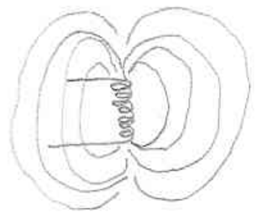
Thus $B = \begin{cases} \frac{4\pi}{c} nI & \text{inside} \quad \text{downward} \\ 0 & \text{outside} \end{cases}$

For finite lenght B small outside



2. Find a general solution for $\vec{B}(\vec{r})$ given $\vec{j}(\vec{r})$

a) Recall for $\vec{E}$:

$\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla} \varphi$

$\vec{\nabla} \cdot \vec{E} = 4\pi \rho \Rightarrow \vec{\nabla}^2 \varphi = -4\pi \rho$

$\varphi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|r-r'|}$

$\vec{E}(\vec{r}) = -\vec{\nabla} \varphi = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|^3} (\vec{r}-\vec{r}')$

b) for $\vec{B}$ use $\vec{\nabla} \cdot \vec{B} = 0$

claim $\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$

for some vector potential $\vec{A}$

02/04/2021

Two arguments that suggest this is reasonable:

a) If $\vec{\nabla} \cdot \vec{B} = 0$ then for any volume V

$0 = \int_V \vec{\nabla} \cdot \vec{B} \, d^3r = \int_{\partial V} \hat{n} \cdot \vec{B} \, ds = \int_{\partial V} \hat{n} \cdot (\vec{\nabla} \times \vec{A}) \, ds$

$= \oint_{\partial(\partial V)} \vec{A} \cdot d\vec{l} = 0$ because the surface of V has no boundary!

b) Compute $\vec{\nabla} \cdot (\vec{\nabla} \times A)$

$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} A_z - \frac{\partial}{\partial z} A_y \right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} A_x - \frac{\partial}{\partial x} A_z \right)$

$+ \frac{\partial}{\partial z}\left(\frac{\partial}{\partial x} A_y - \frac{\partial}{\partial y} A_x \right)$

terms cancel in pairs $\vec{c} \cdot (\vec{c} \times \vec{A}) = 0$

note $\frac{\partial}{\partial x} \frac{\partial}{\partial y} f(x,y) = \frac{\partial}{\partial y} \frac{\partial}{\partial x} f(x,y)$

unless there is singular behavior as $x \to y$

Two arguments that suggest this is reasonable:

a) If $\vec{\nabla}\cdot\vec{B}=0$ then for any volume V

$$0=\int_V \vec{\nabla}\cdot\vec{B}\, d^3r=\int_{\partial V}\hat{n}\cdot\vec{B}\, dS=\int_{\partial V}\hat{n}\cdot(\vec{\nabla}\times\vec{A})\, dS$$

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$$=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}A_z-\frac{\partial}{\partial z}A_y\right)+\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z}A_x-\frac{\partial}{\partial x}A_z\right)$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}A_y-\frac{\partial}{\partial y}A_x\right)$$

terms cancel in pairs $\vec{C}\cdot(\vec{C}\times\vec{A})=0$

note $\frac{\partial}{\partial x}\frac{\partial}{\partial y}f(x,y)=\frac{\partial}{\partial y}\frac{\partial}{\partial x}f(x,y)$

unless there is singular behavior as $x\to y$

Important puzzle:

$\underbrace{\vec{\nabla}\times\vec{E}=0}_{\text{two condition}} \Rightarrow \vec{E}=-\vec{\nabla}\varphi$

3 function

one function is needed!

$\underbrace{\vec{\nabla}\cdot\vec{B}=0}_{\text{one condition}}$ $\vec{B}=\vec{\nabla}\times\vec{A}$

3 functions

3 functions are needed??

Three variables in $\vec{A}(\vec{r})$ are NOT needed

$\vec{A}(\vec{r})$ is redundant: $\vec{A}$ & $\vec{A}+\vec{\nabla}\Lambda$ give the same $\vec{B}$ because $\vec{\nabla}\times(\underbrace{\vec{\nabla}\Lambda}_{})=0$

e.g. $\vec{\nabla}\Lambda$ is conservative

Thus, the benefits of introducing $\vec{A}$ with $\vec{B}=\vec{\nabla}\times\vec{A}$ are less obvious. ~~Cha~~ Changing $\vec{A}\to\vec{A}+\vec{\nabla}\Lambda$ is called changing the gauge, a "gauge transformation" - may make the problem easier?

Important puzzle:

$\underbrace{\vec\nabla\times\vec E=0}_{\text{two condition}} \Rightarrow \vec E = -\vec\nabla\varphi$

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Use this freedom to find $\vec A(\vec r)$

$$\frac{4\pi}{c}\vec j = \vec\nabla\times\vec B = \vec\nabla\times(\vec\nabla\times\vec A)$$

$$= \vec\nabla[\vec\nabla\cdot A] - [\vec\nabla\cdot\vec\nabla]\vec A$$

$$\left[\ \vec A\times(\vec B\times\vec C) = \vec B[\vec A\cdot\vec C] - \vec C[A\cdot B]\ \right]$$

Thus, $\vec\nabla[\vec\nabla\cdot A] - \vec\nabla^2\vec A = \frac{4\pi}{c}\vec j$

write out x-component:

$$\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)A_x = \frac{4\pi}{c}j_x$$

This would be easy to solve if we could remove left term. Replace $\vec A$ by $\underbrace{\vec A+\vec\nabla\Lambda}_{\vec A'}$ & require $\vec\nabla\cdot(\vec A+\vec\nabla\Lambda)=0$.

Easy to find Λ: $\vec\nabla\cdot\vec A+\vec\nabla^2\Lambda=0$

$$\Rightarrow \vec\nabla^2\Lambda = -\vec\nabla\cdot\vec A \Rightarrow \Lambda(\vec r) = \frac{1}{4\pi}\int\frac{1}{|\vec r-\vec r'|}\vec\nabla\cdot\vec A(\vec r')\,d^3r'$$

usual electrostatics solution for $\varphi(\vec r)$ in terms of $4\pi\rho$

Use this freedom to find $\vec{A}(\vec{r})$

$$\frac{4\pi}{c}\vec{J} = \vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

$$= \vec{\nabla}[\vec{\nabla}\cdot A] - [\vec{\nabla}\cdot\vec{\nabla}]\vec{A}$$

$$\left[\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}[\vec{A}\cdot\vec{C}] - \vec{C}[A\cdot B]\right]$$

Thus, $\vec{\nabla}[\vec{\nabla}\cdot A] - \vec{\nabla}^2\vec{A} = \frac{4\pi}{c}\vec{J}$

write out x-component:

$$\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)A_x = \frac{4\pi}{c}J_x$$

This would be easy to solve if we could remove left term. Replace $\vec{A}$ by $\underbrace{\vec{A}+\vec{\nabla}\Lambda}_{\vec{A}'}$ & require $\vec{\nabla}\cdot(\vec{A}+\vec{\nabla}\Lambda) = 0$.

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usual electrostatics solution for $\phi(\vec{r})$ in terms of $4\pi\rho$

Now Ampere's law $\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J}$

becomes $\vec{\nabla}^2 A_i = -\frac{4\pi}{c}J_i \quad i = 1, 2, 3,$

Like 3 electrostatics problems

$$A_i(\vec{r}) = \frac{1}{c}\int\frac{1}{|\vec{r}-\vec{r}'|}J_i(\vec{r}')\,d^3r'$$

and we have found $\vec{A}$!

$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = -\frac{1}{c}\int\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\times\vec{J}(\vec{r}')\,d^3r'$$

We have solved the general problem given $a(\vec{r})$ & $\vec{b}(\vec{r})$ find $\vec{W}(\vec{r})$ obeying

$$\vec{\nabla}\cdot\vec{W} = a(r) \qquad \vec{\nabla}\times\vec{W} = \vec{b}(\vec{r})$$

write $\vec{W} = \vec{W}_{trans} + \vec{W}_{long}$ where

$$\vec{\nabla}\cdot\vec{W}_{trans} = 0 \quad \& \quad \vec{\nabla}\times\vec{W}_{long} = 0$$

$$\vec{W}_{long}(\vec{r}) = \frac{1}{4\pi}\int\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}a(\vec{r}')\,d^3r'$$

$$\vec{W}_{trans}(\vec{r}) = -\frac{1}{4\pi}\int\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\times\vec{b}(\vec{r}')\,d^3r'$$

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Like 3 electrostatics problems

$$A_i(\vec{r}) = \frac{1}{c}\int \frac{1}{|\vec{r}-\vec{r}'|} j_i(\vec{r}')\, d^3r'$$

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$$\vec{B}(\vec{r}) = \vec{\nabla}\times\vec{A}(\vec{r}) = -\frac{1}{c}\int \frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|^3}\times\vec{j}(\vec{r}')\, d^3r'$$

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given $a(\vec{r})$ & $\vec{b}(\vec{r})$ find $\vec{W}(\vec{r})$ obeying

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write $\vec{w} = \vec{w}_{trans} + \vec{w}_{long}$ where

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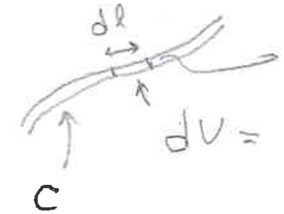
An important special case for the equation for $\vec{B}$: assume $\vec{j}(\vec{r})$ flows in a wire describle by a curve C:



$$dV = dA\cdot dl$$

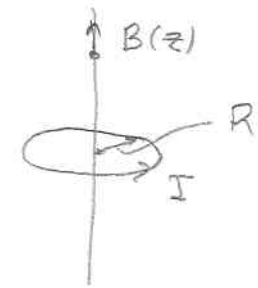
$$\vec{j}(\vec{r})\, dV = \underbrace{|\vec{j}|\, dA}_{I}\, d\vec{l}$$

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{1}{|\vec{r}-\vec{r}'|}\vec{j}(\vec{r}')\, d^3r' = \frac{1}{c}\int_C \frac{1}{|\vec{r}-\vec{r}'|} I\, d\vec{l}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{I}{c}\int \frac{\vec{r}-\vec{r}'}{c\,|\vec{r}-\vec{r}'|^3}\times d\vec{l}$$

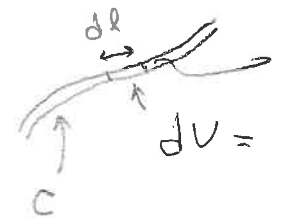
the Biot-Savart Law

Problem



find $\vec{B}(z)$ along the axis of a loop of radius R carry a current I

An important special case for the equation for $\vec{B}$: assume $\vec{J}(\vec{r})$ flows in a wire describible by a curve C:



crosssection dA

$dV = dA \cdot dl$

$$\vec{J}(\vec{r})\, dV = \underbrace{|\vec{J}|\, dA}_{I}\, d\vec{l}$$

$$\vec{A}(\vec{r}) = \frac{1}{c}\int \frac{1}{|\vec{r}-\vec{r}\,'|}\, \vec{J}(\vec{r}\,')\, d^3r' = \frac{1}{c}\int_C \frac{1}{|\vec{r}-\vec{r}\,'|}\, I\, d\vec{l}$$

$$\vec{B} = \vec{\nabla}\times\vec{A} = -\frac{I}{c}\int_C \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\times d\vec{l}$$

the Biot-Savart Law

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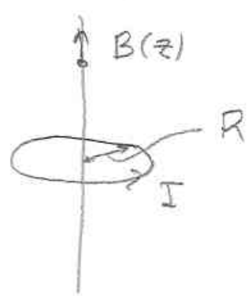
3. Problem



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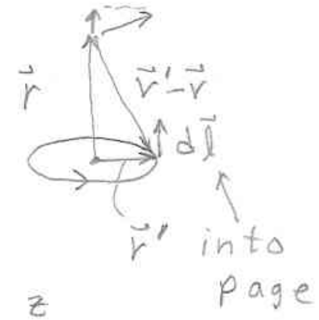
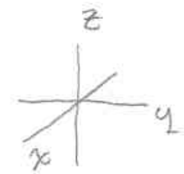




$$\vec{B} = -\frac{I}{c}\oint_C \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\times d\vec{l}$$

$$\vec{B} = \hat{z}\,\frac{I}{c}\,\frac{1}{z^2+R^2}\times\frac{R}{\sqrt{z^2+R^2}}\times 2\pi R$$

$$= \frac{2\pi I}{c}\,\frac{R^2}{(z^2+R^2)^{3/2}}\,\hat{z}$$

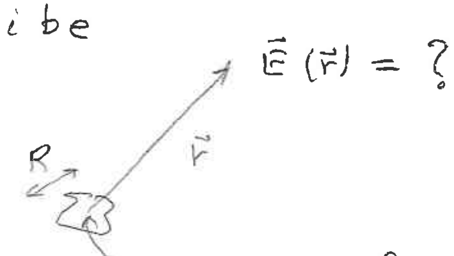
---

4. Multipole expansion

The behavior of $\vec{E}(\vec{r})$ & $\vec{B}(\vec{r})$ far from localized charges and currents which produced the is simple to describe

a) Easiest for $\vec{E}(\vec{r})$

$$\vec{E}(\vec{r}) = \int d^3r'\, \rho(r')\, \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}$$



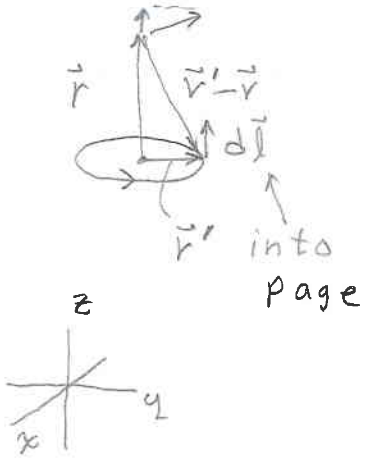
region of $\rho(\vec{r}) \neq 0$

for r large, $r \gg R$

$$\approx \frac{\vec{r}}{|\vec{r}|^3}\underbrace{\int d^3r'\, \rho(r')}_{Q}$$

$$= Q\,\frac{\vec{r}}{|\vec{r}|^3}$$

Looks like a point charge at large distance



$$\vec B = -\frac{I}{c}\oint_C \frac{\vec r - \vec r'}{|\vec r - \vec r'|^3} \times d\vec l$$

$$\vec B = \hat z \, \frac{I}{c} \, \frac{1}{z^2+R^2} \times \frac{R}{\sqrt{z^2+R^2}} \times 2\pi R$$

$$= \frac{2\pi I}{c} \, \frac{R^2}{(z^2+R^2)^{3/2}} \, \hat z$$

4. Multipole expansion

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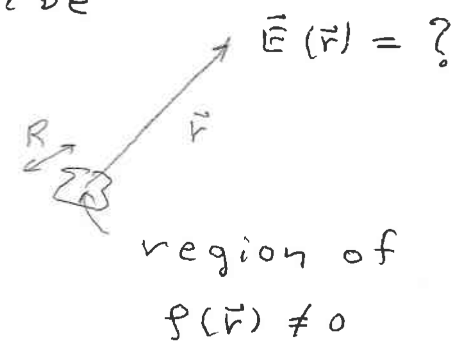


Looks like a point charge at large distance

Find next term in r/R expansion

$$\frac{1}{|\vec r - \vec r'|^3} = \frac{1}{[\vec r^{\,2} - 2\vec r \cdot \vec r' + \vec r'^{\,2}]^{3/2}} \approx \frac{1}{r^3} \frac{1}{\left[1 - \frac{2\vec r \cdot \vec r'}{r^2}\right]^{3/2}}$$

$$\approx \frac{1}{r^2}\left[1 + \frac{3}{2} \frac{2\vec r \cdot r'}{r^2} + \cdots\right]$$

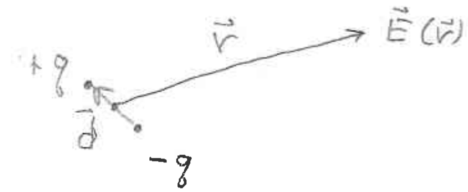
$$\Rightarrow \vec E(\vec r) = \int d^3r' \frac{\vec r - \vec r'}{|\vec r - \vec r'|^3} \approx \int d^3r' \frac{\vec r - \vec r'}{r^3}\left[1 + 3\frac{\vec r \cdot \vec r'}{r^2}\right]\rho(\vec r')$$

$$= \int d^3r' \rho(\vec r') \left\{ \frac{\vec r}{r^3} + \vec r \, \frac{3\,\vec r \cdot \vec r'}{r^5} - \frac{\vec r'}{r^3} + \cdots \right.$$

use $Q = \int d^3r' \rho(\vec r')$ charge; $\vec D = \int d^3r' \, \rho(\vec r') \, \vec r'$ electric dipole moment

$$\vec E(r) = \underbrace{Q\frac{\vec r}{r^3}}_{\text{monopole}} + \underbrace{\frac{3\hat r(\hat r \cdot \vec D) - \vec D}{r^3}}_{\text{dipole}} + \text{quadrupole}$$

Easy example of a dipole



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$$\approx \frac{1}{r^2}\left[1+\frac{3}{2}\,\frac{2\vec r\cdot\vec r'}{r^2}+\dots\right]$$

$$\Rightarrow\quad \vec E(\vec r) = \int d^3r'\,\frac{\vec r-\vec r'}{|\vec r-\vec r'|^3} \approx \int d^3r'\,\frac{\vec r-\vec r'}{r^3}\left[1+3\,\frac{\vec r\cdot\vec r'}{r^2}\right]\rho(\vec r')$$

$$= \int d^3r'\,\rho(\vec r')\left\{\frac{\vec r}{r^3}+\vec r\,\frac{3\,\vec r\cdot\vec r'}{r^5}-\frac{\vec r'}{r^3}+\dots\right.$$

use $Q = \int d^3r'\,\rho(\vec r')$ — charge

$\vec D = \int d^3r'\,\rho(\vec r')\,\vec r'$ — electric dipole moment

$$\vec E(r) = \underbrace{Q\,\frac{\vec r}{r^3}}_{\text{monopole}} + \underbrace{\frac{3\hat r(\hat r\cdot\vec D)-\vec D}{r^3}}_{\text{dipole}} + \text{quadrupole}$$

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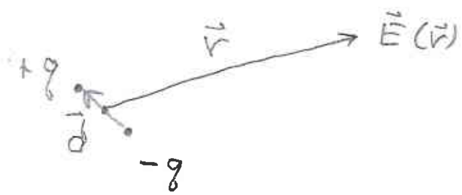

$$E(\vec r) = q\,\frac{\vec r-\frac{\vec d}{2}}{|\vec r-\frac{\vec d}{2}|^3} - q\,\frac{\vec r+\frac{\vec d}{2}}{|\vec r+\frac{\vec d}{2}|^3}$$

$$\approx q\left(\vec r-\frac{\vec d}{2}+3\,\frac{\vec r}{r^3}\,\frac{\vec r\cdot\vec d/2}{r^2}\right) - q\left(\vec r+\frac{\vec d}{2}-3\,\frac{\vec r}{r^3}\,\frac{\vec r\cdot\vec d/2}{r^2}\right)$$

$$= q\,\frac{1}{r^3}\left[3\hat r\,(\hat r\cdot\vec d)-\vec d\right]$$

$\Rightarrow$ dipole moment $\vec D = q\vec d$

b) Since $\vec E$ & $\vec B$ both obey the same equations when $\rho(\vec r)=0$ & $\vec j(\vec r)=0$ the same expansion must hold for $\vec B(\vec r)$ for large $\vec r$

$$\vec B(\vec r) = q^*\,\frac{\vec r}{r^3} + \frac{3\hat r\,(\vec\mu\cdot\hat r)-\vec\mu}{r^3}$$

where $\vec\mu$ is the magnetic dipole moment

$$E(\vec r) = q\,\frac{\vec r - \frac{\vec d}{2}}{\left|\vec r - \frac{\vec d}{2}\right|^3} - q\,\frac{\vec r + \frac{\vec d}{2}}{\left|\vec r + \frac{\vec d}{2}\right|^3}$$

$$\approx q\left(\vec r - \frac{\vec d}{2} + 3\frac{\vec r}{r^3}\frac{\vec r\cdot\vec d/2}{r^2}\right)$$
$$- q\left(\vec r + \frac{\vec d}{2} - 3\frac{\vec r}{r^3}\frac{\vec r\cdot\vec d/2}{r^2}\right)$$

$$= q\,\frac{1}{r^3}\left[3\hat r\,(\hat r\cdot\vec d) - \vec d\right]$$

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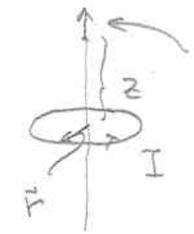
Find $\vec\mu$ for our loop of current



$$B(z) = \hat z\,\frac{2\pi I}{c}\frac{R^2}{r^3}$$

$$= \left.\frac{3\,\vec\mu\cdot\hat r\;\hat r - \vec\mu}{r^3}\right|_{\vec r = r\hat z}$$

symmetry requires $\vec\mu = \mu\hat z$

$$\frac{2\pi I}{c}\frac{R^2}{r^3} = \frac{2\mu}{r^3} \Rightarrow \mu = \frac{\pi R^2 I}{c}$$

Example: A mass M with axial symmetry has uniform mass and charge density & rotates about its axis of symmetry with angular velocity $\vec\omega$, then $\vec\mu$ & $\vec L$ are proportional.

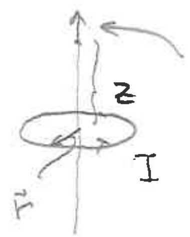
First consider our ring:

$$\mu = \frac{\pi R^2 I}{c} \qquad L = MR^2\omega$$

$$I = \frac{Q}{\tau_{\text{one rotation}}} = \frac{Q}{2\pi/\omega}$$

$$\mu = \frac{\pi R^2}{c}\frac{Q}{2\pi}\omega = \frac{Q}{2Mc}(MR^2\omega) = \underbrace{\frac{Q}{2Mc}}_{\text{gyromagnetic ratio}} L$$

Find $\vec{\mu}$ for our loop of current



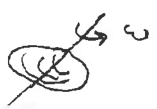
$$B(z) = \hat{z}\,\frac{2\pi I}{c}\,\frac{R^2}{r^3}$$

$$= \left.\frac{3\vec{\mu}\cdot\hat{r}\,\hat{r} - \vec{\mu}}{r^3}\right|_{\vec{r} = r\hat{z}}$$

symmetry requires $\vec{\mu} = \mu\,\hat{z}$

$$\frac{2\pi I}{c}\,\frac{R^2}{r^3} = \frac{2\mu}{r^3} \Rightarrow \mu = \frac{\pi R^2 I}{c}$$

Example: A mass M with axial symmetry has uniform mass and charge density & rotates about its axis of symmetry with angular velocity $\vec{\omega}$, then $\vec{\mu}$ & $\vec{L}$ are proportional.



First consider our ring:

$$\mu = \frac{\pi R^2 I}{c} \qquad L = MR^2\omega$$

$$I = \frac{Q}{\tau_{\text{one rotation}}} = \frac{Q}{2\pi/\omega}$$

$$\mu = \frac{\pi R^2}{c}\,\frac{Q}{2\pi}\,\omega = \frac{Q}{2Mc}(MR^2\omega) = \overbrace{\frac{Q}{2Mc}}^{\text{gyromagnetic ratio}} L$$

This will also be true for our axially symetric solid with proportional mass and charge density – divide it into rings!

For an electron

$$\vec{\mu}_{e^-} = \left[2 + \frac{\alpha}{\pi} + \dots\right]\frac{e}{2m_{e^-}c}\,\vec{L}$$

$$\alpha \approx \frac{1}{137}$$

mixture of e^- & $2e^- + e^+$

$$\rho_{mass} \neq \rho_{charge}$$

For a proton

$$\vec{\mu}_p = 5.583\,\frac{e}{2m_p c}\,\vec{L} \qquad \rho_{mass} \neq \rho_{charge}$$

proton ~ u u d

$$q_u = \frac{2}{3}e$$

$$q_d = -\frac{1}{3}e$$