3/9/21 (304)

Quantum mechanics - summary to date

- We are discussing a simple system with angular momentum with magnitude $j\hbar$.
- j is an integer or half integer
  for example $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$ are all OK
- allowed values of J_z (the angular momentum in the z-direction) are $\hbar m_z$. The m_z are $2j+1$ numbers separated by integers & range from $-j$ to $+j$
- If $j = \frac{3}{2}$, $m_z = -\frac{3}{2}, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}$
  $2j+1 = 4$ values
- Each value of m_z corresponds to an independent vector $|m_z\rangle$ in a $2j+1$ dimensional vector space

(305)

- A general quantum state is a vector $|\psi\rangle$ in this $2j+1$ dimensional space

$$|\psi\rangle = \sum_{m_z=-j}^{+j} \psi_{m_z} |m_z\rangle$$

- require $\sum_{m_z=-j}^{j} |\psi_{m_z}|^2 = 1$
- interpret $|\psi_{m_z}|^2$ as the probability that if the system is in the state $|\psi\rangle$ and J_z is measured, we will find the value $m_z\hbar$

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Two new challenging elements:

- The precise statement $|\psi\rangle = \sum_{m=-j}^{j} \psi_m |m\rangle$ implies an uncertain result. Only probabilites are predicted.
- We **must** work in a complex vector space. In contrast to SHM or AC circuits this is not a mathematical trick, it is real physics!

This structure $|\psi\rangle = \sum_{m=-j}^{j} \psi_m |m\rangle$ is very general. When we discuss position we will use $|\psi\rangle = \int d^3\vec{r}\, \psi(\vec{r})\, |\vec{r}\rangle$

$|\vec{r}\rangle$ state with particle at $\vec{r}$

$|\psi(\vec{r})|^2$ probability (density) of finding the particle at $\vec{r}$

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2. It is essential to become comfortable with complex vector spaces,

① if A is a vector and α a complex number αA is also a vector

② $\alpha(A+B) = \alpha A + \alpha B$

③ $(\alpha_1 + \alpha_2) A = \alpha_1 A + \alpha_1 B$

④ etc.

We also define a complex inner product $(A, \alpha B) = \alpha (A, B)$

$$(A, B) = (B, A)^*$$

this implies (A, A) is real

and $(\alpha A, B) = [(B, \alpha A)]^* = [\alpha (B, A)]^*$

$$= \alpha^* (A, B)$$

require $(A,A) > 0$ unless $A = 0$

Every thing else is similar to the real case

$\{A_1, A_2, \dots A_N\}$ is linearly independent if $\sum_{i=1}^{N} c_n A_n = 0 \Rightarrow c_n = 0,\ 1 \le n \le N$

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if $\sum_{i=1}^{N} C_n A_n = 0 \Rightarrow C_n = 0, \quad 1 \leq n \leq N$

We will require that our physically distinct states $|m_z\rangle$ and $|m_z'\rangle$ with $m_z' \neq m_z$ must be orthogonal.

$$(|m_z'\rangle, |m_z\rangle) = 0$$

or using Dirac's full notation allowing m_z & m_z' to be general

$$(|m_z'\rangle, |m_z\rangle) = \langle m_z' | m_z\rangle = \delta_{m_z' m_z}$$

$$= \begin{cases} 1 & m_z' = m_z \\ 0 & m_z' \neq m_z \end{cases}$$

Note for $|\psi_m\rangle = \sum_{m=-j}^{j} \psi_m |m\rangle$

$$(|\psi\rangle, |\psi\rangle) = \left(\sum_{m'=-j}^{j} \psi_{m'} |m'\rangle, \sum_{m=-j}^{j} \psi_m |m\rangle\right)$$

$$= \sum_{m=-j}^{j} \psi_m^* \psi_m = \sum_{m=-j}^{j} |\psi_m|^2$$

Total probability $= 1$ $\equiv \sum_{m=-j}^{j} |\psi_m|^2 = 1 \equiv \langle\psi|\psi\rangle = 1$

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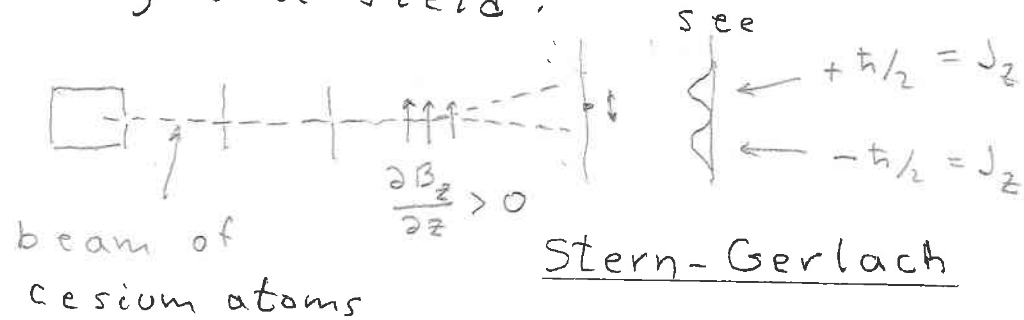
3. How can we measure J_z?

Examine individual, neutral atoms with quantum angular momentum $\vec{J}$ and magnetic moment $\vec{\mu} = \gamma \vec{J}$.

Create a beam by heating a gas of atoms in an oven which escape through a hole, pass through slits and are separated according to μ_z values by an inhomogeneous magnetic field:



<u>Stern-Gerlach</u>

Note $E = -\vec{\mu}\cdot\vec{B} \approx -\mu_z B_z$ ← $a + bz$

force $-\frac{\partial E}{\partial z} = +\mu_z \frac{\partial B_z}{\partial z}$ moves

$\mu_z = +\gamma\hbar m_z$ up for $m_z > 0$, down if $m_z < 0$

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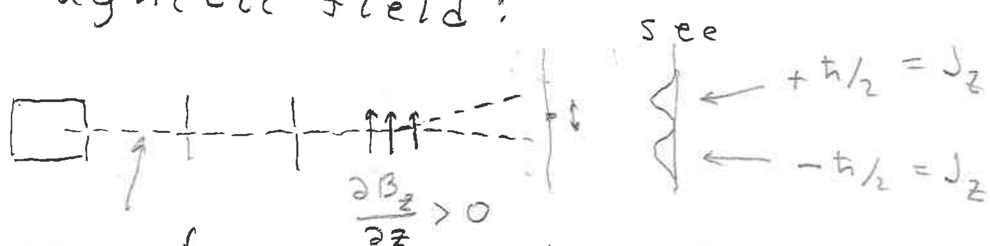

beam of cesium atoms

<u>Stern-Gerlach</u>

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For $j = 1/2$; $|\psi\rangle = \psi_{+1/2}|+1/2\rangle + \psi_{-1/2}|-1/2\rangle$

If each atom is in the state $|\psi\rangle$ and N atoms are analyzed, then

$|\psi_{+1/2}|^2 N$ atoms will be in $+z$ peak

$|\psi_{-1/2}|N$ " " " $-z$ peak

Since these are probabilities these numbers become more accurate as N becomes large.

Once J_z is determined later measurements of J_z will return the same value as was measured

$$|\psi\rangle = \sum_{m_z=-j}^{+j} \psi_{m_z}|m_z\rangle \xrightarrow[J_z = -3/2]{\text{measure}} |\psi\rangle = |-3/2\rangle$$

Called "the reduction of the wave function". Somewhat strange?

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4. Linear operators on the space of states (Hilbert space)

a) Such an operator O must obey

$$O(|A\rangle + |B\rangle) = O|A\rangle + O|B\rangle$$

$$O(\alpha|A\rangle) = \alpha O|A\rangle$$

Given an orthonormal basis $\{|m\rangle\}_{-j\le m\le j}$

we can associate O with an $(2j+1)\times(2j+1)$ matrix of complex numbers

$$O|m\rangle = \sum_{m'=-j}^{j} O_{m'm}|m'\rangle$$

Simply write $O|m\rangle$ in terms of our basis

Note $\langle m''|O|m\rangle = \left(|m''\rangle, \sum_{m'=-j}^{j} O_{m'm}|m'\rangle\right)$

$$= \sum_{m'=-j}^{j} O_{m'm}(|m''\rangle, |m'\rangle)$$

$$= O_{m'',m}$$

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Next allow O to act on a general vector $|\psi\rangle$: $\quad |\psi'\rangle = O|\psi\rangle$

We can express $|\psi\rangle$ & $|\psi'\rangle$ in terms of our basis $\{|m\rangle\}_{-j\le m\le j}$:

$$|\psi\rangle = \sum_{m=-j}^{j} \psi_m|m\rangle \quad \text{and} \quad |\psi'\rangle = \sum_{m'=-j}^{j} \psi'_{m'}|m'\rangle$$

How are $\psi'_{m'}$ & ψ_m related?

$$|\psi'\rangle = O\sum_{m=-j}^{j}\psi_m|m\rangle = \sum_{m=-j}^{j}\psi_m \underbrace{O|m\rangle}_{\sum_{m'=-j}^{j} O_{m'm}|m'\rangle}$$

$$= \sum_{m=-j}^{j}\left\{\sum_{m'=-j}^{j} O_{m'm}|m'\rangle\right\}\psi_m$$

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regular matrix multiplication

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$$\psi'_{m'} = \sum_{m=-j}^{j} O_{m'm}\psi_m \qquad \text{regular matrix multiplication}$$

$$\begin{pmatrix}\psi'_j\\ \psi'_{j-1}\\ \vdots\\ \psi'_{-j}\end{pmatrix} = \begin{pmatrix} O_{jj} & O_{j,j-1} & \cdots & O_{j,-j}\\ O_{j-1,j} & O_{j-1,j-1} & & O_{j-1,-j}\\ \vdots & & & \vdots\\ O_{-j,j} & O_{-j,j-1} & \cdots & O_{-j,-j}\end{pmatrix}\begin{pmatrix}\psi_j\\ \psi_{j-1}\\ \vdots\\ \psi_{-j}\end{pmatrix}$$

b) Associate an operator J_z^{op} with the angular momentum in the z direction:

$$J_z^{op}|m_z\rangle = +\hbar m_z |m_z\rangle$$

Thus
$$J_z\sum_{m=-j}^{j}\psi_m|m\rangle = \sum_{m=-j}^{j}\psi_m \underbrace{J_z|m\rangle}_{m\hbar|m\rangle}$$

$$= \sum_{m=-j}^{j}\underbrace{(m\hbar\,\psi_m)}_{\psi'_m}|m\rangle$$

$$J_z \equiv \begin{pmatrix}\hbar j & 0 & \cdots & 0\\ 0 & \hbar(j-1) & & \vdots\\ \vdots & & \ddots & \\ 0 & & & \hbar(-j)\end{pmatrix} \quad \text{a diagonal matrix}$$

$$\begin{pmatrix} \varphi'_j \\ \varphi'_{j-1} \\ \vdots \\ \varphi'_{-j} \end{pmatrix} = \begin{pmatrix} O_{j,j} & O_{j,j-1} & \cdots & O_{j,-j} \\ O_{j-1,j} & O_{j-1,j-1} & & O_{j-1,-j} \\ \vdots & & & \vdots \\ O_{-j,j} & O_{-j,j-1} & \cdots & O_{-j,-j} \end{pmatrix} \begin{pmatrix} \varphi_j \\ \varphi_{j-1} \\ \vdots \\ \varphi_{-j} \end{pmatrix}$$

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Thus $J_z \sum_{m=-j}^{j} \varphi_m |m\rangle = \sum_{m=-j}^{j} \varphi_m \underbrace{J_z |m\rangle}_{m\hbar |m\rangle}$

$$= \sum_{m=-j}^{j} \underbrace{(m\hbar \varphi_m)}_{\varphi'_m} |m\rangle$$

$$J_z \equiv \begin{pmatrix} \hbar j & 0 & \cdots & 0 \\ 0 & \hbar(j-1) & & \vdots \\ \vdots & & \ddots & \\ 0 & & & \hbar(-j) \end{pmatrix}$$ a diagonal matrix

This can be run backward:

Given J_z^{op} we can find the states $|m\rangle$ & diagonal matrix elements $\hbar m$.

In general, given an operator O if $O|\lambda\rangle = \lambda|\lambda\rangle$ we call $|\lambda\rangle$ an eigenvector of O and λ an eigenvalue of O.

Starting with an operator J_z^{op} its eigenstates are states with definite values of J_z and the eigenvalues are those values

Symmetry among x, y and z implies we should have J_x^{op}, J_y^{op}, J_z^{op} and states $|m\rangle_x$, $|m\rangle_y$ and $|m\rangle_z$.

obeying: $J_x^{op} |m\rangle_x = \hbar m |m\rangle_x$, $J_y^{op} |m\rangle = \hbar m |m\rangle_y$

and $J_z^{op} |m\rangle_z = \hbar m |m\rangle_z$ $\quad -j \le m \le j$