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# Practice Midterm

1. A long straight wire of radius $R$ carries a total current $I$. Assume that the current $I$ is distributed uniformly over the crossection of the wire.

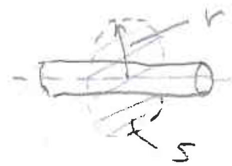

   (a) What is the current density $\vec{j}$ within the wire? [5 points]

   (b) Find the magnetic field $\vec{B}$ outside the wire, $r \geq R$. [14 points]

   (c) Find the magnetic field $\vec{B}$ inside the wire, $r \leq R$. [14 points]

(a) $j = \frac{I}{\pi R^2}$ parallel to wire

(b) $\underbrace{\int_S (\vec{\nabla} \times \vec{B}) \cdot \hat{n} \, ds}$ $= \frac{4\pi}{c} I$




$2\pi r \; B(r) = \frac{4\pi}{c} I$

$B = \frac{2I}{cr}$ vector is tangential to circle concentric with wire, pointing clockwise if viewed in the direction of current flow.

(c) For $r < R$

$2\pi r \; B(r) = \frac{4\pi}{c} \int_S \vec{j} \cdot \hat{n} \, ds = \frac{4\pi}{c} \frac{I}{\pi R^2} \cdot \pi r^2$

or $B(r) = \frac{2I}{cR^2} \cdot r$, same direction as (b)

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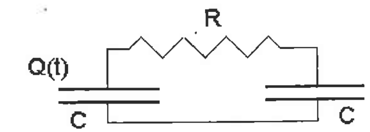
2. Two identical capacitors and a resistor are connected together as shown at the right. Assume that at $t = 0$ the left capacitor has a charge $Q_0$ and the right capacitor is uncharged. Find the charge $Q(t)$ on the left capacitor as a function of time. [33 points]




As charge flows from the left capacitor to the right, the charge on the right capacitor will be $Q_0 - Q(t)$ and the voltage accross the resistor will be $\frac{Q(t)}{C} - \frac{Q_0 - Q(t)}{C}$.

The current given by Ohm's law will be $I(t) R = \frac{Q(t)}{C} - \frac{Q_0 - Q(t)}{C} = \frac{2Q(t) - Q_0}{C}$

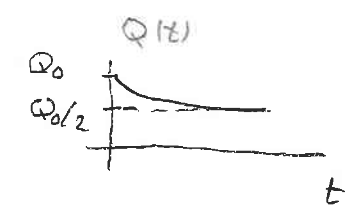
$\therefore \frac{dQ}{dt} = -I(t) = -\frac{2Q(t) - Q_0}{RC}$ use $\tilde{Q} = Q(t) - \frac{Q_0}{2}$

$\frac{d\tilde{Q}}{dt} = -\frac{2}{RC} \tilde{Q} \Rightarrow \tilde{Q} = A e^{-\frac{2}{RC} t}$

$Q(t) = \tilde{Q}(t) + \frac{Q_0}{2} = A e^{-\frac{2}{RC} t} + \frac{Q_0}{2}$

$Q(0) = Q_0 \Rightarrow A = \frac{Q_0}{2}$

$\therefore Q(t) = \frac{Q_0}{2}\left(1 + e^{-\frac{2}{RC} t}\right)$



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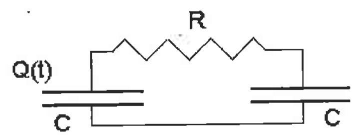


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$$I(t)R = \frac{Q(t)}{C} - \frac{Q_0 - Q(t)}{C} = \frac{2Q(t) - Q_0}{C}$$

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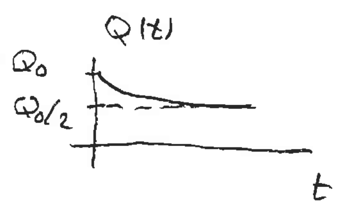
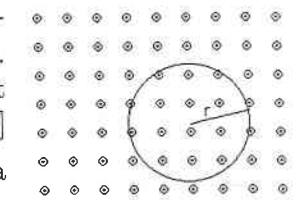





3. Consider a proton with rest energy $mc^2 = 939MeV$ and charge $e = 4.8\ 10^{-10}esu$.

(a) A proton with energy $20\,TeV$ $(20 \times 10^{12}eV)$ moves in a region of uniform magnetic field $\vec{B}$ of magnitude $50KGauss$. If the proton's velocity is perpendicular to $\vec{B}$, show that it moves in a circle and find that circle's radius. [18 points]




(b) A proton with initial velocity $\vec{v}_0$ enters a region with a uniform electric and magnetic field $\vec{E}$ and $\vec{B}$. If the vectors $\vec{E}$ and $\vec{B}$ are orthogonal, what condition must $\vec{v}$ satisfy if the proton is to move through the region without being deflected? [15 points]

(a) $\frac{d\vec{p}}{dt} = e\frac{\vec{v}}{c} \times \vec{B} = -\frac{e}{mc\gamma}\vec{B} \times \vec{p}$ & $\vec{\omega} = -\frac{e}{mc\gamma}\vec{B}$

for circular motion $\vec{v} = \vec{\omega} \times \vec{r}$

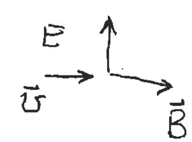
so $v = \omega r = \frac{eB}{mc\gamma} r$, $\mathcal{E} \approx mc^2\gamma$

$$r = \frac{v}{\omega} \approx \frac{c}{\omega} = \frac{mc^2\gamma}{eB} = \frac{mc^2\gamma}{eB} = \frac{\mathcal{E}}{eB}$$

$$= \frac{2 \times 10^{13}\,eV}{e \times 5 \times 10^4 \frac{\text{Stat volt}}{\text{cm}} \times \frac{300\,V}{\text{statvolt}}}$$

$$= \frac{4}{3}10^6\,cm = 13{,}3\,km$$

(b)




$$e\frac{\vec{v}}{c} \times \vec{B} = -e\vec{E}$$

$\vec{v} \parallel \vec{E} \times \vec{B}$ & $|\vec{v}| = c\frac{E}{B}$

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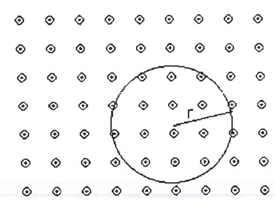


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(a) $$\frac{d\vec{p}}{dt} = e\frac{\vec{v}}{c} \times \vec{B} = -\frac{e}{mc\gamma}\vec{B} \times \vec{p} \quad \& \quad \vec{\omega} = -\frac{e}{mc\gamma}\vec{B}$$

for circular motion $\vec{v} = \vec{\omega} \times \vec{r}$

$$\text{So} \quad v = \omega r = \frac{eB}{mc\gamma} r \quad , \quad \mathcal{E} \approx mc^2\gamma$$

$$r = \frac{v}{\omega} \approx \frac{c}{\omega} = \frac{mc^2\gamma}{eB} = \frac{mc^2\gamma}{eB} = \frac{\mathcal{E}}{eB}$$

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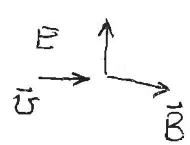


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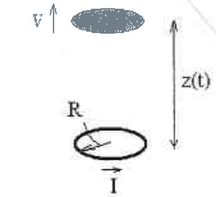
4. A fixed, horizontal, conducting loop of radius $R$ carries a constant current $I$. A uniformly charged, non-conducting disk, also horizontal and of radius $R$, is a distance $z(t)$ directly above the first. You may assume that $z(t) \gg R$. The disk carries charge density $\sigma$.




(a) Find the magnetic field due to the current in the fixed loop at the height $z(t)$. [15 points]

(b) Find the torque $\vec{\tau}$ exerted on the charged disk if it is is moving upward with velocity $v = \dot{z}(t)$. [18 points]

(a) Start with Biot-Savart Law

$$\vec{B}(\vec{r}) = \frac{I}{c}\int \frac{\vec{r} - \vec{r}'(l)}{|\vec{r} - \vec{r}'(l)|^3} \times d\vec{l}$$

$\hat{z}$


$$\vec{B}(z) = \hat{z}\,\frac{I}{c} \cdot \frac{1}{[z^2 + R^2]} \cdot \frac{R}{[z^2 + R^2]^{1/2}} \cdot 2\pi R \approx \hat{z}\,\frac{2\pi R^2 I}{c z^3}$$

(b)

charged dist

$$\Phi_B(r, z) = \frac{2\pi R^2 I}{c z^3} \cdot \pi r^2$$

loop $C$ of radius $r$

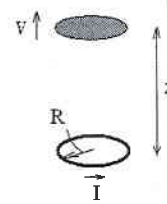
$$\oint_C \vec{E} \cdot d\vec{l} = -\frac{1}{c}\frac{d\Phi(r, z(t))}{dt} = +\frac{1}{c}\,\frac{2\pi R^2 I}{c}\,\pi r^2\,\frac{3}{z^4}\,\underbrace{\frac{dz}{dt}}_{v}$$

$$2\pi r E \Rightarrow E(r) = 3\,\frac{\pi R^2 r I v}{c^2 z^4}$$

$$\vec{\tau} = \hat{z}\int_0^R [2\pi r\, dr\, \sigma] \times E(r) \times r$$

$$= \hat{z}\int_0^R 2\pi\sigma\, 3\,\frac{\pi R^2 I v}{c^2 z^4}\, r^3\, dr = \hat{z}\,\frac{3\pi^2 R^6 I v}{2 c^2 z^4}\,\sigma$$

4. A fixed, horizontal, conducting loop of radius $R$ carries a constant current $I$. A uniformly charged, non-conducting disk, also horizontal and of radius $R$, is a distance $z(t)$ directly above the first. You may assume that $z(t) \gg R$. The disk carries charge density $\sigma$.

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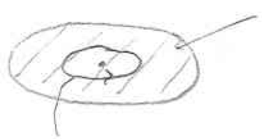

charged dist

loop $C$ of radius $r$

$$\Phi_B(r,z) = \frac{2\pi R^2 I}{c z^3}\cdot \pi r^2$$

$$\underbrace{\oint_C \vec{E}\cdot d\vec{l}}_{=2\pi r E} = -\frac{1}{c}\frac{d\Phi(r,z(t))}{dt} = +\frac{1}{c}\frac{2\pi R^2 I}{c}\pi r^2\frac{3}{z^4}\overbrace{\frac{dz}{dt}}^{v=}$$

$$\Rightarrow E(r) = 3\,\frac{\pi R^2 r I v}{c^2 z^4}$$

$$\vec{\tau} = \hat{z}\int_0^R [2\pi r\,dr\,\sigma]\times E(r)\times r$$

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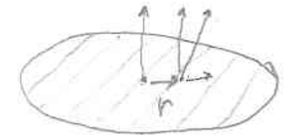
(b) Solution in rest frame of the loop not the disk




$\uparrow v$ must be radial component of $\vec{B}$ for $r \neq 0$ which produces circular force $\vec{v}\times\vec{B}$

$$B_z = \frac{2\pi R^2 I}{c z^3} \qquad \frac{\partial B_z}{\partial z} + \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = -\frac{\partial B_z}{\partial z} = +3\,\frac{2\pi R^2 I}{c z^4}$$

$$\Rightarrow \vec{B}_r = \frac{6\pi R^2 I}{c z^4}\cdot(x, y, 0)\,\frac{1}{2}$$

$$\vec{\tau} = \hat{z}\int_0^R (2\pi r)\,dr\,\sigma\cdot\frac{v}{c}\,B_r\cdot r$$

$$= \hat{z}\int_0^R (2\pi r)\,dr\,\sigma\,\frac{v}{c}\,\frac{3\pi R^2 I}{c z^4}\,r\cdot r$$

$$= \hat{z}\,\frac{6\pi^2\sigma v I R^2}{c^2 z^4}\int_0^R r^3\,dr$$

$$= \hat{z}\,\frac{3\pi^2 R^6 I v}{2c^2 z^4}\,\sigma \quad \text{as before}$$