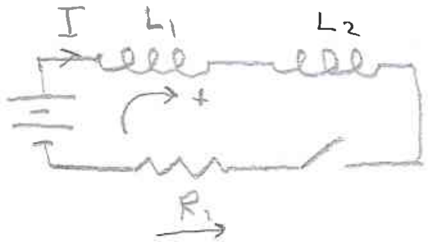
12. Combine inductors



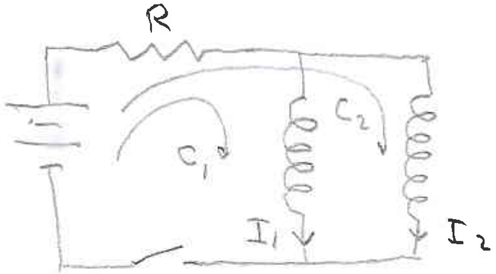
$$\oint \vec{E}\cdot d\vec{l} = -\frac{1}{c}\frac{d\Phi}{dt}$$

$$= -\frac{1}{c}\frac{d\Phi_1}{dt} - \frac{1}{c}\frac{d\Phi_2}{dt}$$

$$-V + IR = -L_1\frac{dI}{dt} - L_2\frac{dI}{dt}$$

$$\text{or} \quad V = IR + \underbrace{(L_1+L_2)}_{L}\frac{dI}{dt}$$

Inductors in series add.



$$I = I_1 + I_2$$

$$\oint_{C_1} \vec{E}\cdot d\vec{r} = -L_1\frac{dI_1}{dt}$$

$$-V + RI = -L_1\frac{dI_1}{dt} \qquad -V + RI = -L_2\frac{dI_2}{dt}$$

divide by L_1 divide by L_2

add:

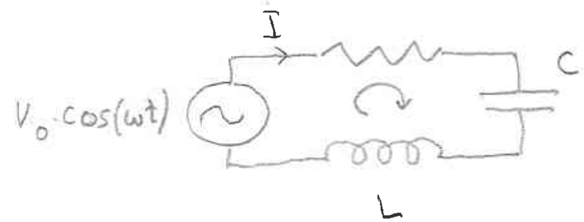
$$(-V+RI)\left(\frac{1}{L_1}+\frac{1}{L_2}\right) = -\frac{dI_1}{dt} - \frac{dI_2}{dt} = -\frac{dI}{dt}$$

$$V = RI + \underbrace{\frac{1}{\frac{1}{L_1}+\frac{1}{L_2}}}_{L}\frac{dI}{dt}$$

Inductors in parallel combine like resistors in parallel

D. AC circuits 02/18/2021

1. Consider circuits driven by a simple oscillating voltage $V(t) = V_0 \cos\omega t$. Could derive from Con Ed an incident radio wave, ...



$$\oint \vec{E}\cdot d\vec{l} = -L\frac{dI}{dt}$$

$$-V_0\cos\omega t + IR + \frac{Q}{C} = -L\frac{dI}{dt}$$

$$\text{or} \quad L\frac{dI}{dt} + RI + \frac{Q}{C} = V_0\cos\omega t$$

$$\text{or} \quad L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0\cos\omega t$$

the same equation as forced S.H.M.

$$m\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + kx = F_0\cos\omega t$$

easily solve by introducing a second equation

$$L\frac{d^2Q_i}{dt^2} + R\frac{dQ_i}{dt} + \frac{1}{C}Q_i = V_0\sin\omega t$$

2/18/21

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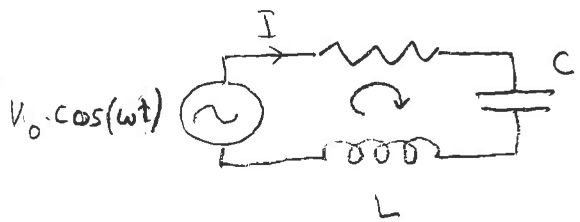

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Calling the solution to the first equation Q_r, multiplying the second equation by "i" & adding

$$L\frac{d^2}{dt^2}\underbrace{(Q_r+iQ_i)}_{Q} + R\frac{d}{dt}(Q_r+iQ_i) + \frac{1}{C}(Q_r+iQ_i)$$

$$= V_0\underbrace{(\cos\omega t + i\sin\omega t)}_{e^{i\omega t}}$$

Thus, solve

$$L\frac{d^2}{dt^2}Q + R\frac{d}{dt}Q + \frac{1}{C}Q = V_0e^{i\omega t}$$

and find the real part of Q.

As in the case of the forced simple harmonic oscillator, this is easily solved by $Q(t) = Q_0e^{i\omega t}$

$$L(i\omega)^2Q_0e^{i\omega t} + R(i\omega)Q_0e^{i\omega t} + \frac{1}{C}Q_0e^{i\omega t} = V_0e^{i\omega t}$$

we have a solution valid for all t if

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However, it is more convenient & conventional to work with current

$$I(t) = \frac{dQ}{dt} = \frac{d}{dt}Q_0 e^{i\omega t} = \underbrace{i\omega Q_0}_{I_0} e^{i\omega t}$$

so use $Q_0 = \dfrac{I_0}{i\omega}$

$$i\omega L I_0 + R I_0 + \frac{C}{i\omega}I_0 = V_0$$

or $$I_0 = \frac{1}{i\omega L + R + \frac{C}{i\omega}} V_0$$

$$I_r(t) = \mathrm{Re}\left\{\frac{1}{i\omega L + R + \frac{C}{i\omega}} V_0 e^{i\omega t}\right\}$$

This is the steady state solution.

Recall we can also add any solution to the homogeneous equation

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = 0$$

solved by $Q(t) = e^{\lambda t}$

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or $\lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$

and $\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$

If $\frac{R}{2L} < \frac{1}{LC}$, underdamped

Use $\omega_1 = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$ and

$$I(t) = \underbrace{Re\left\{ \frac{e^{i\omega t}}{i\omega L + R + \frac{1}{i\omega C}} V_0 \right\}}_{\text{steady-state}} + \underbrace{A e^{-\frac{R}{L}t} \cos(\omega_1 t + \phi)}_{\text{transient}}$$

choose A & ϕ to obey initial conditions

We will usually work at large time and drop the transient solution $e^{-t\frac{R}{L}} \longrightarrow 0$

2. Interpret the math as a complex version of Ohms law

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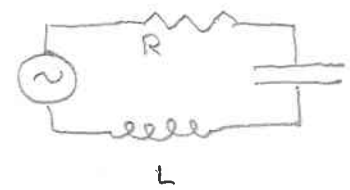
Replace

resistance $R \rightarrow$ impedance Z



$Z = Z_L + Z_R + Z_C$

$Z_L = i\omega L$, $Z_R = R$, $Z_C = \frac{1}{i\omega C}$

Now look at steady-state solution:

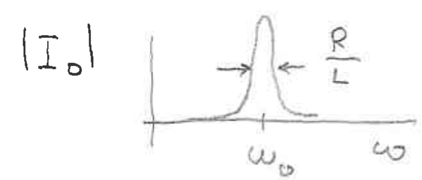
$$I(t) = Re\left\{\frac{V_0 e^{i\omega t}}{i\omega L + R + \frac{1}{i\omega C}}\right\}$$

$$i\omega L + R + \frac{1}{i\omega C} = \left|i\omega L + R - \frac{i}{\omega C}\right| e^{i\phi}$$

where $\tan\phi = \dfrac{\omega L - \frac{1}{\omega C}}{R}$

$$\left|i\omega L - i\frac{1}{\omega C} + R\right| = \sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}$$

$$I(t) = \frac{V_0}{\sqrt{\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2}} \cos(\omega t - \phi)$$



$\omega_0 L - \frac{1}{\omega_0 C} = 0$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$$\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2 = L^2\left[(\omega - \omega_0)^2 + \left(\frac{R}{L}\right)^2\right]$$

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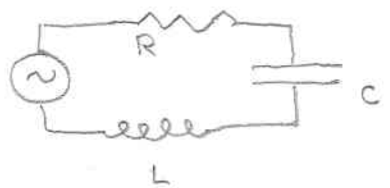

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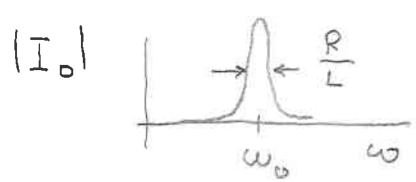

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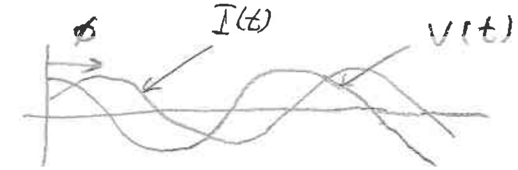
$$\left(\omega L - \frac{1}{\omega C}\right)^2 + R^2 = L^2\left[(\omega - \omega_0)^2 + \left(\frac{R}{L}\right)^2 \right]$$

Such a resonant circuit is at the heart of a radio receiver

- tuning ω_0 will choose the frequency which produces the largest I
- as $R \rightarrow 0$ this response becomes stronger & the selectivity (width of peak) becomes sharper

Describe how I depends on Z:

$$Z = i\omega L + R - \frac{i}{\omega C}$$



a) L Large $\phi > 0$ current lags voltage

b) R large $\phi = 0$ current & voltage in phase

c) $\frac{1}{C}$ large $\phi < 0$ current leads voltage

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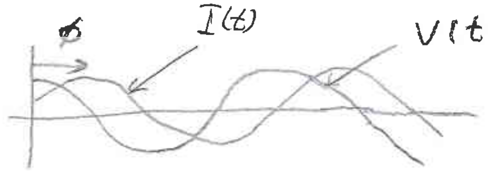

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This same strategy of writing Kirchhoff's laws for real charges:

① $\sum\limits_{\text{closed loops}}$ "V_i" $= 0$ + ② $\sum\limits_{\text{entering node}} I_j = 0$

with real currents and "voltages"

"V_r" $= V_i \cos(\omega t + \phi_i)$ or "V_r" $= L\frac{d^2}{dt^2}Q_r$

or "V_r" $= R\frac{dQ_r}{dt}$ or "V" $= \frac{1}{C}Q_r$

adding imaginary currents & voltages with $Q_r \to iQ_i$ & $V_i\cos(\omega t+\phi) \to iV_i\sin(\omega t+\phi)$

dividing type ① equations by $i\omega$ and adding will give complex Kirchhoff laws with $R_i \to Z_i$ if we hypothesize $Q_r + iQ_i = Q_0 e^{i\omega t}$ for each case.

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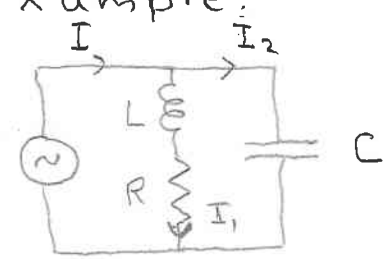
3. Example:



$$Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}}$$

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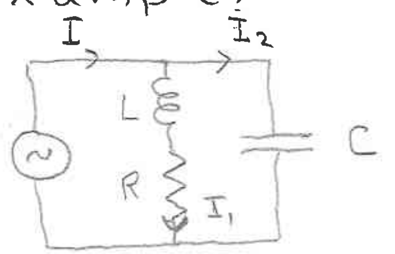


$$Z = \frac{1}{\frac{1}{Z_1} + \frac{1}{Z_2}}$$

$$= \frac{1}{\frac{1}{i\omega L + R} + i\omega C}$$

$$Z = \frac{i\omega L + R}{1 + i\omega C(i\omega L + R)}$$

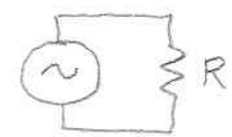
$$|I| = \left|\frac{V_0}{Z}\right| = |V_0| \left[\frac{(1-\omega^2 LC)^2 + \omega^2 R^2 C^2}{R^2 + \omega^2 L^2}\right]^{1/2}$$

for $R \to 0$ + $\omega \to \frac{1}{\sqrt{LC}}$ $\quad Z \to \infty$

The oscillating LC circuit will exactly create applied voltage and no current flows into circuit. When $R \neq 0$ current does flow to compensate power lost in R.

4. Discuss power in AC circuits

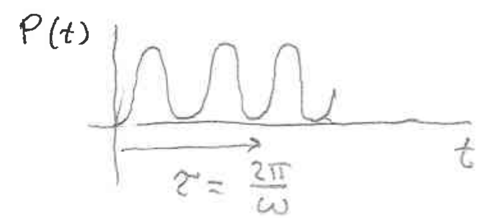
a)



$V = V_0 \cos\omega t$

$I = \frac{V_0}{R} \cos\omega t$

$$P(t) = V(t)\, I(t) = \frac{V_0^2}{R} \cos^2\omega t$$



To calculate average power consumed integrate over one period

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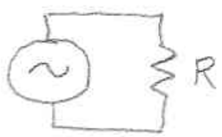
a) 

$$V = V_0 \cos\omega t$$

$$I = \frac{V_0}{R} \cos\omega t$$

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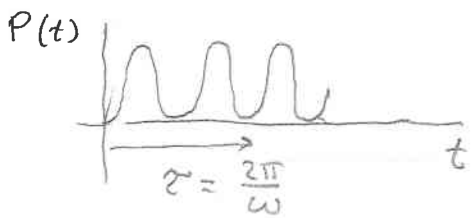

To calculate average power consumed integrate over one period

$$\bar{P} = \frac{1}{\tau} \frac{V_0^2}{R} \int_0^\tau \cos^2[\omega t]$$

$$= \frac{1}{\tau} \frac{V_0^2}{R} \underbrace{\int_0^\tau \frac{1}{2}\left(\cos^2[\omega t] + \sin^2(\omega t)\right) dt}_{\frac{1}{2}\tau}$$

$$= \frac{1}{2} \frac{V_0^2}{R} = \frac{1}{2} V_0 I_0$$

note averaging over many cycles will give the same result.

To avoid the $\frac{1}{2}$ factor AC current and voltage is usually give as the "root mean squared" value

$$V^{RMS} = \left[\int_0^\tau V_0^2 \cos^2 \omega t\right]^{1/2} = \frac{1}{\sqrt{2}} V_0$$

the $\bar{P} = V^{rms} I^{rms}$

For an average over a long time T with $n\tau \leq T < (n+1)\tau$

$$\bar{P} = \frac{1}{T} \int_0^T V_0 I_0 \cos^2 \omega t \, dt$$

$$\approx \frac{1}{n\tau} \underbrace{\int_0^{n\tau} V_0 I_0 \cos^2 \omega t}_{\frac{1}{2} n\tau} = \frac{1}{2} V_0 I_0$$