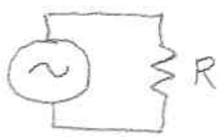
$$Z = \frac{i\omega L + R}{1 + i\omega C(i\omega L + R)}$$

$$|I| = \left|\frac{V_0}{Z}\right| = |V_0| \left[\frac{(1-\omega^2 LC)^2 + \omega^2 R^2 C^2}{R^2 + \omega^2 L^2}\right]^{1/2}$$

for $R \to 0$ & $\omega \to \frac{1}{\sqrt{LC}}$ $\quad Z \to \infty$

The oscillating LC circuit will exactly create applied voltage and no current flows into circuit. When $R \neq 0$ current does flow to compensate power lost in R.

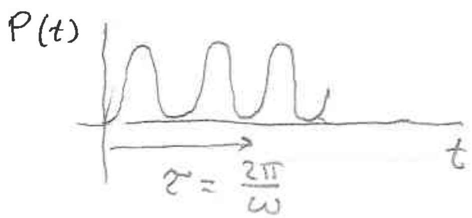
4. Discuss power in AC circuits

a) 

$V = V_0 \cos\omega t$

$I = \frac{V_0}{R} \cos\omega t$

$$P(t) = V(t)\, I(t) = \frac{V_0^2}{R} \cos^2\omega t$$



To calculate average power consumed integrate over one period

$$\bar{P} = \frac{1}{\tau}\frac{V_0^2}{R}\int_0^\tau \cos^2[\omega t]$$

$$= \frac{1}{\tau}\frac{V_0^2}{R}\underbrace{\int_0^\tau \frac{1}{2}\left(\cos^2[\omega t] + \sin^2(\omega t)\right) dt}_{\frac{1}{2}\tau}$$

$$= \frac{1}{2}\frac{V_0^2}{R} = \frac{1}{2} V_0 I_0$$

note averaging over many cycles will give the same result.

To avoid the $\frac{1}{2}$ factor AC current and voltage is usually give as the "root mean squared" value

$$V^{RMS} = \left[\int_0^\tau V_0^2 \cos^2\omega t\right]^{1/2} = \frac{1}{\sqrt{2}} V_0$$

the $\bar{P} = V^{rms} I^{rms}$

For an average over a long time T with $n\tau \le T < (n+1)\tau$

$$\bar{P} = \frac{1}{T}\int_0^T V_0 I_0 \cos^2\omega t\, dt$$

$$\approx \frac{1}{n\tau}\underbrace{\int_0^{n\tau} V_0 I_0 \cos^2\omega t}_{\frac{1}{2}n\tau} = \frac{1}{2} V_0 I_0$$

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02/23/2021

Next consider the case of a complex impedance

$$V_0 = I_0 R + \frac{1}{i\omega C} I_0$$

$$\Rightarrow I_0 = \frac{V_0}{R + \frac{1}{i\omega C}} = \frac{V_0}{\sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}} e^{-i\phi}$$

$$\tan\phi = \frac{-1/\omega C}{R} = -\frac{1}{\omega C R}$$

$$I(t) = re\left[I_0 e^{i\omega t}\right] = \frac{V_0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \cos(\omega t - \phi)$$

$$V(t) = re\left[V_0 e^{i\omega t}\right] = V_0 \cos(\omega t)$$

$$P(t) = \frac{V_0^2}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}}} \cos(\omega t) \underbrace{\cos(\omega t - \phi)}_{\cos(\omega t)\cos\phi + \sin\omega t \sin\phi}$$

Again average over one period

$$\int_0^\tau \cos^2 \omega t \, dt = \frac{\tau}{2}$$

$$\int_0^\tau \sin \omega t \cos \omega t \, dt = \frac{1}{2}\int_0^\tau \sin(2\omega t) \, dt$$

$$= 0$$

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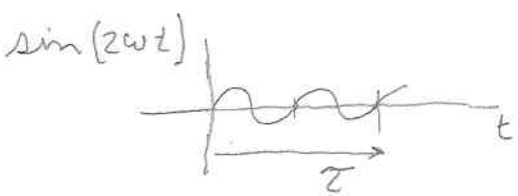

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as $R \to 0$, $\tan\phi = -\frac{1}{R\omega C} \to \infty$

$\phi \to \frac{\pi}{2}$ $\quad \cos\phi \to 0$ $\quad$ & $\bar{P} \to 0$!

Thus, as $R \to 0$ there are no Ohmic losses $P(t)$ alternates between positive and negative. Energy flows into C but later flows out.

This is bad for the company providing the power since there will be energy loss in transmission lines but no power consumed by the consumer and a small electric bill. Con Ed limits a reactive load, $|\phi| \leq \phi_{max}$ or charges extra!

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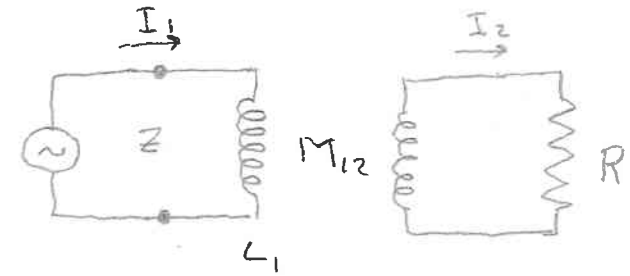
5. Final example



$$V_0 = i\omega L_1 I_1 + i\omega M_{12} I_2$$

$$0 = i\omega L_2 I_2 + R I_2 + i\omega M_{21} I_1$$

$$I_2 = \frac{-i\omega M_{21}}{i\omega L_2 + R} I_1$$

$$\Rightarrow V_0 = \left\{ i\omega L_1 + i\omega M_{12}\frac{-i\omega M_{21}}{i\omega L_2 + R} \right\} I_1$$

$$Z = \frac{V_0}{I_0} = i\omega L_1 + \frac{\omega^2 M_{21}^2}{i\omega L_2 + R}$$

and for small ω the transformed load looks resistive

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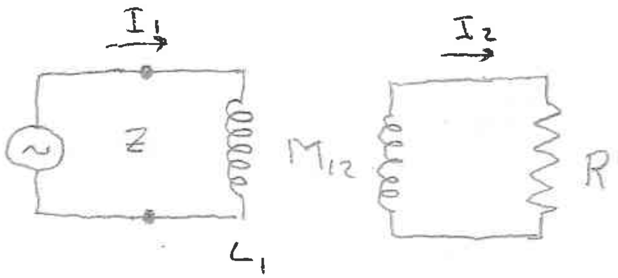

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E. Electromagnetic waves

1. Study Maxwell's equations in a region where $\vec{j}(\vec{r},t) = 0$ & $\rho(\vec{r},t) = 0$

$$\vec\nabla \times \vec B = -\frac{1}{c}\frac{\partial \vec E}{\partial t} \qquad \vec\nabla \times \vec E = \frac{1}{c}\frac{\partial \vec B}{\partial t}$$

$$\vec\nabla \cdot \vec B = 0 \qquad \vec\nabla \cdot \vec E = 0$$

Take the curl of :

$$\underbrace{\vec\nabla \times (\vec\nabla \times \vec E)}_{\vec\nabla(\underbrace{\vec\nabla\cdot\vec E}_{0}) - (\vec\nabla^2)\vec E} = \frac{1}{c}\frac{\partial}{\partial t}(\vec\nabla \times \vec B) = -\frac{1}{c^2}\frac{\partial^2 \vec E}{\partial t^2}$$

$$\Rightarrow \quad \vec\nabla^2 \vec E(\vec r,t) = \frac{1}{c^2}\frac{\partial^2 \vec E(\vec r,t)}{\partial t^2}$$

This is an example of a wave equation and is easily solved by a traveling wave:

try $\vec E(x,y,z,t) = \hat z E_z(x,t)$

$$\vec\nabla^2 E_i(x,y,z,t) = \frac{1}{c^2}\frac{\partial^2 E_i(x,y,z,t)}{\partial t} \quad i=1,2,3$$

becomes

$$\frac{\partial^2}{\partial x^2} E_z(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} E_z(x,t)$$

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Take the curl of :

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becomes

$$\frac{\partial^2}{\partial x^2}E_z(x,t) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}E_z(x,t)$$

One solution is $E_z(x,t) = f(x-ct)$

since $\frac{\partial^2}{\partial x^2}f(x-ct) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}f(x-ct)$

wave moves to the right with speed c.

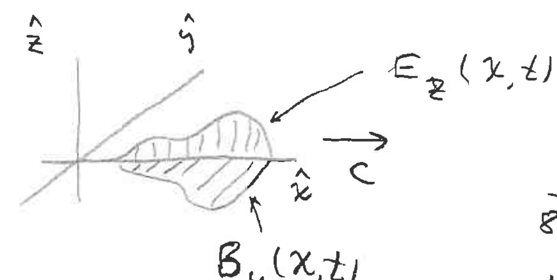
$f(x+ct)$ will be similar but move to the left.

What about $\vec{B}$?

$$\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} = -\hat{x}\frac{\partial}{\partial x}\times\hat{z}\,E_z(x,t)$$

$$= \hat{y}\,\frac{\partial f(x-ct)}{\partial x}$$

$$\Rightarrow \quad \vec{B} = -\hat{y}\,f(x-ct)$$



pulse of light with energy density $\frac{1}{8\pi}\left[\vec{E}^2(x,t)+\vec{B}^2(x,t)\right]$ moving to the right at the speed of light!

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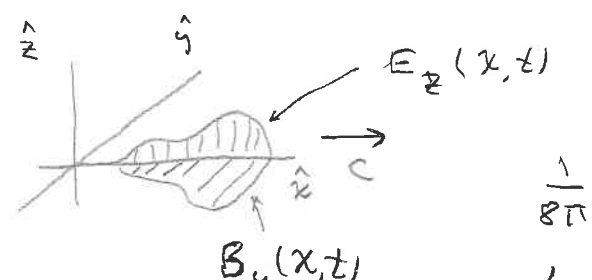

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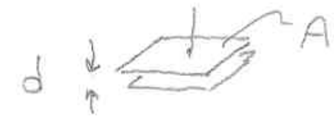
Note for this example $\vec{E}$ & $\vec{B}$ do not depend on y or z: pulse of light extend across the entire y-z plane moving in the +x direction.

---

2. The speed of this wave could have been predicted from simple measurements made with a ruler and clock.

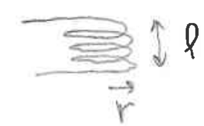
ⓐ Build a capacitor C with measured area and separation



$$V = dE = d\,4\pi\,Q/A = \frac{Q}{C}$$

$$C = A/4\pi d$$

ⓑ Build an inductor



N turns radius R

$$L = \frac{1}{c^2}\,\frac{4\pi^2 r^2 N}{\ell}$$

ⓒ

with clock measure $\omega = \frac{1}{\sqrt{LC}}$

or period $\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{LC}$

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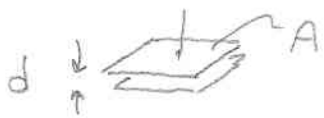

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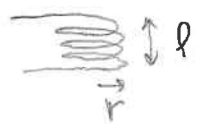
(b) Build an inductor



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(c)

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$$\tau = 2\pi\left[\frac{1}{c^2}\cdot\frac{4\pi^2 r^2 N}{\ell}\cdot\frac{A}{4\pi d}\right]^{1/2}$$

$$\text{\&}\quad c = \underset{\substack{\uparrow \\ 1/\text{time}}}{\frac{2\pi}{\tau}}\left[\underset{\substack{\uparrow \\ \text{length}^2}}{\frac{\pi r^2 A N}{\ell d}}\right]^{1/2}$$

Measure dimensions and period, predict the speed of light!

3. A different perspective:

Look at $\frac{\partial^2 \vec{E}}{\partial t^2} = c^2(\vec{\nabla}^2)\vec{E}$

or $\frac{\partial^2 E_z}{\partial t^2} = c^2 \frac{\partial^2}{\partial x^2} E_z$

as similar to $\frac{d^2}{dt^2} q = -\omega^2 q$ for S.H.M.

except $c^2 \frac{\partial^2}{\partial x^2} \neq -\omega^2$

However, if we choose $E_z(x,t) = \cos kx\, E_z(t)$

the $c^2 \frac{\partial^2}{\partial x^2} = -c^2 k^2 = -\omega^2$!

and $\frac{\partial^2}{\partial t^2}\left[\cos kx\, E_z(t)\right] = -c^2k^2\left[\cos kx\, E_z(t)\right]$

is easy to solve: $E_z(t) = E_0 \cos(kct + \phi)$

$$\tau = 2\pi\left[\frac{1}{c^2}\cdot\frac{4\pi^3 r^2 N}{\ell}\cdot\frac{A}{4\pi d}\right]^{1/2}$$

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and we have a new more specific solution: $E_z(x,t) = E_0 \cos(kx)\cos(ckt+\phi)$.

How is this related to our earlier solution? Not hard to see

$E_z(x,t) = \frac{1}{2}E_0 \cos(kx + ckt + \phi) + \frac{1}{2}\cos(kx - ckt - \phi)$

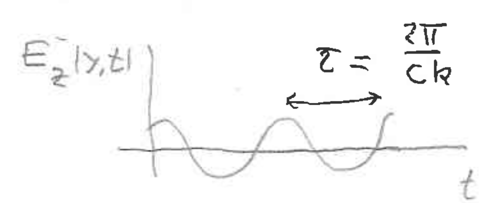
$E_0 \cos kx \cos(ckt + \phi)$ is a "standing wave"

it does not move: keep same x-dependence but oscillates in amplitude

Can be constructed from two traveling waves!

Plot $E_x(x,t) = E_0 \cos(\overbrace{kx - ckt}^{k(x-ct)} - \phi)$

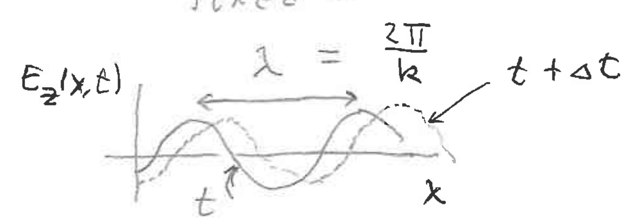
fixed x



$\omega = ck$, $\tau = \frac{2\pi}{\omega}$

$\nu = \frac{1}{\tau}$

fixed t



$c = \frac{\omega}{k} = \frac{\lambda}{\tau}$

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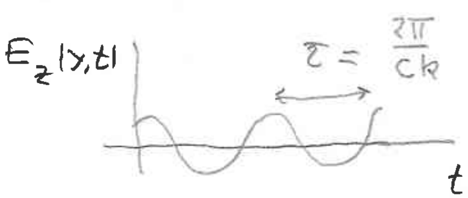
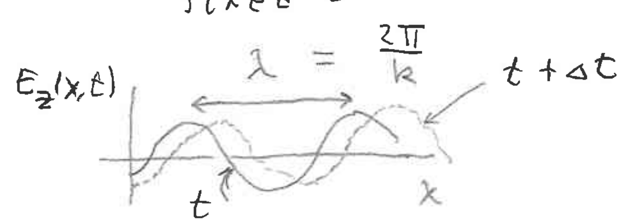


fixed t



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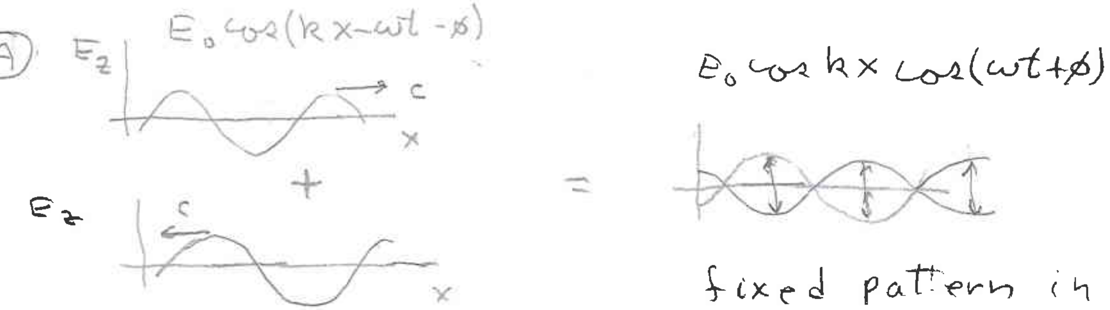
Two important facts:

(A)

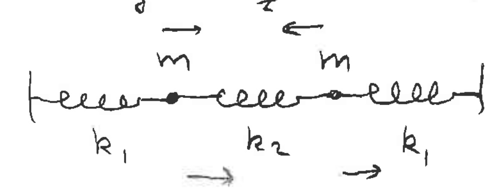


$E_0 \cos(kx+\omega t+\phi)$

fixed pattern in space, amplitude oscillates

(B) Wave equation $\frac{d^2 E_z}{dx^2} = +c^2 \frac{d^2}{dx^2} E_z$

becomes a SHO problem if we choose $E_z(x,t) = E_z(t)\cos(kx)$ for any k! The wave equation contains an infinite number of SHO problems. These coupled SHO behaviors, each with different frequencies can be separated by using $E_z(t)\cos(kx)$! Similar to



two modes each with a different frequency