3/18/21

Overview of angular momentum and rotation

1. Simple systems of a few particles (atoms, nuclei, molecule, electrons etc) with minimum energy are described by a $2j+1$ dimensional complex vector space. [j an integer or half integer]

2. States $|m\rangle_z$, $-j \le m \le j$ with definite values, $\hbar m$ of J_z form a basis for the space

3. In addition to the linear operator J_z^{op}, defined by $J_z^{op}|m\rangle_z = \hbar m|m\rangle_z$ there are J_x^{op} and J_y^{op} each defining a different basis of eigenstates

$$J_x|m\rangle_x = \hbar m|m\rangle_x \quad J_y|m\rangle_y = \hbar m|m\rangle_y$$

4. For each rotation $\mathcal{R}(\hat{n},\theta)$ possible in space, there must be a corresponding linear operator $R(\hat{n},\theta)$ that carries out this rotation on $2j+1$-dim space of quantum states:

$$\mathcal{R}(\hat{n},\theta) \to R(\hat{n},\theta)$$

must obey:

if $\mathcal{R}(\hat{n},\theta)\,\mathcal{R}(\hat{n}',\theta') = \mathcal{R}(\hat{n}'',\theta'')$

then $R(\hat{n},\theta)\,R(\hat{n}',\theta') = R(\hat{n}'',\theta'')$

5. $R(\hat{n},\theta)$ preserves the inner product $\Rightarrow$ $R^{+}(\hat{n},\theta) = R(\hat{n},\theta)^{-1}$

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then $R(\hat{n},\theta)\,R(\hat{n}',\theta') = R(\hat{n}'',\theta'')$

5. $R(\hat{n},\theta)$ preserves the inner product $\Rightarrow$ $R^{\dagger}(\hat{n},\theta) = R(\hat{n},\theta)^{-1}$

6. For small $\delta\phi$

$$R(\hat{x},\delta\theta) \approx I - iK_x\,\delta\phi + O(\delta\phi^2)$$
$$R(\hat{y},\delta\theta) \approx I - iK_y\,\delta\phi + O(\delta\phi^2)$$
$$R(\hat{z},\delta\theta) \approx I - iK_z\,\delta\phi + O(\delta\phi^2)$$

7. Since rotations around z cannot change the value of J_z

$$K_z\,|m_z\rangle_z \propto |m_z\rangle_z$$

8. For this simple system in its ground state our $2j+1$ states describe NOTHING but the system's angular momentum.

Thus K_z & J_z must be proportional

$$K_x = \frac{1}{\hbar}J_x, \quad K_y = \frac{1}{\hbar}J_z, \quad K_z = \frac{1}{\hbar}J_z$$

9. Claim

$$R(\hat{n},\delta\phi) = I - \frac{i}{\hbar}n_x\,\delta\phi\,J_x - \frac{i}{\hbar}n_y\,\delta\phi\,J_y - \frac{i}{\hbar}n_z\,\delta\phi\,J_z$$

6. For small $\delta\phi$

$$R(\hat{x},\delta\theta) \approx I - iK_x\,\delta\phi + O(\delta\phi^2)$$
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$$R(\hat{n},\delta\phi) = I - \frac{i}{\hbar}n_x\delta\phi\,J_x - \frac{i}{\hbar}n_y\delta\phi\,J_y - \frac{i}{\hbar}n_z\delta\phi\,J_z$$

Proof:

Look at ordinary rotations in space: $R(\hat{n},\delta\phi)[\vec{r}] = \vec{r} + \delta\phi\,\hat{n}\times\vec{r}$

$$= \vec{r} + \delta\phi\,n_x\,\hat{x}\times\vec{r} + \delta\phi\,n_y\,\hat{y}\times\vec{r} + \delta\phi\,n_z\,\hat{z}\times\vec{r}$$
$$\approx [I + \delta\phi n_x\,\hat{x}\times][I + \delta\phi n_y\,\hat{y}\times][I + \delta\phi n_z\,\hat{z}\times]\vec{r}$$
$$= R(\hat{x},\delta\phi n_x)\,R(\hat{y},\delta\phi n_y)\,R(\hat{z},\delta\phi n_z)$$

Therefore

$$R(\hat{n},\delta\phi) \approx R(\hat{x},\delta\phi n_x)\,R(\hat{y},\delta\phi n_y)\,R(\hat{z},\delta\phi n_z)$$
$$\approx \left[I - \frac{i}{\hbar}\delta\phi n_x J_x\right]\left[I - \frac{i}{\hbar}\delta\phi n_y J_y\right]\left[I - \frac{i}{\hbar}\delta\phi n_z J_z\right]$$
$$\approx I - \frac{i}{\hbar}\delta\phi n_x J_x - \frac{i}{\hbar}\delta\phi n_y J_y - \frac{i}{\hbar}\delta\phi n_z J_z \quad \text{QED}$$
$$= I - \frac{i}{\hbar}\delta\phi\,\hat{n}\cdot\vec{J}$$

Construct rotations from generators

$$R(\hat{n},\phi+\delta\phi) = R(\hat{n},\delta\phi)\,R(\hat{n},\phi)$$
$$R(\hat{n},\phi) + \delta\phi\frac{d}{d\phi}R(\hat{n},\phi) = \left[I - \frac{i}{\hbar}\delta\phi\,\vec{J}\cdot\hat{n}\right]R(\hat{n},\phi)$$

Proof:

Look at ordinary rotations in space: $R(\hat{n}, \delta\phi)[\vec{r}] = \vec{r} + \delta\phi\, \hat{n} \times \vec{r}$

$$= \vec{r} + \delta\phi\, n_x\, \hat{x} \times \vec{r} + \delta\phi\, n_y\, \hat{y} \times \vec{r} + \delta\phi\, n_z\, \hat{z} \times \vec{r}$$

$$\approx [I + \delta\phi\, n_x\, \hat{x} \times][I + \delta\phi\, n_y\, \hat{y} \times][I + \delta\phi\, n_z\, \hat{z} \times]\vec{r}$$

$$= R(\hat{x}, \delta\phi\, n_x)\, R(\hat{y}, \delta\phi\, n_y)\, R(\hat{z}, \delta\phi\, n_z)$$

Therefore

$$R(\hat{n}, \delta\phi) \approx R(\hat{x}, \delta\phi\, n_x)\, R(\hat{y}, \delta\phi\, n_y)\, R(\hat{z}, \delta\phi\, n_z)$$

$$\approx \left[I - \frac{i}{\hbar}\delta\phi\, n_x J_x\right]\left[I - \frac{i}{\hbar}\delta\phi\, n_y J_y\right]\left[I - \frac{i}{\hbar}\delta\phi\, n_z J_z\right]$$

$$\approx I - \frac{i}{\hbar}\delta\phi\, n_x J_x - \frac{i}{\hbar}\delta\phi\, n_y J_y - \frac{i}{\hbar}\delta\phi\, n_z J_z \qquad \text{QED}$$

$$= I - \frac{i}{\hbar}\delta\phi\, \hat{n} \cdot \vec{J}$$

Construct rotations from generators

$$R(\hat{n}, \phi + \delta\phi) = R(\hat{n}, \delta\phi)\, R(\hat{n}, \phi)$$

$$R(\hat{n}, \phi) + \delta\phi \frac{d}{d\phi} R(\hat{n}, \phi) = \left[I - \frac{i}{\hbar}\delta\phi\, \vec{J} \cdot \hat{n}\right] R(\hat{n}, \phi)$$

Equate $\delta\phi$ terms:

$$\frac{d}{d\phi} R(\hat{n}, \phi) = -\frac{i}{\hbar}\vec{J} \cdot \hat{n}\, R(\hat{n}, \phi) \qquad R(\hat{n}, 0) = I$$

$$\boxed{R(\hat{n}, \phi) = e^{-\frac{i}{\hbar}\hat{n} \cdot \vec{J}\phi}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left[-\frac{i}{\hbar}\phi\, \vec{J} \cdot \hat{n}\right]^n$$

Sum converges quicky with $\frac{1}{n!}$

Find the matrices J_x, J_y, J_z

Consider

$$R(\hat{x}, \theta)\left[1 - i\frac{1}{\hbar} J_y \delta\phi\right] R(\hat{x}, -\theta) = 1 - \frac{i}{\hbar}\delta\phi\, \hat{n}(\theta) \cdot \vec{J}$$

Find $\hat{n}(\theta)$ for $\theta = \delta\theta$ small

Again use rotations in space:

$$[I + \delta\theta\, \hat{x} \times][I + \delta\phi\, \hat{y} \times][I - \delta\theta\, \hat{x} \times]\, \vec{r}$$

$$= \vec{r} + \delta\phi\, \hat{y} \times \vec{r} + \delta\theta\, \delta\phi\, [\hat{x} \times (\hat{y} \times \vec{r}) - \hat{y} \times (\hat{x} \times \vec{r})]$$

$$= \vec{r} + \delta\phi\, \hat{y} \times \vec{r} + \delta\theta\, \delta\phi\, [\hat{y}(\hat{x} \cdot \vec{r}) - \cancel{\vec{r}(\hat{x} \cdot \hat{y})} - \hat{x}(\hat{y} \cdot \vec{r}) + \cancel{\vec{r}(\hat{y} \cdot \hat{x})}]$$

$$= \vec{r} + \delta\phi\, \hat{y} + \delta\theta\, \delta\phi\, \underbrace{(\hat{x} \times \hat{y})}_{\hat{z}} \times \vec{r}$$

Equate $\delta\phi$ terms:

$$\frac{d}{d\phi} R(\hat{n},\phi) = -\frac{i}{\hbar} \vec{J}\cdot\hat{n}\, R(\hat{n},\phi) \qquad R(\hat{n},0) = I$$

$$\boxed{R(\hat{n},\phi) = e^{-\frac{i}{\hbar}\hat{n}\cdot\vec{J}\phi}} \equiv \sum_{n=0}^{\infty} \frac{1}{n!}\left[-\frac{i}{\hbar}\phi\, \vec{J}\cdot\hat{n}\right]^n$$

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$$= \vec{r} + \delta\phi\, \hat{y}\times\vec{r} + \delta\theta\,\delta\phi\, \hat{z}\times\vec{r}$$

$\Rightarrow$

$$\left(I - \frac{i}{\hbar}\delta\theta J_x\right)\left(I - \frac{i}{\hbar}\delta\phi J_y\right)\left(I + \frac{i}{\hbar}\delta\theta J_z\right)$$

$$= I - \frac{i}{\hbar}[\delta\phi J_y + \delta\theta\,\delta\phi J_z]$$

$$-\frac{1}{\hbar^2}\delta\theta\,\delta\phi\,[J_x J_y - J_y J_x] = -\frac{i}{\hbar}\delta\theta\,\delta\phi\, J_z$$

$$\underbrace{J_x J_y - J_y J_x}_{\equiv [J_x, J_y]} = i\hbar J_z$$

in general $[J_i, J_j] = i\hbar \underbrace{\sum_k \varepsilon_{ijk} J_k}_{(\hat{e}_i\times\hat{e}_j)\cdot\vec{J}}$

or $[J_x, J_y] = i\hbar J_z$

$[J_y, J_z] = i\hbar J_x$

$[J_z, J_x] = i\hbar J_y$

we can find all of the finite dimensional matrices which obey this!

$$= \vec{r} + \delta\phi\, \hat{y} \times \vec{r} + \delta\theta\, \delta\phi\, \hat{z} \times \vec{r}$$

$\Longrightarrow$

$$\left(I - \frac{i}{\hbar}\delta\theta J_x\right)\left(I - \frac{i}{\hbar}\delta\phi J_y\right)\left(I + \frac{i}{\hbar}\delta\theta J_z\right)$$

$$= I - \frac{i}{\hbar}\left[\delta\phi J_y + \delta\theta\,\delta\phi J_z\right]$$

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$[J_z, J_x] = i\hbar J_y$

we can find all of the finite dimensional matrices which obey this!

These conditions a very restrictive and completely determine all possible matrices. As is shown in appendix A of the online notes the possible eigenvalues of $J_x^2 + J_y^2 + J_z^2$ are $\hbar^2 j(j+1)$ and in that case the J_i are $(2j+1)\times(2j+1)$ matrices where j is an integer or half-integer and if a basis is chosen to diagonalize J_z then its diagonal elements (the $2j+1$ eigenvalues of J_z) are $\hbar m_z$ with $m_z = j, j-1, j-2, \ldots -j+1, -j$.

Focus on the easiest and most interesting case: $j = 1/2$

spin-$1/2$

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Focus on the easiest and most interesting case: $j = 1/2$

spin-$1/2$

$2j+1 = 2$ so our space of states has two basis elements. Choose them to be $|{+1/2}\rangle_z$ & $|{-1/2}\rangle_z$

Then

$$J_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

can guess the matrices for J_x and J_y. The conventional choices are

$$J_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \& \quad J_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

These choices are often written

$$\vec{J} = \frac{\hbar}{2}\vec{\sigma} \quad \text{where}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \& \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices with simple properties

$2j+1=2$ so our space of states has two basis elements. Choose them to be $|+\tfrac{1}{2}\rangle_z$ & $|-\tfrac{1}{2}\rangle_z$

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$$J_z \equiv \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

can guess the matrices for J_x and J_y. The conventional choices are

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are the Pauli matrices with simple properties

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} I + i \varepsilon_{ijk} \sigma_k$$

$$\sigma_x \sigma_y = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = \begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} \overset{\checkmark}{=} i\sigma_z$$

Then $J_x J_y - J_y J_x = \frac{\hbar^2}{4}(\sigma_x \sigma_y - \sigma_y \sigma_x)$

$$= \frac{\hbar^2}{2} i \sigma_z = i\hbar \frac{\hbar}{2}\sigma_z$$

$$\overset{\checkmark}{=} i\hbar J_z$$

Now we can answer the question "Which states have definite values of J_y?

$$\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix}a\\ b\end{pmatrix}$$

$|+\tfrac{1}{2}\rangle_y$

$$-ib = a \qquad ia = b$$

$$|+\tfrac{1}{2}\rangle_y = \begin{pmatrix}a\\ ia\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\end{pmatrix} \qquad |-\tfrac{1}{2}\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\end{pmatrix}$$

choose $a = \frac{1}{\sqrt{2}}$ $\qquad \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{i}{\sqrt{2}}\right|^2 = 1$

$$\sigma_x^2 = \sigma_y^2 = \sigma_z^2 = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_i \sigma_j = \delta_{ij} I + i \varepsilon_{ijk} \sigma_k$$

$$\sigma_x \sigma_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \overset{\checkmark}{=} i \sigma_z$$

Then $J_x J_y - J_y J_x = \frac{\hbar^2}{4}(\sigma_x \sigma_y - \sigma_y \sigma_x)$

$$= \left(\frac{\hbar}{2}\right)^2 i \sigma_z = i\hbar \frac{\hbar}{2} \sigma_z$$

$$\overset{\checkmark}{=} i\hbar J_z$$

Now we can answer the question "Which states have definite values of J_y?

$$\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = + \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$|+1/2\rangle_y$

$-ib = a \qquad ia = b$

$$|+1/2\rangle_y = \begin{pmatrix} a \\ ia \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \qquad |-1/2\rangle_y = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

choose $a = \frac{1}{\sqrt{2}}$ $\qquad \left|\frac{1}{\sqrt{2}}\right|^2 + \left|\frac{i}{\sqrt{2}}\right|^2 = 1$

Thus $|\pm 1/2\rangle_y = \frac{1}{\sqrt{2}}|+1/2\rangle_z \pm \frac{i}{\sqrt{2}}|-1/2\rangle$ $\quad (*)$

By requiring J_z & J_y to act on the same 2-dim vector space we make it impossible for a quantum state to have definite values of J_y & J_z at the same time:

$$|+1/2\rangle_y = \frac{1}{\sqrt{2}}|+1/2\rangle_z + \frac{i}{\sqrt{2}}|-1/2\rangle_z$$

only $J_y = +\hbar/2$

prob $= \left|\frac{1}{\sqrt{2}}\right|^2 = 1/2 \qquad J_z = \frac{\hbar}{2}$

prob $= \left|\frac{i}{\sqrt{2}}\right|^2 = 1/2 \qquad J_z = -\frac{\hbar}{2}$

J_z is completely uncertain:
example of Heisenberg's uncertainty principle!

Combine $(*)$ at the top of page

add: $|+1/2\rangle_y + |-1/2\rangle_y = \frac{2}{\sqrt{2}}|+1/2\rangle_z$

$|+1/2\rangle_y - |-1/2\rangle_y = \frac{2i}{\sqrt{2}}|-1/2\rangle_z$

Thus $|\pm 1/2\rangle_y = \frac{1}{\sqrt{2}}|+1/2\rangle_z \pm \frac{i}{\sqrt{2}}|-1/2\rangle$ (*)

By requiring J_z & J_y to act on the same 2-dim vector space we make it impossible for a quantum state to have definite values of J_y & J_z at the same time:

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prob $= |\frac{1}{\sqrt{2}}| = 1/2$ $\quad J_z = \frac{\hbar}{2}$

prob $= |\frac{i}{\sqrt{2}}|^2 = 1/2$ $\quad J_z = -\frac{\hbar}{2}$

J_z is completely uncertain: example of Heisenberg's uncertainty principle!

Combine (*) at the top of page

add: $|+1/2\rangle_y + |-1/2\rangle_y = \frac{2}{\sqrt{2}}|+1/2\rangle_z$

$$|+1/2\rangle_y - |-1/2\rangle_y = \frac{2i}{\sqrt{2}}|-1/2\rangle_z$$

$\Rightarrow$ $|+1/2\rangle_z = \frac{1}{\sqrt{2}}|+1/2\rangle_y + \frac{1}{\sqrt{2}}|-1/2\rangle_y$

$$|-1/2\rangle_z = \frac{-i}{\sqrt{2}}|+1/2\rangle_y + \frac{i}{\sqrt{2}}|-1/2\rangle_y$$

Measurement theory

prepare $J_z = +\frac{\hbar}{2}$		$|1/2\rangle_z = \frac{1}{\sqrt{2}}|1/2\rangle_y + \frac{1}{\sqrt{2}}|-1/2\rangle_y$
measure $J_y = -\frac{\hbar}{2}$	(prob 1/2)	$|+1/2\rangle_y = \frac{1}{\sqrt{2}}|1/2\rangle_z - \frac{i}{\sqrt{2}}|-1/2\rangle_z$
measure $J_z = -\frac{\hbar}{2}$	(prob 1/2)	$|-1/2\rangle_z = \frac{-i}{\sqrt{2}}|+1/2\rangle_y + \frac{i}{\sqrt{2}}|-1/2\rangle_y$

- measuring J_y changed the J_z value of our state.
- the state went from one with

  $\mathrm{Prob}_{J_z=\hbar/2} = 1 \quad \mathrm{Prob}_{J_z=-\hbar/2} = 0$

  to one with

  $\mathrm{Prob}_{J_z=\hbar/2} = \frac{1}{2} \quad \mathrm{Prob}_{J_z=-\frac{\hbar}{2}} = \frac{1}{2}$

$$\Rightarrow \quad |+\tfrac{1}{2}\rangle_z = \frac{1}{\sqrt{2}}|+\tfrac{1}{2}\rangle_y + \frac{1}{\sqrt{2}}|-\tfrac{1}{2}\rangle_y$$

$$|-\tfrac{1}{2}\rangle_z = \frac{-i}{\sqrt{2}}|+\tfrac{1}{2}\rangle_y + \frac{i}{\sqrt{2}}|-\tfrac{1}{2}\rangle_y$$

Measurement theory

prepare $J_z = +\frac{\hbar}{2}$		$|\tfrac{1}{2}\rangle_z = \frac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle_y + \frac{1}{\sqrt{2}}|-\tfrac{1}{2}\rangle_y$
measure $J_y = -\frac{\hbar}{2}$	(prob ½)	$|+\tfrac{1}{2}\rangle_y = \frac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle_z - \frac{i}{\sqrt{2}}|-\tfrac{1}{2}\rangle_z$
measure $J_z = -\frac{\hbar}{2}$	(prob ½)	$|-\tfrac{1}{2}\rangle_z = \frac{-i}{\sqrt{2}}|+\tfrac{1}{2}\rangle_y + \frac{i}{\sqrt{2}}|-\tfrac{1}{2}\rangle_y$

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- the state went from one with

$$\text{Prob}_{J_z = \hbar/2} = 1 \qquad \text{Prob}_{J_z = -\hbar/2} = 0$$

to one with

$$\text{Prob}_{J_z = \hbar/2} = \frac{1}{2} \qquad \text{Prob}_{J_z = -\frac{\hbar}{2}} = \frac{1}{2}$$

A consequence of relation between operator:

$$[J_y, J_z] = J_y J_z - J_z J_y = i\hbar J_x \neq 0$$

Understand properties of a general spin ½ state

$$|\psi\rangle = a|+\tfrac{1}{2}\rangle_z + b|-\tfrac{1}{2}\rangle_z \qquad |a|^2 + |b|^2 = 1$$

Prepare $|\psi\rangle$ multiple times and for each trial measure J_x or J_y or J_z

$$\langle J_z\rangle = \langle\psi|J_z|\psi\rangle = (a^* \; b^*)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\cdot\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}(|a|^2 - |b|^2)$$

$$\langle J_x\rangle = \langle\psi|J_x|\psi\rangle = (a^* \; b^*)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\cdot\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}(a^* b + b^* a) = \hbar\, \mathrm{re}(a^* b)$$

$$\langle J_y\rangle = \langle\psi|J_y|\psi\rangle = (a^* \; b^*)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}(-i a^* b + i b^* a) = \hbar\, \mathrm{im}(a^* b)$$

A consequence of relation between operator:

$$[J_y, J_z] = J_y J_z - J_z J_y = i\hbar J_x \neq 0$$

Understand properties of a general spin ½ state

$$|\psi\rangle = a|+\tfrac{1}{2}\rangle_z + b|-\tfrac{1}{2}\rangle_z \qquad |a|^2+|b|^2=1$$

Prepare $|\psi\rangle$ multiple times and for each trial measure J_x or J_y or J_z

$$\langle J_z\rangle = \langle\psi|J_z|\psi\rangle = (a^* \;\; b^*)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\cdot\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}\left(|a|^2-|b|^2\right)$$

$$\langle J_x\rangle = \langle\psi|J_x|\psi\rangle = (a^* \;\; b^*)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\cdot\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}\left(a^*b + b^*a\right) = \hbar\,\mathrm{re}(a^*b)$$

$$\langle J_y\rangle = \langle\psi|J_y|\psi\rangle = (a^* \;\; b^*)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}\left(-i\,a^*b + i\,b^*a\right) = \hbar\,\mathrm{im}(a^*b)$$

Smart way to write a & b:

$$a = \cos(\theta/2)\, e^{-i\phi/2} \qquad b = \sin\theta/_2\; e^{+i\phi/2}$$

$$|a|^2+|b|^2 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

$$\langle J_z\rangle = \frac{\hbar}{2}\left(|a|^2-|b|^2\right) = \frac{\hbar}{2}\left(\cos^2\theta/_2 - \sin^2\theta/_2\right)$$

$$= \frac{\hbar}{2}\cos\theta$$

$$\langle J_x\rangle = \frac{\hbar}{2}\left(a^*b + b^*a\right)$$

$$= \frac{\hbar}{2}\cos\theta/_2\sin\theta/_2\left(e^{i\phi}+e^{-i\phi}\right)$$

$$= \frac{\hbar}{2}\sin\theta\cos\phi$$

$$\langle J_y\rangle = \frac{\hbar}{2}\left(-i\,a^*b + i\,ab^*\right)$$

$$= \frac{\hbar}{2}\cos\theta/_2\sin\theta/_2\left(-i\,e^{i\phi}+i\,e^{-i\phi}\right)$$

$$= \frac{\hbar}{2}\sin\theta\sin\phi$$

$\langle\psi|\vec{J}|\psi\rangle$ is a vector of length $\frac{\hbar}{2}$ with polar coordinates θ, ϕ