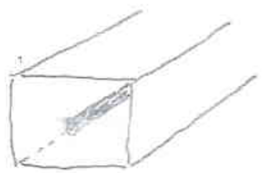
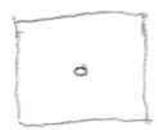
c) Diversion: numerical electrostatics

Consider a long conducting tube with square cross section with a central wire carrying charge density λ per unit length



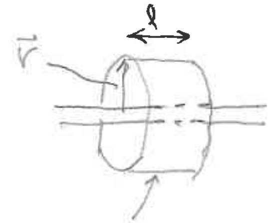


Find the potential $\varphi(x,y)$ everywhere inside the tube. Solve in two parts

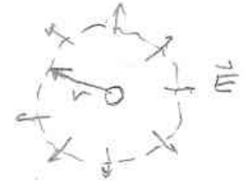
1. Transform into solution of Laplace equation.
2. Solve Laplace equation numerically by relaxation

Assume central charge is distributed uniformly on surface of rod with radius d.

1. 1st solve easier problem without the tube



Gaussian surface



$$2\pi r \ell E = 4\pi\lambda\ell$$

$$\vec{E} = \frac{2\lambda}{r}\hat{r}$$

$$\varphi_0(r) = \begin{cases} -2\lambda \ln(r/d) & r \geq d \\ 0 & r \leq d \end{cases}$$

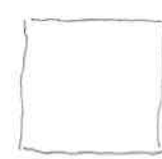
$$E_x \overset{?}{=} -\frac{\partial}{\partial x}\varphi_0(r) = 2\lambda \frac{1}{r}\frac{\partial}{\partial x}\sqrt{x^2+y^2} = \frac{2\lambda}{r^2}x \checkmark$$

Write solution to complete problem as

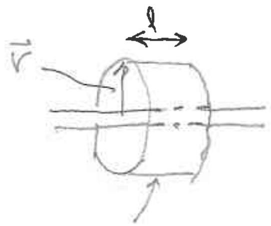
$$\varphi(\vec{r}) = \tilde{\varphi}(r) + \varphi_0(r)$$

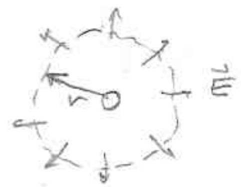
Then $\vec{\nabla}^2\tilde{\varphi}(r) = 0$

& $\tilde{\varphi}(r)\Big|_{\text{on tube}} = +2\lambda\ln(r/d)$

 $\leftarrow \tilde{\varphi}(\vec{r}) = 2\lambda\ln(r/d)$

① 1^{st} solve easier problem without the tube





$$2\pi r l \, E = 4\pi \lambda l$$

$$\vec{E} = \frac{2\lambda}{r}\hat{r}$$

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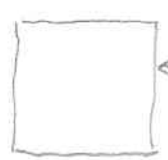
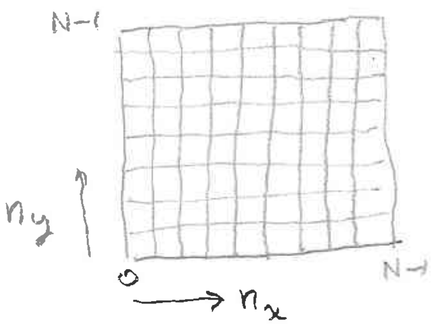


$\leftarrow \tilde{\varphi}(\vec{r}) = 2\lambda \ln(r/d)$

② Solve numerically by iterative relaxation method

(i) Introduce a grid and assign values of $\varphi(r)$ to the grid points

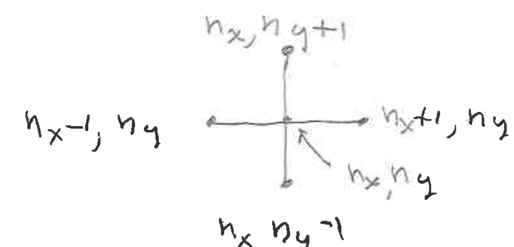


(ii) Start with $\varphi^{(0)}(n_x, n_y)$

where $\varphi^{(0)}(n_x, n_y) = \begin{cases} 0 & \text{inside} \\ -\varphi_0(n_x, n_y) & \text{on boundary} \end{cases}$

(iii) Iterate choosing $k+1$ trial from k^{th} trial by averaging neighbors:

$$\varphi^{(k+1)}(n_x, n_y) = \frac{1}{4}\Big[\varphi(n_x+1, n_y) + \varphi(n_x-1, n_y) + \varphi(n_x, n_y+1) + \varphi(n_x, n_y-1)\Big]$$



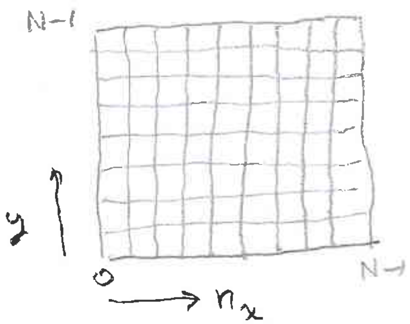
This should approach Laplace solution

where $\varphi(r) = \frac{1}{4\pi R^2}\int \varphi(r')\, ds'$

↖ integrate over sphere of radius R

② Solve numerically by iterative relaxation method

(i) Introduce a grid and assign values of $\Phi(r)$ to the grid points



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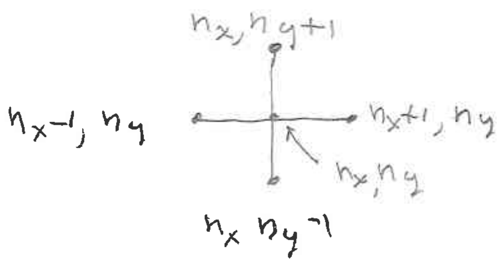

This should approach Laplace solution

where $\Phi(r) = \frac{1}{4\pi R^2}\int \Phi(r')\, dS'$ ← integrate over sphere of radius R

02/02/2021

$$\lim_{k\to\infty} \Phi^{(k)}(n_x, n_y) \doteq \tilde{\Phi}(n_x, n_y)$$

+ errors from finite grid

Easy to prove convergence since we are actually minimizing the energy:

$$\mathcal{E} = \frac{\Delta^2}{8\pi}\sum_{n_x=0}^{N-2}\sum_{n_y=0}^{N-2}\Big\{\big[\Phi(n_x+1, n_y) - \Phi(n_x, n_y)\big]^2 + \big[\Phi(n_x, n_y+1) - \Phi(n_x, n_y)\big]^2\Big\}$$

$\Phi(n_x, n_y)$ appears in 8 terms

$$4\big[\Phi(n_x, n_y)\big]^2 - 2\,\Phi(n_x, n_y)\big[\Phi(n_x+1, n_y) + \Phi(n_x-1, n_y) + \Phi(n_x, n_y+1) + \Phi(n_x, n_y-1)\big]$$

$$\therefore\ \frac{\partial \mathcal{E}}{\partial \Phi(n_x, n_y)} = 8\,\Phi(n_x, n_y) - 2\big[\Phi(n_x+1, n_y) + \Phi(n_x-1, n_y) + \Phi(n_x, n_y+1) + \Phi(n_x, n_y-1)\big]$$

at each step we are choosing $\Phi^{(k)}(n_x, n_y)$ to minimize $\mathcal{E}$. Since $\mathcal{E}$ is positive this must converge!

$$\lim_{k\to\infty} \varphi^{(k)}(n_x,n_y) = \tilde{\varphi}(n_x,n_y)$$
+ errors from finite grid

Easy to prove convergence since we are actually minimizing the energy:

$$\mathcal{E} = \frac{\Delta^2}{8\pi}\sum_{n_x=0}^{N-2}\sum_{n_y=0}^{N-2}\left\{\left[\varphi(n_x+1,n_y)-\varphi(n_x,n_y)\right]^2 + \left[\varphi(n_x,n_y+1)-\varphi(n_x,n_y)\right]^2\right\}$$

$\varphi(n_x,n_y)$ appears in 4 terms

$$4\left[\varphi(n_x,n_y)\right]^2 - 2\varphi(n_x,n_y)\left[\varphi(n_x+1,n_y)+\varphi(n_x-1,n_y) + \varphi(n_x,n_y+1)+\varphi(n_x,n_y-1)\right]$$

$$\therefore \frac{\partial \mathcal{E}}{\partial \varphi(n_x,n_y)} = 8\varphi(n_x,n_y) - 2\left[\varphi(n_x+1,n_y)+\varphi(n_x-1,n_y) + \varphi(n_x,n_y+1)+\varphi(n_x,n_y-1)\right]$$

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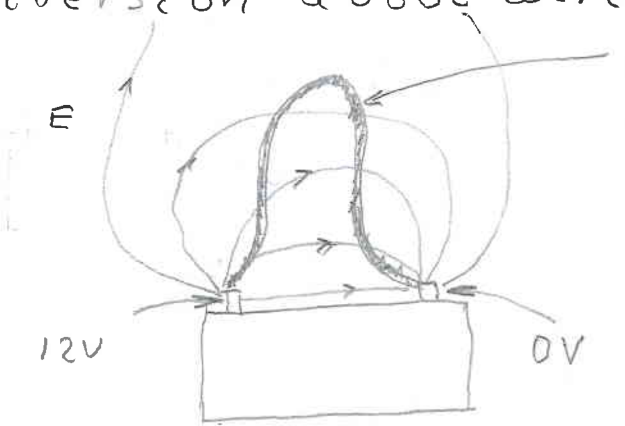
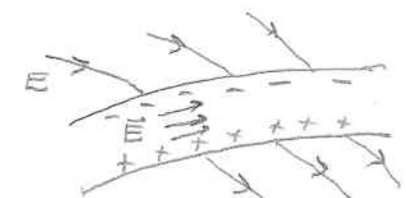
[Discuss Python example: dipole.ipynb]

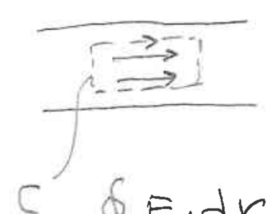
d) Diversion about wires



insert wire so current flows from +12V to 0

segment of (enlarged)



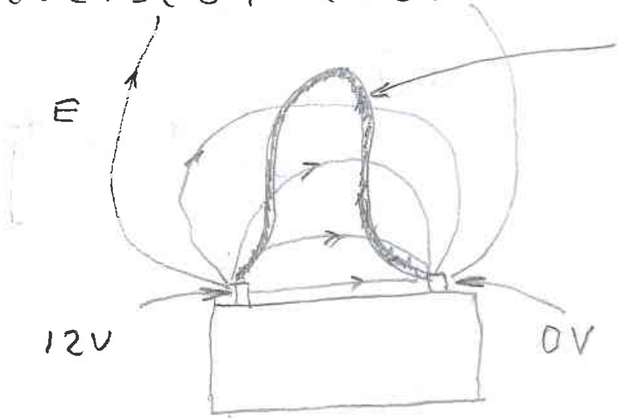


$$\oint_C \vec{E}\cdot d\vec{r} = 0$$

- surface charge removes $E_\perp$ to surface
- inside expect $\vec{E}$ // to wire and uniform accross cross section
- wire disturbs E close by may have little effect farther away?

[Discuss Python example:
dipole.ipynb]

d) Diversion about wires



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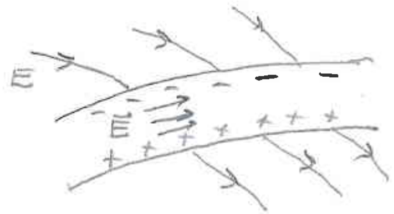

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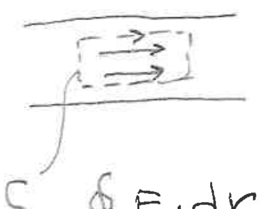
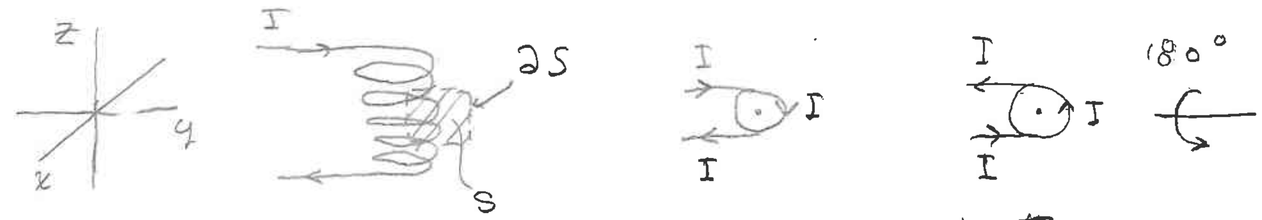


$$\oint_C E \cdot dr = 0$$

Return to Magnetostatics

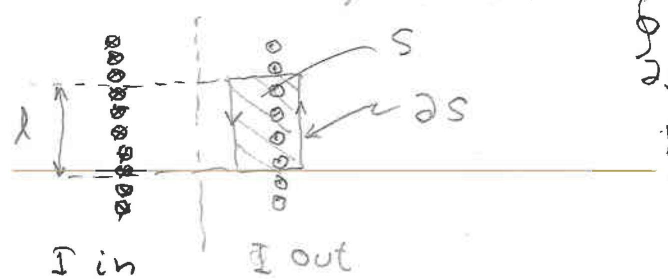
e) $\vec{B}$ from a long coil or solenoid



side top bottom

Three direction to consider for $\vec{B}$:

1. radially outward? Not consistent with above 180° rotation $I \to -I$ but "outward" remains outward
2. circular like I? No $I \parallel B$ but relation must involve right hand
3. must be axial // to central axis of cylinder:



$$\oint_{\partial S} \vec{B} \cdot d\ell = \frac{4\pi}{c} \int \vec{J} \cdot \hat{n}\, ds$$

$$B\ell = \frac{4\pi}{c} I \cdot N_{turns}$$

$N_{turns} = n \cdot \ell$

n: # turns per length

Return to Magnetostatics

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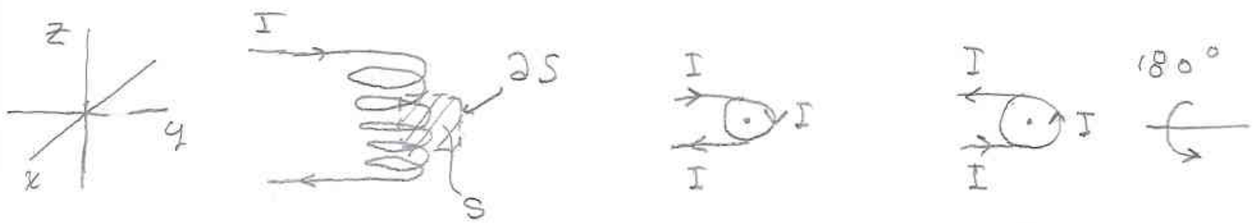

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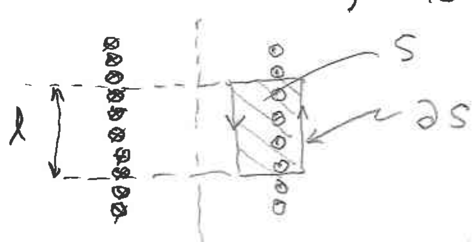


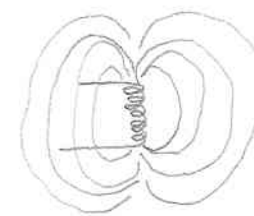
$$\oint_{\partial S} \vec{B}\cdot d\vec{l} = \frac{4\pi}{c}\int \vec{J}\cdot\hat{n}\, ds$$

$$Bl = \frac{4\pi}{c} I \cdot \underbrace{N_{turns}}_{= n\cdot l}$$

n: # turns per length

Thus $B = \begin{cases} \frac{4\pi}{c} n I & \text{inside downward} \\ 0 & \text{outside} \end{cases}$

For finite lenght B small outside



2. Find a general solution for $\vec{B}(\vec{r})$ given $\vec{J}(\vec{r})$

a) Recall for $\vec{E}$:

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}\Phi$$

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \Rightarrow \vec{\nabla}^2\Phi = -4\pi\rho$$

$$\Phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|r-r'|}$$

$$\text{+ } \vec{E}(\vec{r}) = -\vec{\nabla}\Phi = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3}(\vec{r}-\vec{r}')$$

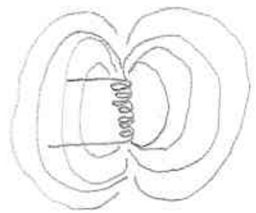
b) for $\vec{B}$ use $\vec{\nabla}\cdot\vec{B} = 0$

claim $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$

for some vector potential $\vec{A}$

Thus $B = \begin{cases} \frac{4\pi}{c} nI & \text{inside} \quad \text{downward} \\ 0 & \text{outside} \end{cases}$

For finite lenght B small outside



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a) Recall for $\vec{E}$:

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$\varphi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|r-r'|}$

$\vec{E}(\vec{r}) = -\vec{\nabla}\varphi = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3} (\vec{r}-\vec{r}')$

b) for $\vec{B}$ use $\vec{\nabla} \cdot \vec{B} = 0$

claim $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$

for some vector potential $\vec{A}$

Two arguments that suggest this is reasonable:

a) If $\vec{\nabla} \cdot \vec{B} = 0$ then for any volume V

$$0 = \int_V \vec{\nabla} \cdot \vec{B} \, d^3r = \int_{\partial V} \hat{n} \cdot \vec{B} \, dS = \int_{\partial V} \hat{n} \cdot (\vec{\nabla} \times \vec{A}) \, dS$$

$$= \oint_{\partial(\partial V)} \vec{A} \cdot d\ell = 0$$

because the surface of V has no boundary!

b) Compute $\vec{\nabla} \cdot (\vec{\nabla} \times A)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x\right)$$

terms cancel in pairs $\vec{c} \cdot (\vec{c} \times \vec{A}) = 0$

note $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x,y)$

unless there is singular behavior as $x \to y$