b) Examine properties of $R(\hat{n}, \theta)$ for small θ

generator of rotations about $\hat{z}$

$$R(\hat{z}, \delta\theta) \approx I - iK_z\,\delta\theta + O(\delta\theta^2)$$

definition of operator K_z

$$R(\hat{z}, \delta\theta)^\dagger R(\hat{z}, \delta\theta) = I$$

$$(I + iK_z\,\delta\theta)^\dagger (I - iK_z\,\delta\theta) = I$$

$$I + i(K_z^\dagger - K_z)\,\delta\theta + O(\delta\theta^2) = I$$

$$\Rightarrow K_z = K_z^\dagger$$

operator K_z is hermitian or self-adjoint

c) Relation between K_z and J_z^{op}:

$$R(\hat{z}, \phi)\,|m_z\rangle_z = e^{ic_{m_z}\phi}\,|m_z\rangle_z$$

Since a rotation about the $\hat{z}$ axis should not change the z component of the angular momentum

Thus

$$[I - i\,\delta\phi\,K_z]\,|m_z\rangle_z = \underbrace{e^{-ic_{m_z}\delta\phi}}_{1 - ic_{m_z}\delta\phi}\,|m_z\rangle_z$$

$$K_z\,|m_z\rangle_z = c_{m_z}\,|m_z\rangle_z$$

The basis $|m_z\rangle_z$ diagonalizes both K_z and J_z^{op}

note: $R(\hat{z}, \phi_1)\,R(\hat{z}, \phi_2) = R(\hat{z}, \phi_1 + \phi_2)$

requires that $R(\hat{z}, \phi)\,|m_z\rangle_z = e^{ic_{m_z}\phi}\,|m_z\rangle_z$

since we need $e^{ic_{m_z}\phi_1}\,e^{ic_{m_z}\phi_2} = e^{ic_{m_z}(\phi_1 + \phi_2)}$

Further $R(\hat{z}, 2\pi) = I$

$$\Rightarrow e^{ic_{m_z}\cdot 2\pi} = 1$$

$$= c_{m_z} = \text{an integer}$$

This would all be very simple if $c_{m_z} = m_z$!

Thus

$$[I - i\,\delta\phi K_z]\,|m_z\rangle_z = \underbrace{e^{-ic_{m_z}\delta\phi}}_{1 - ic_{m_z}\delta\phi}\,|m_z\rangle_z$$

$$K_z|m_z\rangle_z = c_{m_z}|m_z\rangle_z$$

The basis $|m_z\rangle_z$ diagonalizes both K_z and J_z^{op}

<u>note</u>: $R(\hat{z},\phi_1)R(\hat{z},\phi_2) = R(\hat{z},\phi_1+\phi_2)$

requires that $R(\hat{z},\phi)|m_z\rangle_z = e^{ic_{m_z}\phi}|m_z\rangle_z$

since we need $e^{ic_{m_z}\phi_1}e^{ic_{m_z}\phi_2} = e^{ic_{m_z}(\phi_1+\phi_2)}$

Further $R(\hat{z},2\pi) = I$

$$\Rightarrow e^{ic_{m_z}\cdot 2\pi} = 1$$

$$\Rightarrow c_{m_z} = \text{an integer}$$

This would all be very simple if $c_{m_z} = m_z$!

Hypothesize that

$$K_z = \frac{1}{\hbar}J_z$$

[easy to anticipate once you have learned Hamitonian mechanics]

note when m_z is a half integer $R(\hat{z},2\pi) = -I \neq I$?

d) Consider small rotations to fix the matrices J_x, J_y & J_z.

Rotation symmetry requires

$$K_x = \frac{1}{\hbar}J_x,\ K_y = \frac{1}{\hbar}J_y\ \&\ K_z = \frac{1}{\hbar}J_z$$

so drop K and use $\frac{1}{\hbar}J$ instead

Claim

$$R(\hat{n},\delta\theta) \approx I - i\frac{\delta\theta}{\hbar}(\hat{n}_x J_x + \hat{n}_y J_y + \hat{n}_z J_z)$$

$$= R(\hat{x},\delta\theta n_x)R(\hat{y},\delta\theta n_y)R(\hat{z},\delta\theta n_z) + O(\delta\theta^2)$$

Hypothesize that

$$K_z = \frac{1}{\hbar} J_z$$

[easy to anticipate once you have learned Hamiltonian mechanics]

note when m_z is a half integer $R(\hat z, 2\pi) = -I \neq I$?

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claim

$$R(\hat n, \delta\theta) \approx I - i\frac{\delta\theta}{\hbar}\left(\hat n_x J_x + \hat n_y J_y + \hat n_z J_z\right)$$

$$= R(\hat x, \delta\theta n_x)\, R(\hat y, \delta\theta n_y)\, R(\hat z, \delta\theta n_z) + O(\delta\theta^2)$$

Establish this property of rotations by rotating 3-dim vectors

$$R(\hat n, \delta\theta)\{\vec r\} = \vec r + \delta\theta\, \hat n \times \vec r$$

$$= \vec r + \delta\theta n_x\, \hat x \times \vec r + \delta\theta n_y\, \hat y \times \vec r + \delta\theta n_z\, \hat z \times \vec r$$

$$\overset{\checkmark}{=} R(\hat x, \delta\theta n_x)\, R(\hat y, \delta\theta n_y)\, R(\hat z, \delta\theta n_z)$$

Consider

$$R(\hat x, \theta)\left[1 - i\frac{1}{\hbar} J_y \delta\phi\right] R(\hat x, -\theta)$$

$$= 1 - i\frac{1}{\hbar}\, \hat n(\theta)\cdot\vec J\, \delta\phi$$

determine $\hat n(\theta)$ for small θ

1st for 3-dim rotations

$$[I + \delta\theta\, \hat x \times][1 + \delta\phi\, \hat y \times][I - \delta\theta\, \hat x \times]\, \vec r$$

$$= \vec r + \delta\phi\, \hat y \times \vec r + \delta\theta\delta\phi\left(\hat x \times (\hat y \times \vec r) - \hat y \times (\hat x \times \vec r)\right)$$

$$= \vec r + \delta\phi\, \hat y \times \vec r + \delta\theta\delta\theta\left[\hat y(\hat x\cdot\vec r) - \cancel{\vec r(\hat x\cdot\hat y)} - \hat x(\hat y\cdot\vec r) + \cancel{\vec r(\hat y\cdot\hat x)}\right]$$

$=$