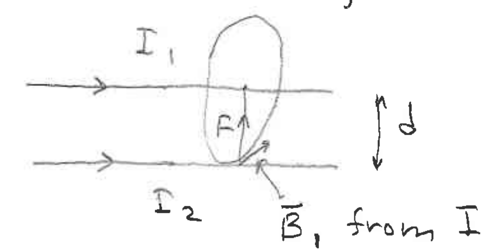
$$= 2\times10^{-3}\ \frac{\text{esu}}{\text{cm}^2} = 2\times10^{-3}\ \text{Gauss}$$

SI

$$B = \mu_0\frac{1}{2\pi}\frac{I}{r} = \frac{4\pi\times10^{-7}}{2\pi}\underbrace{\frac{1\,\text{amp}}{1\,\text{m}}}_{\text{Tesla}}$$

$$\Rightarrow 1\ \text{Tesla} = 10^4\ \text{Gauss}$$

b) Force between two wires carrying current



$$B = \frac{2I_1}{dc}$$

force on segment of 2^{nd} wire of length Δl

$$\vec{F} = \frac{\Delta Q}{c}\vec{v}\times\vec{B}$$

upward direction

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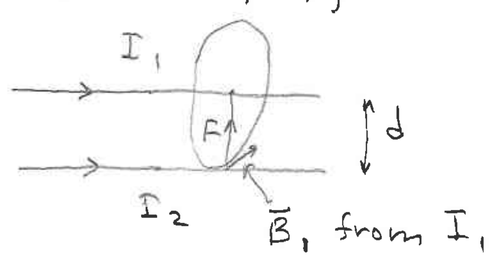

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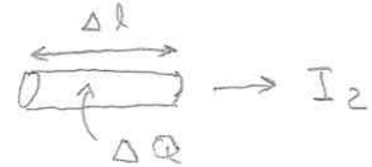


assume all charges move with the same velocity

Let Δt be the time it takes the moving charge ΔQ in the length Δl of the wire to move out of that segment. $v\Delta t = \Delta l$, $I_2 = \frac{\Delta Q}{\Delta t}$

Force per unit length on wire

$$\frac{\Delta F}{\Delta l} = \left(\frac{v}{c}\Delta Q B_1\right)\frac{1}{\Delta l} = \frac{v}{c} I_2 \Delta t B_1 \frac{1}{\Delta l}$$

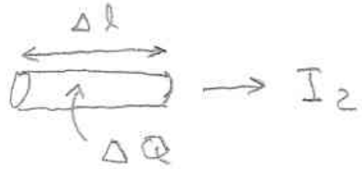
$$= \frac{I_2 B_1}{c} = \frac{I_2}{c}\cdot\frac{2I_1}{dc}$$

$$= \frac{2I_1 I_2}{dc^2}$$

If $I_1 = I_2 = 1$ Amp & $d = 1$ cm

$$\frac{\Delta F}{\Delta l} = \frac{2\left(3\times10^{9}\frac{\text{esu}}{\text{sec}}\right)^2}{1\text{ cm}\times\left(3\times10^{10}\frac{\text{cm}}{\text{sec}}\right)^2} = 0.02\frac{\text{dyne}}{\text{cm}}$$

a very weak force ($\frac{1}{c^2}$ is small)



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$$= \frac{2I_1 I_2}{dc^2}$$

If $I_1 = I_2 = 1\,\text{Amp}$ & $d = 1\,\text{cm}$

$$\frac{\Delta F}{\Delta l} = \frac{2\left(3\times10^9\,\frac{\text{esu}}{\text{sec}}\right)^2}{1\,\text{cm}\times\left(3\times10^{10}\,\frac{\text{cm}}{\text{sec}}\right)^2} = 0.02\,\frac{\text{dyne}}{\text{cm}}$$

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c) Diversion: numerical electrostatics

Consider a long conducting tube with square cross section with a central wire carrying charge density λ per unit length

Find the potential $\varphi(x,y)$ everywhere inside the tube. Solve in two parts

1. Transform into solution of Laplace equation.
2. Solve Laplace equation numerically by relaxation

Assume central charge is distributed uniformly on surface of rod with radius d.

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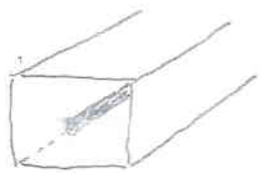

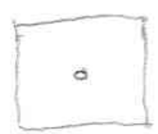

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① Transform into solution of Laplace equation.

② Solve Laplace equation numerically by relaxation

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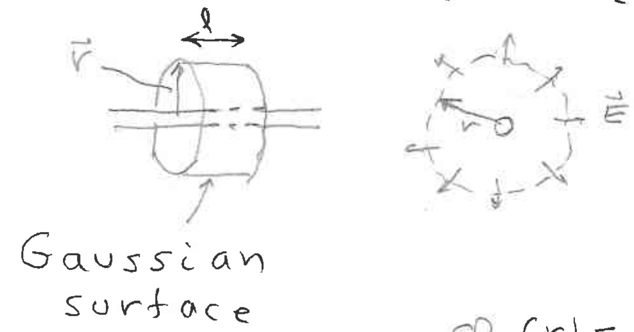
① 1st solve easier problem without the tube



$$2\pi r \ell E = 4\pi \lambda \ell$$

$$\vec{E} = \frac{2\lambda}{r}\hat{r}$$

$$\varphi_0(r) = \begin{cases} -2\lambda \ln(r/d) & r \geq d \\ 0 & r \leq d \end{cases}$$

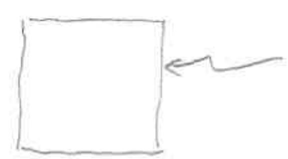
$$E_x \overset{?}{=} -\frac{\partial}{\partial x}\varphi_0(r) = 2\lambda \frac{1}{r}\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{2\lambda}{r^2}x \checkmark$$

Write solution to complete problem as

$$\varphi(\vec{r}) = \tilde{\varphi}(r) + \varphi_0(r)$$

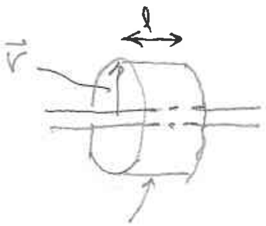
Then $\vec{\nabla}^2\tilde{\varphi}(r) = 0$

& $\tilde{\varphi}(r)\Big|_{\text{on tube}} = +2\lambda \ln(r/d)$

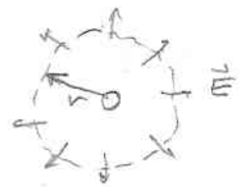


$\tilde{\varphi}(\vec{r}) = 2\lambda \ln(r/d)$

① 1^{st} solve easier problem without the tube



Gaussian surface



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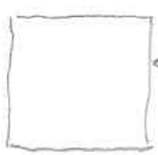
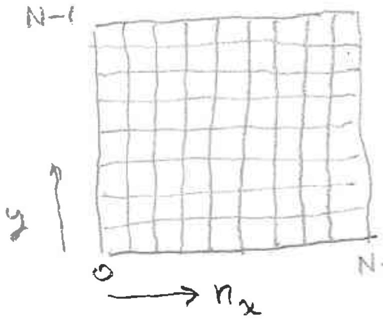


$\leftarrow \tilde{\varphi}(\vec{r}) = 2\lambda\ln(r/d)$

② Solve numerically by iterative relaxation method

(i) Introduce a grid and assign values of $\varphi(r)$ to the grid points

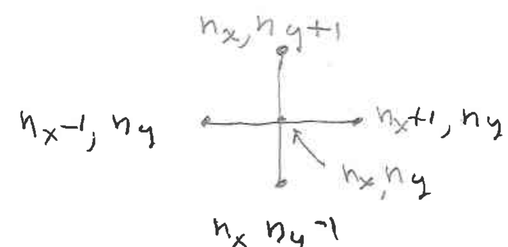


(ii) Start with $\varphi^{(0)}(n_x, n_y)$

where $\varphi^{(0)}(n_x, n_y) = \begin{cases} 0 & \text{inside} \\ -\varphi_0(n_x, n_y) & \text{on boundary} \end{cases}$

(iii) Iterate choosing $k+1$ trial from k^{th} trial by averaging neighbors:

$$\varphi^{(k+1)}(n_x, n_y) = \frac{1}{4}\Big[\varphi(n_x+1, n_y) + \varphi(n_x-1, n_y) + \varphi(n_x, n_y+1) + \varphi(n_x, n_y-1)\Big]$$



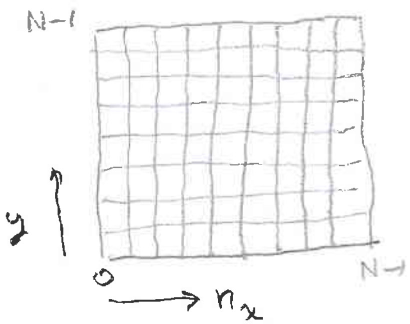
This should approach Laplace solution

where $\varphi(r) = \frac{1}{4\pi R^2}\int \varphi(r')\,ds'$

$\uparrow$ integrate over sphere of radius R

2 Solve numerically by iterative relaxation method

(i) Introduce a grid and assign values of $\varphi(r)$ to the grid points

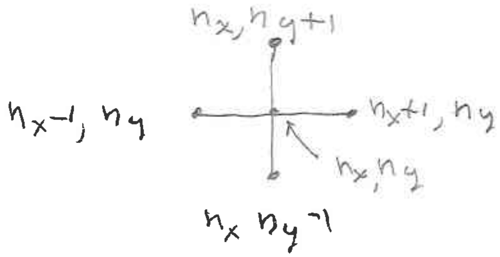


(ii) Start with $\varphi^{(0)}(n_x, n_y)$

where $\varphi^{(0)}(n_x, n_y) = \begin{cases} 0 & \text{inside} \\ \varphi_0(n_x, n_y) & \text{on boundary} \end{cases}$

(iii) Iterate choosing $k+1$ trial from k^{th} trial by averaging neighbors:

$$\varphi^{(k+1)}(n_x, n_y) = \frac{1}{4}\Big[\varphi(n_x+1, n_y) + \varphi(n_x-1, n_y) + \varphi(n_x, n_y+1) + \varphi(n_x, n_y-1)\Big]$$



This should approach Laplace solution

where $\varphi(r) = \frac{1}{4\pi R^2}\int \varphi(r')\, dS'$ integrate over sphere of radius R

$$\lim_{k\to\infty} \varphi^{(k)}(n_x, n_y) \doteq \tilde{\varphi}(n_x, n_y)$$

+ errors from finite grid

Easy to prove convergence since we are actually minimizing the energy:

$$\mathcal{E} = \frac{\Delta^2}{8\pi}\sum_{n_x=0}^{N-2}\sum_{n_y=0}^{N-2}\Big\{\big[\varphi(n_x+1, n_y) - \varphi(n_x, n_y)\big]^2 + \big[\varphi(n_x, n_y+1) - \varphi(n_x, n_y)\big]^2\Big\}$$

$\varphi(n_x, n_y)$ appears in 8 terms

$$4\big[\varphi(n_x, n_y)\big]^2 - 2\,\varphi(n_x, n_y)\big[\varphi(n_x+1, n_y) + \varphi(n_x-1, n_y) + \varphi(n_x, n_y+1) + \varphi(n_x, n_y-1)\big]$$

$$\therefore\ \frac{\partial \mathcal{E}}{\partial \varphi(n_x, n_y)} = 8\,\varphi(n_x, n_y) - 2\big[\varphi(n_x+1, n_y) + \varphi(n_x-1, n_y) + \varphi(n_x, n_y+1) + \varphi(n_x, n_y-1)\big]$$

at each step we are choosing $\varphi^{(k)}(n_x, n_y)$ to minimize $\mathcal{E}$. Since $\mathcal{E}$ is positive this must converge!