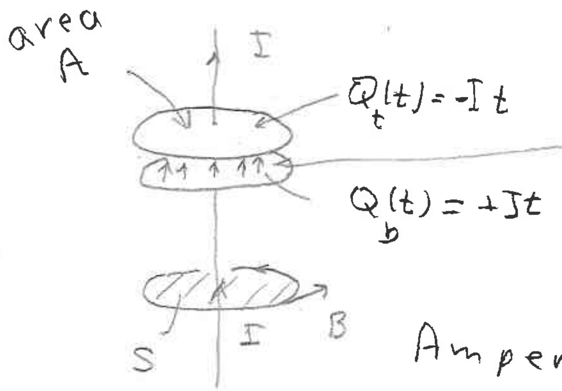
C Now consider changing fields

1. Maxwell's displacement current



$$E = \frac{Q_b(t)}{A} \cdot 4\pi = \frac{4\pi I t}{A}$$

Ampere's law $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j}$

$$\oint_{\partial S} \vec{B} \cdot d\vec{l} = \frac{4\pi}{c} \int_S \hat{n} \cdot \vec{j}\, ds = \frac{4\pi}{c} I$$

However, in this problem we could choose S to pass between capacitor plates where $\vec{j} = 0$ and find a contradiction: $\vec{B} = 0$??

But we forgot $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$!

$$\oint_{\partial S'} \vec{B} \cdot d\vec{l} = \frac{1}{c} \int \frac{d\vec{E}}{dt} \cdot \hat{n}\, ds = \frac{1}{c} \frac{d}{dt} \left(\frac{4\pi I}{A} t \right) A$$

$$= \frac{4\pi}{c} I$$ Problem solved

This consistency is guaranteed!

$$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial E}{\partial t}$$

take the divergence of both sides

$$0 = \vec{\nabla} \cdot \left[\vec{\nabla} \times \vec{B} \right] = \frac{4\pi}{c} \vec{\nabla} \cdot \vec{j} + \frac{1}{c} \underbrace{\vec{\nabla} \frac{\partial \vec{E}}{\partial t}}_{\frac{\partial}{\partial t} \underbrace{\vec{\nabla} \cdot \vec{E}}_{4\pi\rho}}$$

$$= \frac{4\pi}{c} \underbrace{\left(\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} \right)}_{= 0 \checkmark}$$

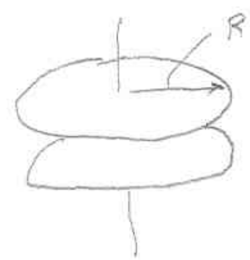
$\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$

is required for

$\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{\partial E}{\partial t}$ to make sense!

---

3. Find $\vec{B}$ field between plates of charging capacitor



top view

$\vec{B}$ will be tangent to circles concentric wit symmetry axis

$$\oint_{\partial S} \vec{B} \cdot d\vec{r} = \frac{1}{c} \int_S \frac{\partial \vec{E}}{\partial t} \cdot \hat{n}\, ds$$

$$2\pi r\, B(r) = \frac{1}{c} \frac{I}{\pi R^2} \times 4\pi \; \pi r^2$$

$$B(r) = \frac{2I}{R^2 c} r$$

This consistency is guaranteed!

$$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}$$

take the divergence of both sides

$$0 = \vec{\nabla}\cdot\left[\vec{\nabla}\times\vec{B}\right] = \frac{4\pi}{c}\vec{\nabla}\cdot\vec{J} + \frac{1}{c}\underbrace{\vec{\nabla}\cdot\frac{\partial\vec{E}}{\partial t}}_{\frac{\partial}{\partial t}\underbrace{\vec{\nabla}\cdot\vec{E}}_{4\pi\rho}}$$

$$= \frac{4\pi}{c}\underbrace{\left(\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t}\right)}_{=0\ \checkmark}$$

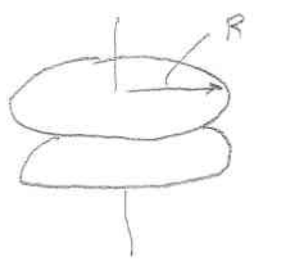
$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

is required for

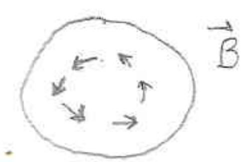
$\vec{\nabla}\times\vec{B} = \frac{4\pi}{c}\vec{J} + \frac{\partial\vec{E}}{\partial t}$ to make sense!

---

3. Find $\vec{B}$ field between plates of charging capacitor



top view



$\vec{B}$ will be tangent to circles concentric wit symmetry axis

$$\oint_{\partial S}\vec{B}\cdot\vec{d}r = \frac{1}{c}\int_S \frac{\partial\vec{E}}{\partial t}\cdot\hat{n}\,dS$$

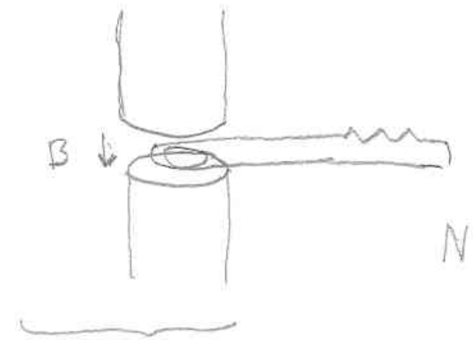
$$2\pi r\,B(r) = \frac{1}{c}\,\frac{I}{\pi R^2}\times 4\pi\ \ \pi r^2$$

$$B(r) = \frac{2I}{R^2 c}\,r$$

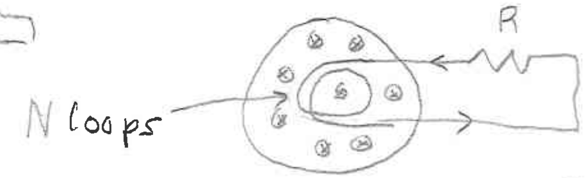
2. Faraday's law

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$$

changing $\vec{B}$ produces $\vec{E}$



top view



$$\oint_{\partial S}\vec{E}\cdot d\vec{l} = -\frac{1}{c}\int_S \frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,dS$$

If the conductivity of the resistor is much smaller than in the wire $\vec{J} = \sigma\vec{E}$ implies $\vec{E}$ in resistor is much larger than $\vec{E}$ in wire

$$\oint_{\partial S}\vec{E}\cdot d\vec{l} \equiv V_{resistor} = IR$$

$$\frac{1}{c}\oint\frac{\partial\vec{B}}{\partial t}\cdot\hat{n}\,dS = \frac{N}{c}\,\underbrace{\dot{\Phi}_B}_{\text{rate of change of flux through one loop}}$$

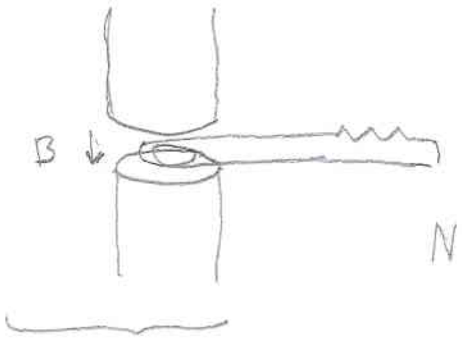
If $\dot{\Phi} > 0$ direction of I is shown & resists change of Φ: <u>Lentz's Law</u>

2. Faraday's Law

2/11/21

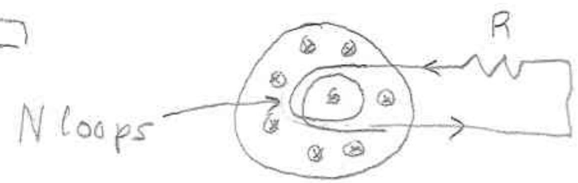
$$\vec{\nabla} \times \vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t}$$

changing $\vec{B}$ produces $\vec{E}$



large magnet

top view



$$\oint_{\partial S} \vec{E}\cdot d\vec{l} = -\frac{1}{c}\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, ds$$

If the conductivity of the resistor is much smaller than in the wire $\vec{j} = \sigma \vec{E}$ implies $\vec{E}$ in resistor is much larger than $\vec{E}$ in wire

$$\oint_{\partial S} \vec{E}\cdot d\vec{l} \equiv V_{resistor} = IR$$

$$\frac{1}{c}\oint \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, ds = \frac{N}{c}\dot{\Phi}_B$$

rate of change of flux through one loop

If $\dot{\Phi} > 0$ direction of I is shown & resists change of Φ: Lentz's Law

Try some concrete numbers for this set up: $r = 1$ cm

$N = 1000$ turns, $B = 1$ kGauss/sec

$R = 10$ Ohms find I

$$I = \frac{1}{R}\frac{1}{c}\pi r^2 \dot{B} N$$

$10^3 \frac{\text{stat volt}}{\cancel{\text{cm}}}$

$$= \frac{1}{10\ \text{Ohm}} \times \frac{1}{3\times 10^{10}\ \frac{\text{cm}}{\text{sec}}}\ \pi\,(1\,\text{cm})^2\ \frac{1\ \text{kG}}{\text{sec}} \times 1000$$

$\frac{\text{Volt}}{\text{Amp}} \leftarrow \frac{1}{300}$ stat volt

$$= \text{Amp} \times \frac{300\times\pi\times 10^3\times 10^3}{10\times 3\times 10^{10}}$$

$$= 3.14\ \text{mA}$$

4. In addition to changing $\vec{B}$ field we can also have a moving loop (as in the armature of a generator

$$\oint_C \vec{E}'\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\overbrace{\left\{\int_{S(t)} \vec{B}(t)\cdot\hat{n}\, ds\right\}}^{\Phi_B(t)}$$

in rest frame of loop

Try some concrete numbers for this set up: $r = 1$ cm

$N = 1000$ turns, $\dot{B} = 1$ kGauss/sec

$R = 10$ Ohms find I

$$I = \frac{1}{R}\frac{1}{c}\pi r^2 \dot{B} N$$

$10^3 \frac{\text{stat volt}}{\cancel{\text{cm}}}$

$$= \frac{1}{10\text{ Ohm}} \times \frac{1}{3\times 10^{10}\frac{\text{cm}}{\text{sec}}}\, \pi\,(1\,\text{cm})^2\, \frac{1\text{ kG}}{\text{sec}} \times 1000$$

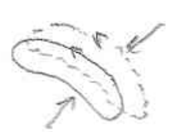
$\frac{\text{Volt}}{\text{Amp}} \leftarrow \frac{1}{300}$ stat volt

$$= \text{Amp} \times \frac{300\times\pi\times 10^3\times 10^3}{10\times 3\times 10^{10}}$$

$$= 3.14 \text{ mA}$$

---

4. In addition to changing $\vec{B}$ field we can also have a moving loop (as in the armature of a generator



$$\oint_C \vec{E}'\cdot d\vec{l} = -\frac{1}{c}\frac{d}{dt}\left\{\overbrace{\int_{S(t)} \vec{B}(t)\cdot\hat{n}\, dS}^{\Phi_B(t)}\right\}$$

in rest frame of loop

proof:

$$\Delta t \frac{d\Phi_B}{dt} = \int_{S(t+\Delta t)} \vec{B}(t+\Delta t)\cdot\hat{n}\, dS - \int_{S(t)} \vec{B}(t)\cdot\hat{n}\, dS$$

I

$$= \int_{S(t+\Delta t)} \left[\vec{B}(t+\Delta t) - \vec{B}(t)\right]\cdot\hat{n}\, dS$$

II

$$+ \underbrace{\int_{S(t+\Delta t)} \vec{B}(t)\cdot\hat{n}\, dS - \int_{S(t)} \vec{B}(t)\cdot\hat{n}\, dS}_{= -\int_{\text{strip}} \vec{B}\cdot\hat{n}\, dS}$$

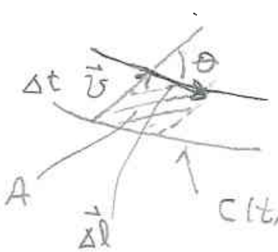
from Gauss' theorem since

$S(t+\Delta t) \cup \text{strip} \cup S(t) =$ boundary of $\mathcal{V}$

$$\& \int_{\partial\mathcal{V}} \vec{B}\cdot\hat{n}\, dS = \int_{\mathcal{V}} \vec{\nabla}\cdot\vec{B}\, dV = 0$$

Look at the strip $\hat{n}\,\Delta A = \Delta t\, v\, dl \sin\theta\, \hat{n}$

$$= d\vec{l}\times\vec{v}\,\Delta t$$



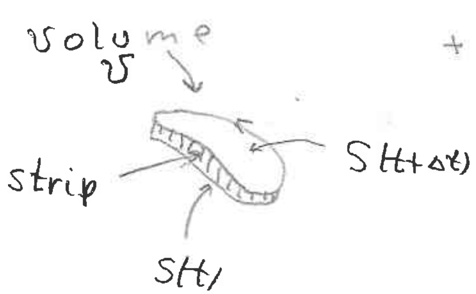
divide by $c\Delta t$

proof:

$$\Delta t \, \frac{d\Phi_B}{dt} = \int_{S(t+\Delta t)} \vec{B}(t+\Delta t)\cdot \hat{n}\, dS - \int_{S(t)} \vec{B}(t)\cdot\hat{n}\, dS$$

$$= \overbrace{\int_{S(t+\Delta t)} \left[\vec{B}(t+\Delta t) - \vec{B}(t)\right]\cdot\hat{n}\, dS}^{\mathrm{I}}$$

$$+ \overbrace{\underbrace{\int_{S(t+\Delta t)} \vec{B}(t)\cdot\hat{n}\, dS - \int_{S(t)} \vec{B}(t)\cdot\hat{n}\, dS}_{= -\int_{strip} \vec{B}\cdot\hat{n}\, dS}}^{\mathrm{II}}$$



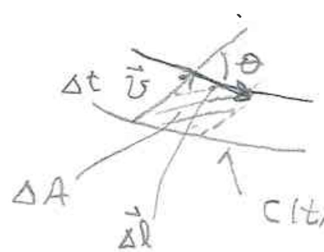
from Gauss' theorem since

$S(t+\Delta t) \cup strip \cup S(t) =$ boundary of $\mathcal{V}$

$$\oint_{\partial\mathcal{V}} \vec{B}\cdot\hat{n}\, dS = \int_{\mathcal{V}} \vec{\nabla}\cdot\vec{B}\, dV = 0$$

Look at the strip $\quad \hat{n}\,\Delta A = \Delta t\, v\, dl \sin\theta\, \hat{n}$

$$= d\vec{l}\times\vec{v}\,\Delta t$$



divide by $c\,\Delta t$

$$\frac{1}{c}\frac{d\Phi_B}{dt} = \overbrace{\frac{1}{c}\oint_S \frac{d\vec{B}}{dt}\cdot\hat{n}\, dS}^{\mathrm{I}} - \overbrace{\int_C \vec{B}\cdot(d\vec{l}\times\vec{v})\,\frac{1}{c}}^{\mathrm{II}}$$

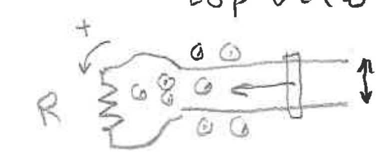
$$= -\int_S (\vec{\nabla}\times\vec{E})\cdot\hat{n}\, dS - \int_C \left(\frac{\vec{v}}{c}\times\vec{B}\right)\cdot d\vec{l}$$

$$= -\int_C \underbrace{\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)}_{\vec{E}'}\cdot\hat{n}\, ds$$

$\vec{E}'$ in rest system of $C(t)$

5. Problem

top view



Bar of mass M slides down two conducting rails making an angle θ with the horizontal. There is a vertical magnetic field through which the conducting bar moves and a resistance R which completes the circuit. What is the bar's terminal velocity?

$$\frac{1}{c}\frac{d\Phi_B}{dt} = \overbrace{\frac{1}{c}\oint_S \frac{d\vec B}{dt}\cdot\hat n\, dS}^{\mathrm{I}} - \overbrace{\int_C \vec B\cdot(d\vec l\times\vec v)\frac{1}{c}}^{\mathrm{II}}$$

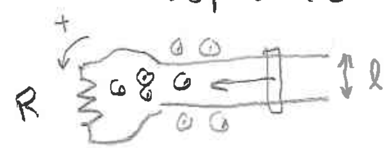
$$= -\int_S (\vec\nabla\times\vec E)\cdot\hat n\, dS - \int_C \left(\frac{\vec v}{c}\times\vec B\right)\cdot d\vec l$$

$$= -\int_C \underbrace{\left(\vec E + \frac{\vec v}{c}\times\vec B\right)}_{\vec E'}\cdot\hat n\, ds$$

$\vec E'$ in rest system of $C(t)$

5. Problem

top view

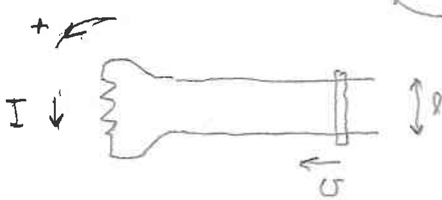


Bar of mass M slides down two conducting rails making an angle θ with the horizontal. There is a vertical magnetic field through which the conducting bar moves and a resistance R which completes the circuit. What is the bar's terminal velocity?

$$\oint \vec E\cdot d\vec l = -\frac{1}{c}\frac{d}{dt}\Phi_B(t) = +\frac{1}{c}\, l v B\cos\theta$$

$$IR = +\frac{1}{c}\, l v B\cos\theta$$

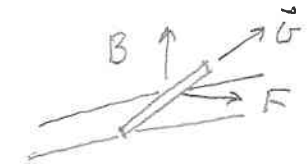
velocity of bar



[I tends to increase Φ_B Lenz]

What is Lorentz force on current in wire? Let Q be total charge in wire flowing at velocity v then $I = \frac{\Delta Q}{\Delta t} = \frac{Q}{l/v}$ or $vQ = lI$

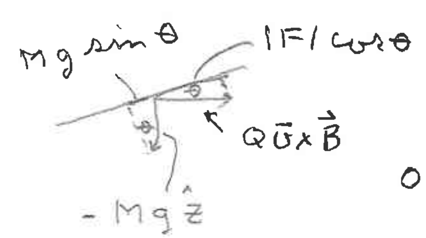
Look at bar



$\vec F = Q\frac{\vec v}{c}\times\vec B$

$\vec F$ is horizontal & $|\vec F| = Q v\frac{B}{c} = lI\frac{B}{c}$

Balance $Mg\sin\theta = |F|\cos\theta$



$$Mg\sin\theta = \cos\theta\, l\,\frac{B}{c}\cdot\left[\frac{lvB}{Rc}\cos\theta\right]$$

or $$v = \frac{Mg\tan\theta\, Rc^2}{l^2B^2\cos\theta}$$