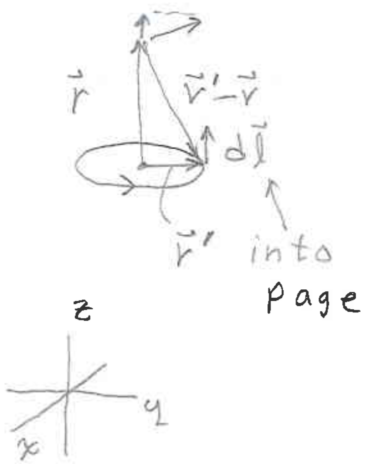
251



$$\vec B = -\frac{I}{c}\oint_C \frac{\vec r - \vec r'}{|\vec r - \vec r'|^3} \times d\vec l$$

$$\vec B = \hat z \,\frac{I}{c}\,\frac{1}{z^2+R^2} \times \frac{R}{\sqrt{z^2+R^2}} \times 2\pi R$$

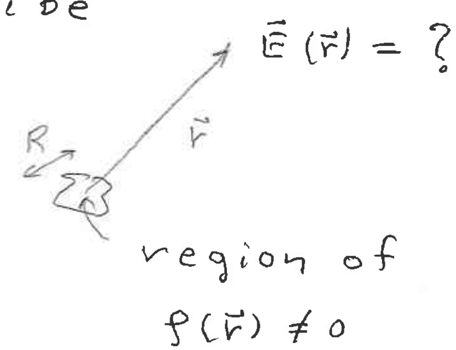
$$= \frac{2\pi I}{c}\,\frac{R^2}{(z^2+R^2)^{3/2}}\,\hat z$$

4. Multipole expansion

The behavior of $\vec E(\vec r)$ & $\vec B(\vec r)$ far from localized charges and currents which produced the is simple to describe

a) Easiest for $\vec E(\vec r)$

$$\vec E(\vec r) = \int d^3r'\,\rho(r')\,\frac{\vec r - \vec r'}{|\vec r - \vec r'|^3}$$



for r large, $r \gg R$

$$\approx \frac{\vec r}{|\vec r|^3}\underbrace{\int d^3r'\,\rho(r')}_{Q}$$

$$= Q\,\frac{\vec r}{|\vec r|^3}$$

Looks like a point charge at large distance

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Find next term in r/R expansion

$$\frac{1}{|\vec r - \vec r'|^3} = \frac{1}{[\vec r^{\,2} - 2\vec r\cdot\vec r' + \vec r'^{\,2}]^{3/2}} \approx \frac{1}{r^3}\,\frac{1}{\left[1 - \frac{2\vec r\cdot\vec r'}{r^2}\right]^{3/2}}$$

$$\approx \frac{1}{r^2}\left[1 + \frac{3}{2}\,\frac{2\vec r\cdot\vec r'}{r^2} + \dots\right]$$

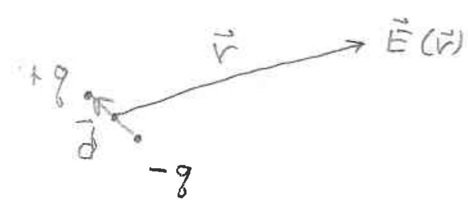
$$\Rightarrow \vec E(\vec r) = \int d^3r'\,\frac{\vec r - \vec r'}{|\vec r - \vec r'|^3} \approx \int d^3r'\,\frac{\vec r - \vec r'}{r^3}\left[1 + 3\,\frac{\vec r\cdot\vec r'}{r^2}\right]\rho(\vec r')$$

$$= \int d^3r'\,\rho(\vec r')\left\{\frac{\vec r}{r^3} + \vec r\,\frac{3\,\vec r\cdot\vec r'}{r^5} - \frac{\vec r'}{r^3} + \dots\right.$$

use $Q = \int d^3r'\,\rho(\vec r')$ charge $\qquad \vec D = \int d^3r'\,\rho(\vec r')\,\vec r'$ electric dipole moment

$$\vec E(r) = \underbrace{Q\,\frac{\vec r}{r^3}}_{\text{monopole}} + \underbrace{\frac{3\hat r(\hat r\cdot\vec D) - \vec D}{r^3}}_{\text{dipole}} + \text{quadrupole}$$

Easy example of a dipole



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$$= \int d^3r'\rho(\vec r')\left\{\frac{\vec r}{r^3}+\vec r\,\frac{3\vec r\cdot\vec r'}{r^5}-\frac{\vec r'}{r^3}+\dots\right.$$

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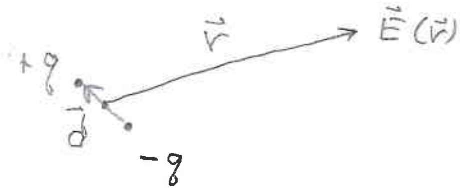

$$E(\vec r) = q\frac{\vec r-\frac{\vec d}{2}}{|\vec r-\frac{\vec d}{2}|^3} - q\frac{\vec r+\frac{\vec d}{2}}{|\vec r+\frac{\vec d}{2}|^3}$$

$$\approx q\left(\vec r-\frac{\vec d}{2}+3\frac{\vec r}{r^3}\frac{\vec r\cdot\vec d/2}{r^2}\right) - q\left(\vec r+\frac{\vec d}{2}-3\frac{\vec r}{r^3}\frac{\vec r\cdot\vec d/2}{r^2}\right)$$

$$= q\frac{1}{r^3}\left[3\hat r(\hat r\cdot\vec d)-\vec d\right]$$

$\Rightarrow$ dipole moment $\vec D = q\vec d$

b) Since $\vec E$ + $\vec B$ both obey the same equations when $\rho(\vec r)=0$ + $\vec J(\vec r)=0$ the same expansion must hold for $\vec B(\vec r)$ for large $\vec r$

$$\vec B(\vec r) = q^*\frac{\vec r}{r^3} + \frac{3\hat r(\vec\mu\cdot\hat r)-\vec\mu}{r^3}$$

where $\vec\mu$ is the magnetic dipole moment

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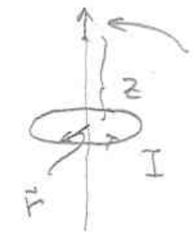
Find $\vec\mu$ for our loop of current



$$B(z) = \hat z\,\frac{2\pi I}{c}\,\frac{R^2}{r^3}$$

$$= \frac{3\,\vec\mu\cdot\hat r\;\hat r - \vec\mu}{r^3}\Bigg|_{\vec r = r\hat z}$$

symmetry requires $\vec\mu = \mu\,\hat z$

$$\frac{2\pi I}{c}\frac{R^2}{r^3} = \frac{2\mu}{r^3} \Rightarrow \mu = \frac{\pi R^2 I}{c}$$

Example: A mass M with axial symmetry has uniform mass and charge density & rotates about its axis of symmetry with angular velocity $\vec\omega$, then $\vec\mu$ & $\vec L$ are proportional.

First consider our ring:

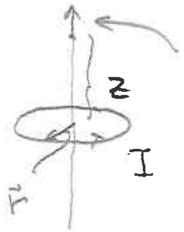
$$\mu = \frac{\pi R^2 I}{c} \qquad L = MR^2\omega$$

$$I = \frac{Q}{\tau_{\text{one rotation}}} = \frac{Q}{2\pi/\omega}$$

gyromagnetic ratio

$$\mu = \frac{\pi R^2}{c}\frac{Q}{2\pi}\omega = \frac{Q}{2Mc}(MR^2\omega) = \underbrace{\frac{Q}{2Mc}}\,L$$

Find $\vec{\mu}$ for our loop of current



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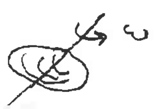

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This will also be true for our axially symetric solid with proportional mass and charge density
- divide it into rings!

For an electron

$$\vec{\mu}_{e^-} = \left[2 + \frac{\alpha}{\pi} + \cdots\right]\frac{e}{2m_{e^-}c}\,\vec{L}$$

$$\alpha \approx \frac{1}{137}$$

mixture of e^- & $2e^- + e^+$

$$\rho_{mass} \neq \rho_{charge}$$

For a proton

$$\vec{\mu}_p = 5.583\,\frac{e}{2m_p c}\,\vec{L} \qquad \rho_{mass} \neq \rho_{charge}$$

proton ~ u u d

$$q_u = \frac{2}{3}e$$

$$q_d = -\frac{1}{3}e$$

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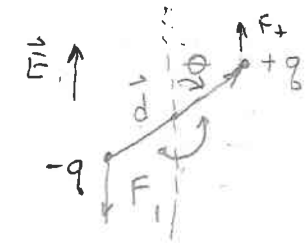
Finally ask what forces act on a dipole if it is place in a uniform field



It experiences a torque $|\vec{\tau}| = \left[\frac{d}{2} \sin\theta \, Eq\right] \times 2$

or $\vec{\tau} = \vec{D} \times \vec{E}$

Since $\vec{E}$, $\vec{D}$ and $\vec{B}$, $\vec{\mu}$ are in close analogy

$$\vec{\tau} = \vec{\mu} \times \vec{B}$$

If $\vec{\mu}$ comes from rotation and angular momentum, then

$$\vec{\mu} = \frac{Q}{2mc} \vec{L} \qquad \vec{\tau} = \vec{\mu} \times B$$

$$\& \ \frac{d\vec{L}}{dt} = \vec{\tau} = \vec{\mu} \times \vec{B} = \frac{Q}{2mc} \vec{L} \times B$$

or $$\frac{d\vec{L}}{dt} = -\left[\frac{Q\vec{B}}{2mc}\right] \times \vec{L} \Rightarrow \vec{\omega}_{prec} = \frac{QB}{2mc}$$

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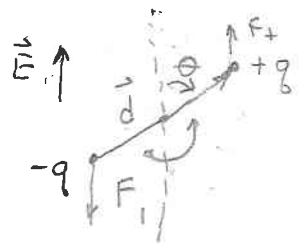

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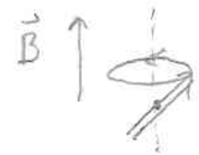
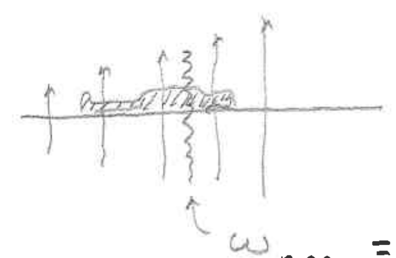


$$\vec{\mu} = \frac{Q}{2mc}\vec{L}$$

Larmor precession

$$\omega_{prec} = \frac{QB}{2mc}$$

can impose a radio frequency E & M field with frequency ω & see energy absorbed when $\omega = \omega_{prec}$. Since most of the atomic nuclei in your body each have $\vec{\mu} \neq 0$ a spatially varying $\vec{B}$ field will lose energy only to those nuclei at $\vec{r}$ with $\frac{QB(\vec{r})}{2mc} = \omega_{RF}$



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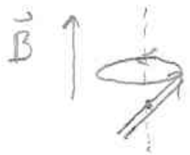
magnetic resonant imaging MRI

use gradients in 3 dimensions and do a lot of image processing



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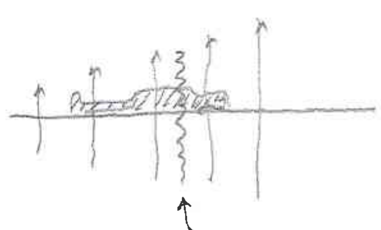

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5. How does a particle with charge q move in a uniform magnetic field $\vec{B}$?

$$\frac{d\vec{p}}{dt} = \frac{q}{c}\vec{v}\times\vec{B}$$

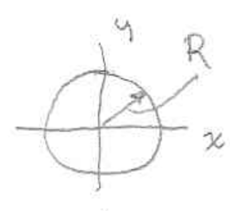
$$= \frac{q}{m\gamma_v c}\vec{p}\times\vec{B} = -\frac{q\vec{B}}{m\gamma_v c}\times\vec{p}$$

Thus, $\vec{p}$ rotates with angular velocity $\vec{\omega}_{cyclotron} = -\frac{q\vec{B}}{m\gamma_v c}$

and the particle moves in a circle

$$x(t) = R\cos\omega_c t \qquad y(t) = R\sin\omega_c t$$

$$v_x = -R\omega_c\sin\omega_c t \qquad v_y = R\omega_c\cos\omega_c t$$



$$v = \omega_c R \qquad p = mv\gamma_v = m\omega_c R\gamma_v$$

or $p = m\cdot\frac{qB}{\cancel{m}\cancel{\gamma_v}c}\cdot R\cancel{\gamma_v}$

$$R = \frac{pc}{qB}$$

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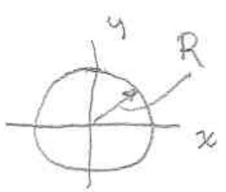

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$$\text{or} \quad p = m \cdot \frac{qB}{m\cancel{\gamma_v} c} \cdot R\cancel{\gamma_v}$$

$$R = \frac{pc}{qB}$$

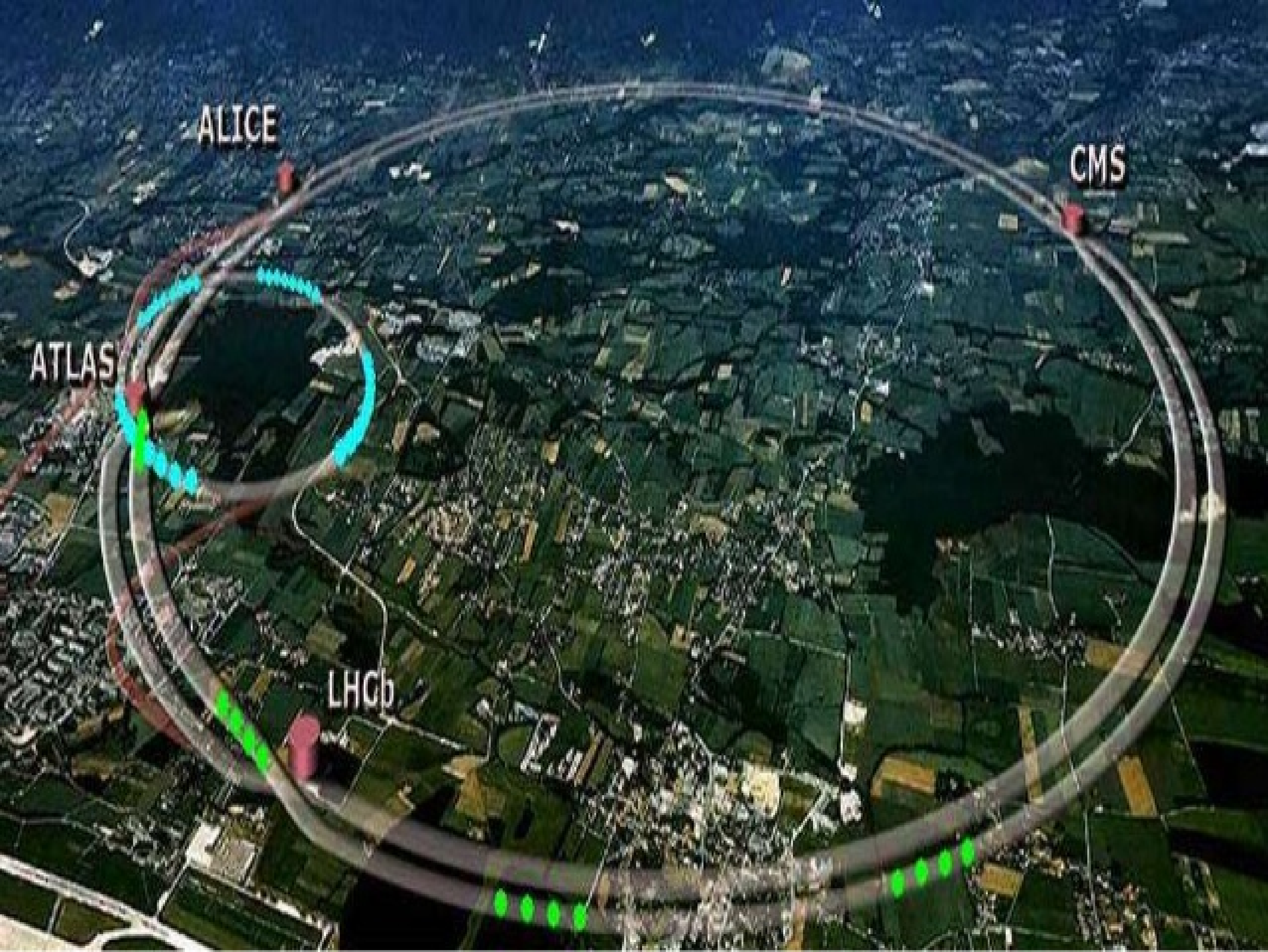
Problem: The Large Hadron Collider (LHC) at CERN uses $B = 8.36$ Tesla and accelerates protons to 7 TeV $= 7\times10^{12}$ eV.

How large is it ?

$$R = \frac{pc}{qB} = \frac{7\times10^{12}\, e \times \overbrace{\frac{1}{300}\,\text{stat volt}}^{\frac{esu}{cm}}}{e \times 8.3\times10^{4}\ \frac{esu}{cm^2}}$$

$$= \frac{7}{3\times8.3} \times 10^{6}\ \text{cm} = 2.8\ \text{km}$$

(actual radius is 4.3 km


ALICE
CMS
ATLAS
LHCb

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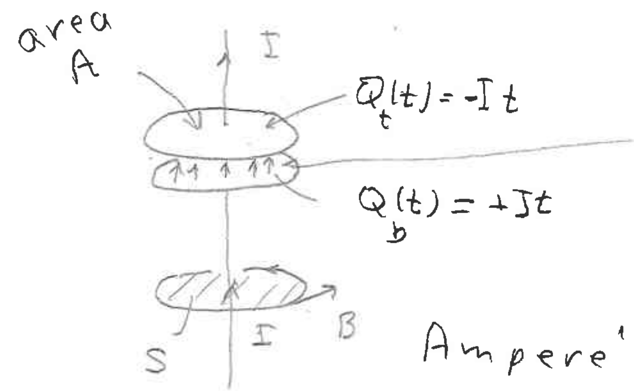
C Now consider changing fields

1. Maxwell's displacement current



$$E = \frac{Q_b(t)}{A} \cdot 4\pi = \frac{4\pi I t}{A}$$

Ampere's law $\vec\nabla \times \vec B = \frac{4\pi}{c}\vec j$

$$\oint_{\partial S} \vec B \cdot d\vec l = \frac{4\pi}{c}\int_S \hat n \cdot \vec j\, ds = \frac{4\pi}{c} I$$

However, in this problem we could choose S to pass between capacitor plates where $\vec j = 0$ and find a contradiction: $\vec B = 0$??

But we forgot $\vec\nabla \times \vec B = \frac{4\pi}{c} j + \frac{1}{c}\frac{\partial \vec E}{\partial t}$!

$$\oint_{\partial S'} \vec B \cdot d\vec l = \frac{1}{c}\int \frac{d\vec E}{dt} \cdot \hat n\, dS = \frac{1}{c}\frac{d}{dt}\left(\frac{4\pi I}{A} t\right) A$$

$$= \frac{4\pi}{c} I$$

Problem solved

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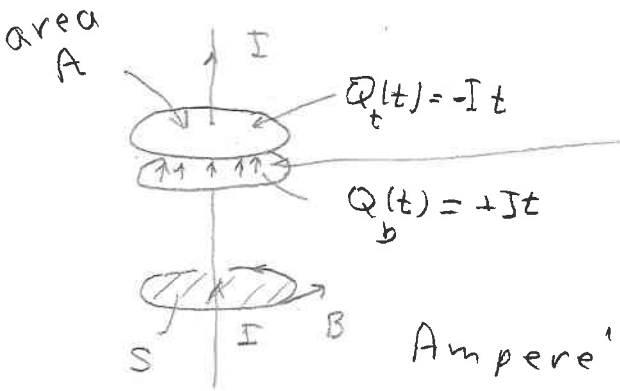

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$$= \frac{4\pi}{c} I \qquad \text{Problem solved}$$

This consistency is guaranteed!

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{1}{c} \frac{\partial E}{\partial t}$$

take the divergence of both sides

$$0 = \vec{\nabla} \cdot [\vec{\nabla} \times \vec{B}] = \frac{4\pi}{c} \vec{\nabla} \cdot \vec{J} + \frac{1}{c} \underbrace{\vec{\nabla} \frac{\partial \vec{E}}{\partial t}}_{\frac{\partial}{\partial t} \underbrace{\vec{\nabla} \cdot \vec{E}}_{4\pi \rho}}$$

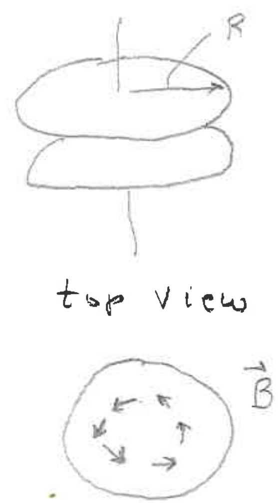
$$= \frac{4\pi}{c} \underbrace{\left(\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right)}_{= 0 \checkmark}$$

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

is required for

$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{J} + \frac{\partial E}{\partial t}$ to make sense!

3. Find $\vec{B}$ field between plates of charging capacitor



$\vec{B}$ will be tangent to circles concentric wit symmetry axis

$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \frac{1}{c} \int_S \frac{\partial \vec{E}}{\partial t} \cdot \hat{n} \, ds$$

$$2\pi r B(r) = \frac{1}{c} \frac{I}{\pi R^2} \times 4\pi \quad \pi r^2$$

$$B(r) = \frac{2I}{R^2 c} r$$