Next allow O to act on a general vector $|\psi\rangle$:

$$|\psi'\rangle = O|\psi\rangle$$

We can express $|\psi\rangle$ & $|\psi'\rangle$ in terms of our basis $\{|m\rangle\}_{-j\le m\le j}$:

$$|\psi\rangle = \sum_{m=-j}^{j} \psi_m |m\rangle \quad \text{and} \quad |\psi'\rangle = \sum_{m'=-j}^{j} \psi'_{m'} |m'\rangle$$

How are $\psi'_{m'}$ & ψ_m related?

$$|\psi'\rangle = O\sum_{m=-j}^{j} \psi_m |m\rangle = \sum_{m=-j}^{j} \psi_m \underbrace{O|m\rangle}_{\sum_{m'=-j}^{j} O_{m'm}|m'\rangle}$$

$$= \sum_{m=-j}^{j} \left\{ \sum_{m'=-j}^{j} O_{m'm} |m'\rangle \right\} \psi_m$$

$$= \sum_{m'=-j}^{j} \underbrace{\left\{ \sum_{m=-j}^{j} O_{m'm}\psi_m \right\}}_{\psi'_{m'}} |m'\rangle$$

$$\psi'_{m'} = \sum_{m=-j}^{j} O_{m'm}\psi_m$$ regular matrix multiplication

$$\begin{pmatrix} \psi'_j \\ \psi'_{j-1} \\ \vdots \\ \psi'_{-j} \end{pmatrix} = \begin{pmatrix} O_{jj} & O_{j,j-1} & \cdots & O_{j,-j} \\ O_{j-1,j} & O_{j-1,j-1} & & O_{j-1,-j} \\ \vdots & & & \vdots \\ O_{-j,j} & O_{-j,j-1} & \cdots & O_{-j,-j} \end{pmatrix} \begin{pmatrix} \psi_j \\ \psi_{j-1} \\ \vdots \\ \psi_{-j} \end{pmatrix}$$

b) Associate an operator J_z^{op} with the angular momentum in the z direction:

$$J_z^{op}|m_z\rangle = +\hbar m_z |m_z\rangle$$

Thus $$J_z \sum_{m=-j}^{j} \psi_m |m\rangle = \sum_{m=-j}^{j} \psi_m \underbrace{J_z|m\rangle}_{m\hbar|m\rangle}$$

$$= \sum_{m=-j}^{j} \underbrace{(m\hbar\,\psi_m)}_{\psi'_m}|m\rangle$$

$$J_z \equiv \begin{pmatrix} \hbar j & 0 & \cdots & 0 \\ 0 & \hbar(j-1) & & \vdots \\ \vdots & & \ddots & \\ 0 & & & \hbar(-j) \end{pmatrix}$$ a diagonal matrix

$$\begin{pmatrix} \psi'_j \\ \psi'_{j-1} \\ \vdots \\ \psi'_{-j} \end{pmatrix} = \begin{pmatrix} O_{j,j} & O_{j,j-1} & \cdots & O_{j,-j} \\ O_{j-1,j} & O_{j-1,j-1} & & O_{j-1,-j} \\ \vdots & & & \vdots \\ O_{-j,j} & O_{-j,j-1} & \cdots & O_{-j,-j} \end{pmatrix} \begin{pmatrix} \psi_j \\ \psi_{j-1} \\ \vdots \\ \psi_{-j} \end{pmatrix}$$

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This can be run backward:

Given J_z^{op} we can find the states $|m\rangle$ & diagonal matrix elements $\hbar m$.

In general, given an operator O if $O|\lambda\rangle = \lambda|\lambda\rangle$ we call $|\lambda\rangle$ an eigenvector of O and λ an eigenvalue of O.

Starting with an operator J_z^{op} its eigenstates are states with definite values of J_z and the eigenvalues are those values

Symmetry among x, y and z implies we should have J_x^{op}, J_y^{op}, J_z^{op} and states $|m\rangle_x$, $|m\rangle_y$ and $|m\rangle_z$ obeying: $J_x^{op}|m\rangle_x = \hbar m |m\rangle_x$, $J_y^{op}|m\rangle = \hbar m |m\rangle_y$

and $J_z^{op}|m\rangle_z = \hbar m |m\rangle_z \qquad -j \le m \le j$

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Important If we prepare the state $|\psi\rangle$ many times and measure J_z on each state what will be the average result?

If we make N trials the average of all the results will be

$$\langle J_z \rangle = \frac{1}{N}\left\{ \sum_{m_z=-j}^{j} \hbar m_z \underbrace{|\psi_{m_z}|^2 N}_{\substack{\text{average}\\ \text{number of times}\\ \text{we find } \hbar m_z}} \right\}$$

where $|\psi\rangle = \sum_{m_z=-j}^{j} \psi_{m_z} |m_z\rangle$.

Note:

$$\langle \psi | J_z^{op} | \psi \rangle = \Big(\sum_{m_z'} \psi_{m_z'} |m_z'\rangle , J_z^{op} \sum_{m_z} \psi_{m_z} |m_z\rangle \Big)$$

$$= \Big(\sum_{m_z'} \psi_{m_z'} |m_z'\rangle , \sum_{m_z} \psi_{m_z} \hbar m_z |m_z\rangle \Big)$$

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5. What about J_x & J_y?

Rotational symmetry requires:

- There should be $2j+1$ states $|m_x\rangle_x$, $-j \le m_x \le j$ with $J_x = \hbar m_x$
- There should be $2j+1$ states $|m_y\rangle_y$, $-j \le m_y \le j$ with $J_y = \hbar m_y$
- These compliment the $2j+1$ states $|m_z\rangle_z$, $-j \le m_z \le j$ with $J_z = \hbar m_z$.

How are these three sets of states related?

$$|\psi\rangle \overset{?}{=} \sum_{m_x=-j}^{j} \sum_{m_y=-j}^{j} \sum_{m_z=-j}^{j} \underbrace{\psi_{m_x m_y m_z}}_{\text{3 index tensor}} \overbrace{\underbrace{|m_x\rangle_x \otimes |m_y\rangle_y \otimes |m_z\rangle_z}_{\text{tensor product of 3 states}}}^{\text{dimension } (2j+1)^3}$$

$|\psi_{m_x m_y m_z}|^2 \overset{?}{=}$ probability that $J_x = \hbar m_x$, $J_y = \hbar m_y$, $J_z = \hbar m_z$

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NO!!

- Much more interesting

$$\{|m_x\rangle_x\}_{-j \le m_x \le j}, \ \{|m_y\rangle_y\}_{-j \le m_y \le j} \ \& \ \{|m_z\rangle_z\}_{-j \le m_z \le j}$$

are each a different set of $2j+1$ basis vectors for the <u>same</u> space of states.

- We can figure out how these must be related by studying rotations that could be used to rotate $\hat{z}$ into $\hat{x}$.
- Assume that a rotation of a quantum state $|\psi\rangle$ about $\hat{n}$ through θ can be carried out by by appling a linear operator $R(\hat{n},\theta)$ to $|\psi\rangle$

$$|\psi\rangle \longrightarrow R(\hat{n},\theta)\,|\psi\rangle$$

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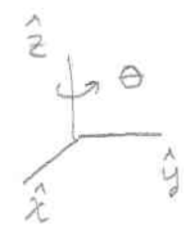
- Recall we already know one example of this for a 3-dim real vector space:

$$\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$$

$$R(\hat{z},\theta)[\vec{r}] = \vec{r}\,' = x'\hat{x} + y'\hat{y} + z'\hat{z}$$



$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

- Figure out some general properties of $R(\hat{n},\theta)$:

a) $R(\hat{n},\theta)$ should preserve the inner product

$$\big(R(\hat{n},\theta)A,\ R(\hat{n},\theta)B\big) = (A,B)$$

Look at

$$\big(R(\hat{n},\theta)|m'\rangle,\ R(\hat{n},\theta)|m\rangle\big) = \langle m'|m\rangle$$

$$\Big(\sum_{\bar{m}'} \underbrace{R(\hat{n},\theta)_{\bar{m}'m'}}_{\text{number}}|\bar{m}'\rangle,\ \sum_{\bar{m}} \underbrace{R(\hat{n},\theta)_{\bar{m}m}}_{\text{number}}|\bar{m}\rangle\Big) = \delta_{m'm}$$

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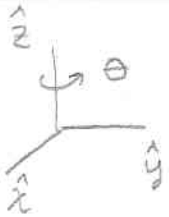

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$$\sum_{\bar{m}} \underbrace{R(\hat{n},\theta)^{*}_{\bar{m}m'}}_{R(\hat{n},\theta)^{T*}_{m'\bar{m}}} R(\hat{n},\theta)_{\bar{m}m} = \delta_{m'm}$$

$$R(\hat{n},\theta)^{T*}_{m'\bar{m}}\, R(\hat{n},\theta)_{\bar{m}m}$$

$$\underbrace{R^{T*}(\hat{n},\theta)}_{R^{\dagger}(\hat{n},\theta)} R(\hat{n},\theta) = I$$

the "adjoint" of R or the "hermitian conjugate" of R

$R^{\dagger}R = I$ $(R^{\dagger} = R^{-1})$ $\Rightarrow$ R is unitary, preserves inner product in a complex vector space

$O^{T}O = I$ $(O^{t} = O^{-1})$ $\Rightarrow$ O is orthogonal, preserved dot product in a real vector space

Note $(O|m'\rangle, |m\rangle) = O^{*}_{mm'} = O^{\dagger}_{m'm}$
$= (|m'\rangle, O^{\dagger}|m\rangle)$

$\Rightarrow$ $(OA, B) = (A, O^{\dagger}B)$ since relation is true for a basis of states

$$\sum_{\bar m} \underbrace{R(\hat n,\theta)^*_{\bar m m'}}_{R(\hat n,\theta)^{T*}_{m'\bar m}} R(\hat n,\theta)_{\bar m m} = \delta_{m'm}$$

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b) Examine properties of $R(\hat n,\theta)$ for small θ

$$R(\hat z, \delta\theta) \approx I - i K_z \delta\theta + O(\delta\theta^2)$$

generator of rotations about $\hat z$

definition of operator K_z

$$R(\hat z,\delta\theta)^\dagger R(\hat z,\delta\theta) = I$$

$$(I + iK_z\delta\theta)^\dagger (I - iK_z\delta\theta) = I$$

$$I + i(K_z^\dagger - K_z)\delta\theta + O(\delta\theta^2) = I$$

$$\Rightarrow K_z = K_z^\dagger$$

operator K_z is hermitian or self-adjoint

c) Relation between K_z and J_z^{op}:

$$R(\hat z,\phi)|m_z\rangle_z = e^{i c_{m_z}\phi}|m_z\rangle_z$$

Since a rotation about the $\hat z$ axis should not change the z component of the angular momentum

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Thus

$$[I - i\,\delta\phi\,K_z]\,|m_z\rangle_z = \underbrace{e^{-ic_{m_z}\delta\phi}}_{1 - ic_{m_z}\delta\phi}\,|m_z\rangle_z$$

$$K_z\,|m_z\rangle_z = c_{m_z}\,|m_z\rangle_z$$

The basis $|m_z\rangle_z$ diagonalizes both K_z and J_z^{op}

note: $R(\hat{z},\phi_1)\,R(\hat{z},\phi_2) = R(\hat{z},\phi_1+\phi_2)$ requires that $R(\hat{z},\phi)\,|m_z\rangle_z = e^{ic_{m_z}\phi}\,|m_z\rangle_z$ since we need $e^{ic_{m_z}\phi_1}\,e^{ic_{m_z}\phi_2} = e^{ic_{m_z}(\phi_1+\phi_2)}$

Further $R(\hat{z},2\pi) = I$

$$\Rightarrow e^{ic_{m_z}\cdot 2\pi} = 1$$

$$\Rightarrow c_{m_z} = \text{an integer}$$

This would all be very simple if $c_{m_z} = m_z$!

Thus

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Hypothesize that

$$K_z = \frac{1}{\hbar}J_z$$

[easy to anticipate once you have learned Hamitonian mechanics]

note when m_z is a half integer $R(\hat{z},2\pi) = -I \neq I$?

d) Consider small rotations to fix the matrices J_x, J_y & J_z.

Rotation symmetry requires

$$K_x = \frac{1}{\hbar}J_x,\quad K_y = \frac{1}{\hbar}J_y \;\&\; K_z = \frac{1}{\hbar}J_z$$

so drop K and use $\frac{1}{\hbar}J$ instead

Claim

$$R(\hat{n},\delta\theta) \approx I - i\frac{\delta\theta}{\hbar}\left(\hat{n}_x J_x + \hat{n}_y J_y + \hat{n}_z J_z\right)$$

$$= R(\hat{x},\delta\theta n_x)\,R(\hat{y},\delta\theta n_y)\,R(\hat{z},\delta\theta n_z) + O(\delta\theta^2)$$

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$$= R(\hat{x}, \delta\theta n_x)\, R(\hat{y}, \delta\theta n_y)\, R(\hat{z}, \delta\theta n_z) + O(\delta\theta^2)$$

Establish this property of rotations by rotating 3-dim vectors

$$R(\hat{n}, \delta\theta)\{\vec{r}\} = \vec{r} + \delta\theta\, \hat{n} \times \vec{r}$$

$$= \vec{r} + \delta\theta n_x\, \hat{x} \times \vec{r} + \delta\theta n_y\, \hat{y} \times \vec{r} + \delta\theta n_z\, \hat{z} \times \vec{r}$$

$$\stackrel{\checkmark}{=} R(\hat{x}, \delta\theta n_x)\, R(\hat{y}, \delta\theta n_y)\, R(\hat{z}, \delta\theta n_z)$$

Consider

$$R(\hat{x}, \theta)\left[1 - i\frac{1}{\hbar} J_y\, \delta\phi\right] R(\hat{x}, -\theta)$$

$$= 1 - i\frac{1}{\hbar}\, \hat{n}(\theta) \cdot \vec{J}\, \delta\phi$$

determine $\hat{n}(\theta)$ for small θ

1st for 3-dim rotations

$$[I + \delta\theta\, \hat{x} \times][1 + \delta\phi\, \hat{y} \times][I - \delta\theta\, \hat{x} \times]\, \vec{r}$$

$$= \vec{r} + \delta\phi\, \hat{y} \times \vec{r} + \delta\theta\,\delta\phi\,\big(\hat{x} \times (\hat{y} \times \vec{r}) - \hat{y} \times (\hat{x} \times \vec{r})\big)$$

$$= \vec{r} + \delta\phi\, \hat{y} \times \vec{r} + \delta\theta\,\delta\theta\big[\hat{y}\,(\hat{x} \cdot \vec{r}) - \cancel{\vec{r}\,(\hat{x} \cdot \hat{y})} - \hat{x}\,(\hat{y} \cdot \vec{r}) + \cancel{\vec{r}\,(\hat{y} \cdot \hat{x})}\big]$$

$=$

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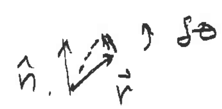

$$= \vec{r} + \delta\theta\, n_x\, \hat{x} \times \vec{r} + \delta\theta\, n_y\, \hat{y} \times \vec{r} + \delta\theta\, n_z\, \hat{z} \times \vec{r}$$

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$$= \vec{r} + \delta\phi\, \hat{y} \times \vec{r} + \delta\theta\, \delta\theta \left[\hat{y}(\hat{x} \cdot \vec{r}) - \cancel{\vec{r}(\hat{x} \cdot \hat{y})} - \hat{x}(\hat{y} \cdot \vec{r}) + \cancel{\vec{r}(\hat{y} \cdot \hat{x})}\right]$$

$=$

$$= \vec{r} + \delta\phi\, \hat{y} \times \vec{r} + \delta\theta\, \delta\phi\, \hat{z} \times \vec{r}$$

$\Rightarrow$

$$\left(I - \tfrac{i}{\hbar}\delta\theta\, J_x\right)\left(I - \tfrac{i}{\hbar}\delta\phi\, J_y\right)\left(I + \tfrac{i}{\hbar}\delta\theta\, J_x\right)$$

$$= I - \tfrac{i}{\hbar}\left[\delta\phi\, J_y + \delta\theta\, \delta\phi\, J_z\right]$$

$$-\tfrac{1}{\hbar^2}\delta\theta\, \delta\phi\left[J_x J_y - J_y J_x\right] = -\tfrac{i}{\hbar}\delta\theta\, \delta\phi\, J_z$$

$$\underbrace{J_x J_y - J_y J_x}_{\equiv [J_x,\, J_y]} = i\hbar J_z$$

in general $[J_i, J_j] = i\hbar \underbrace{\sum_k \varepsilon_{ijk} J_k}_{(\hat{e}_i \times \hat{e}_j) \cdot \vec{J}}$

or $[J_x, J_y] = i\hbar J_z$

$[J_y, J_z] = i\hbar J_x$

$[J_z, J_x] = i\hbar J_y$

we can find all of the finite dimensional matrices which obey this!