$$\Rightarrow \quad |+\tfrac{1}{2}\rangle_z = \frac{1}{\sqrt{2}}|+\tfrac{1}{2}\rangle_y + \frac{1}{\sqrt{2}}|-\tfrac{1}{2}\rangle_y$$

$$|-\tfrac{1}{2}\rangle_z = \frac{-i}{\sqrt{2}}|+\tfrac{1}{2}\rangle_y + \frac{i}{\sqrt{2}}|-\tfrac{1}{2}\rangle_y$$

Measurement theory

prepare $J_z = +\frac{\hbar}{2}$ $\qquad |\tfrac{1}{2}\rangle_z = \frac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle_y + \frac{1}{\sqrt{2}}|-\tfrac{1}{2}\rangle_y$

measure $J_y = -\frac{\hbar}{2}$ (prob ½) $\qquad |+\tfrac{1}{2}\rangle_y = \frac{1}{\sqrt{2}}|\tfrac{1}{2}\rangle_z - \frac{i}{\sqrt{2}}|-\tfrac{1}{2}\rangle_z$

measure $J_z = -\frac{\hbar}{2}$ (prob ½) $\qquad |-\tfrac{1}{2}\rangle_z = \frac{-i}{\sqrt{2}}|+\tfrac{1}{2}\rangle_y + \frac{i}{\sqrt{2}}|-\tfrac{1}{2}\rangle_y$

- measuring J_y changed the J_z value of our state.
- the state went from one with

$$\text{Prob}_{J_z=\hbar/2} = 1 \qquad \text{Prob}_{J_z=-\hbar/2} = 0$$

to one with

$$\text{Prob}_{J_z=\hbar/2} = \frac{1}{2} \qquad \text{Prob}_{J_z=-\frac{\hbar}{2}} = \frac{1}{2}$$

03/23/2021

A consequence of relation between operator:

$$[J_y, J_z] = J_y J_z - J_z J_y = i\hbar J_x \neq 0$$

Understand properties of a general spin ½ state:

$$|\psi\rangle = a|+\tfrac{1}{2}\rangle_z + b|-\tfrac{1}{2}\rangle_z \qquad |a|^2+|b|^2=1$$

Prepare $|\psi\rangle$ multiple times and for each trial measure J_x <u>or</u> J_y <u>or</u> J_z

$$\langle J_z\rangle = \langle\psi|J_z|\psi\rangle = (a^* \;\; b^*)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\cdot\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}(|a|^2 - |b|^2)$$

$$\langle J_x\rangle = \langle\psi|J_x|\psi\rangle = (a^* \;\; b^*)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\cdot\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}(a^*b + b^*a) = \hbar\, \mathrm{re}(a^*b)$$

$$\langle J_y\rangle = \langle\psi|J_y|\psi\rangle = (a^* \;\; b^*)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}(-i\,a^*b + i\,b^*a) = \hbar\, \mathrm{im}(a^*b)$$

A consequence of relation between operator:

$$[J_y, J_z] = J_y J_z - J_z J_y = i\hbar J_x \neq 0$$

Understand properties of a general spin ½ state

$$|\psi\rangle = a|+½\rangle_z + b|-½\rangle_z \qquad |a|^2+|b|^2=1$$

Prepare $|\psi\rangle$ multiple times and for each trial measure J_x or J_y or J_z

$$\langle J_z\rangle = \langle\psi|J_z|\psi\rangle = (a^* \;\; b^*)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\cdot\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}(|a|^2-|b|^2)$$

$$\langle J_x\rangle = \langle\psi|J_x|\psi\rangle = (a^* \;\; b^*)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\cdot\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}\left(a^*b + b^*a\right) = \hbar\, \mathrm{re}(a^*b)$$

$$\langle J_y\rangle = \langle\psi|J_y|\psi\rangle = (a^* \;\; b^*)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix}\frac{\hbar}{2}$$

$$= \frac{\hbar}{2}(-i\, a^*b + i\, b^*a) = \hbar\, \mathrm{im}(a^*b)$$

Smart way to write a & b:

$$a = \cos(\theta/2)\, e^{-i\phi/2} \qquad b = \sin\theta/2\; e^{+i\phi/2}$$

$$|a|^2+|b|^2 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

$$\langle J_z\rangle = \frac{\hbar}{2}(|a|^2-|b|^2) = \frac{\hbar}{2}(\cos^2\theta/2 - \sin^2\theta/2)$$

$$= \frac{\hbar}{2}\cos\theta$$

$$\langle J_x\rangle = \frac{\hbar}{2}(a^*b + b^*a)$$

$$= \frac{\hbar}{2}\cos\theta/2 \sin\theta/2\,(e^{i\phi} + e^{-i\phi})$$

$$= \frac{\hbar}{2}\sin\theta\cos\phi$$

$$\langle J_y\rangle = \frac{\hbar}{2}(-i\, a^*b + i\, a b^*)$$

$$= \frac{\hbar}{2}\cos\theta/2\,\sin\theta/2\,(-i e^{i\phi} + i e^{-i\phi})$$

$$= \frac{\hbar}{2}\sin\theta\sin\phi$$

$\langle\psi|\vec{J}|\psi\rangle$ is a vector of length $\frac{\hbar}{2}$ with polar coordinates θ, ϕ

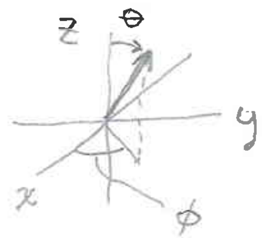
Smart way to write a & b:

$$a = \cos(\theta/2)\, e^{-i\phi/2} \qquad b = \sin\theta/2\, e^{+i\phi/2}$$

$$|a|^2 + |b|^2 = \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1$$

$$\langle J_z \rangle = \frac{\hbar}{2}\left(|a|^2 - |b|^2\right) = \frac{\hbar}{2}\left(\cos^2\theta/2 - \sin^2\theta/2\right)$$

$$= \frac{\hbar}{2}\cos\theta$$



$$\langle J_x \rangle = \frac{\hbar}{2}\left(a^* b + b^* a\right)$$

$$= \frac{\hbar}{2}\cos\theta/2 \sin\theta/2 \left(e^{i\phi} + e^{-i\phi}\right)$$

$$= \frac{\hbar}{2}\sin\theta\cos\phi$$

$$\langle J_y \rangle = \frac{\hbar}{2}\left(-i a^* b + i a b^*\right)$$

$$= \frac{\hbar}{2}\cos\theta/2 \sin\theta/2 \left(-i e^{i\phi} + i e^{-i\phi}\right)$$

$$= \frac{\hbar}{2}\sin\theta\sin\phi$$

$\langle\psi|\vec{J}|\psi\rangle$ is a vector of length $\frac{\hbar}{2}$ with polar coordinates θ, ϕ

Because of the special properties of the Pauli matrices we can easily determine the general rotation operator for spin ½:

$$R(\hat{n},\theta) = e^{-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\,\theta} \qquad \vec{J} = \frac{\hbar}{2}\vec{\sigma}$$

$$= e^{-i\vec{\sigma}\cdot\hat{n}\frac{\theta}{2}}$$

$$= \sum_{k=0}^{\infty}\frac{1}{k!}\left(-i\vec{\sigma}\cdot\hat{n}\frac{\theta}{2}\right)^k$$

$$(\vec{\sigma}\cdot\hat{n})^2 = \left(\sigma_x n_x + \sigma_y n_y + \sigma_z n_z\right)^2$$

$$= n_x^2\sigma_x^2 + n_y^2\sigma_y^2 + n_z^2\sigma_z^2$$

$$+ n_x n_y\left(\sigma_x\sigma_y + \sigma_y\sigma_x\right) + n_x n_z\left(\sigma_x\sigma_z + \sigma_z\sigma_x\right) + n_y n_z\left(\sigma_y\sigma_z + \sigma_z\sigma_x\right)$$

[recall $\sigma_i\sigma_i = I$, $i \neq j$ $\sigma_i\sigma_j = -\sigma_j\sigma_i$]

$$= n_x^2 + n_y^2 + n_z^2 = 1$$

$$e^{-i\vec{\sigma}\cdot\hat{n}\frac{\theta}{2}} = \sum_{\ell=0}^{\infty}\frac{1}{2\ell!}\left(-i\frac{\theta}{2}\right)^{2\ell} + \vec{\sigma}\cdot n\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)!}\left(-i\frac{\theta}{2}\right)^{2\ell+1}$$

Because of the special properties of the Pauli matrices we can easily determine the general rotation operator for spin 1/2 :

$$R(\hat{n},\theta) = e^{-\frac{i}{\hbar}\vec{J}\cdot\hat{n}\,\theta} \qquad \vec{J} = \frac{\hbar}{2}\vec{\sigma}$$

$$= e^{-i\,\vec{\sigma}\cdot\hat{n}\,\frac{\theta}{2}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(-i\,\vec{\sigma}\cdot\hat{n}\,\frac{\theta}{2}\right)^k$$

$$(\vec{\sigma}\cdot\hat{n})^2 = (\sigma_x n_x + \sigma_y n_y + \sigma_z n_z)^2$$

$$= n_x^2\sigma_x^2 + n_y^2\sigma_y^2 + n_z^2\sigma_z^2$$

$$+ n_x n_y(\sigma_x\sigma_y + \sigma_y\sigma_x) + n_x n_z(\sigma_x\sigma_z + \sigma_z\sigma_x)$$

$$+ n_y n_z(\sigma_y\sigma_z + \sigma_z\sigma_x)$$

[recall $\sigma_i\sigma_i = I$, $i \neq j$ $\sigma_i\sigma_j = -\sigma_j\sigma_i$]

$$= n_x^2 + n_y^2 + n_z^2 = 1$$

$$e^{-i\,\vec{\sigma}\cdot\hat{n}\,\frac{\theta}{2}} = \sum_{\ell=0}^{\infty}\frac{1}{2\ell!}\left(-i\,\frac{\theta}{2}\right)^{2\ell} + \vec{\sigma}\cdot n\sum_{\ell=0}^{\infty}\frac{1}{(2\ell+1)!}\left(-i\,\frac{\theta}{2}\right)^{2\ell+1}$$

Where the first sum gives $\cos\frac{\theta}{2}$ and the second $-i\sin\frac{\theta}{2}$

$$e^{-i\,\vec{\sigma}\cdot\hat{n}\,\frac{\theta}{2}} = \cos\frac{\theta}{2} - i\,\vec{\sigma}\cdot\hat{n}\sin\frac{\theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} - i n_z\sin\frac{\theta}{2} & -i(n_x - i n_y)\sin\frac{\theta}{2} \\ -i(n_x + i n_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + i n_z\sin\frac{\theta}{2} \end{pmatrix}$$

complete general rotation matrix!

Recall $|+\tfrac{1}{2}\rangle_z = \begin{pmatrix}1\\0\end{pmatrix}$

$$|+\tfrac{1}{2}\rangle_y = \frac{1}{\sqrt{2}}|+\tfrac{1}{2}\rangle_z + \frac{i}{\sqrt{2}}|-\tfrac{1}{2}\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ i\frac{1}{\sqrt{2}}\end{pmatrix}$$

Try rotating $|+\tfrac{1}{2}\rangle_z$ $-90°$ about x axis

$$e^{-i\frac{J_x}{\hbar}\left(-\frac{\pi}{2}\right)}|\tfrac{1}{2}\rangle_z = e^{+i\sigma_x\frac{\pi}{4}}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\pi}{4} & +i\sin\pi/4 \\ +i\sin\pi/4 & \cos\frac{\pi}{4}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}} & +i\frac{1}{\sqrt{2}} \\ +i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ i\frac{1}{\sqrt{2}}\end{pmatrix} \checkmark$$

Where the first sum gives $\cos\frac{\theta}{2}$

and the second $-i\sin\frac{\theta}{2}$

$$e^{-i\vec{\sigma}\cdot\hat{n}\frac{\theta}{2}} = \cos\frac{\theta}{2} - i\vec{\sigma}\cdot\hat{n}\sin\frac{\theta}{2}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} - in_z\sin\frac{\theta}{2} & -i(n_x - in_y)\sin\frac{\theta}{2} \\ -i(n_x+in_y)\sin\frac{\theta}{2} & \cos\frac{\theta}{2} + in_z\sin\frac{\theta}{2} \end{pmatrix}$$

complete general rotation matrix!

Recall $|+\tfrac{1}{2}\rangle_z = \begin{pmatrix}1\\0\end{pmatrix}$

$$|+\tfrac{1}{2}\rangle_y = \frac{1}{\sqrt{2}}|+\tfrac{1}{2}\rangle_z + \frac{i}{\sqrt{2}}|-\tfrac{1}{2}\rangle = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ i\frac{1}{\sqrt{2}}\end{pmatrix}$$

Try rotating $|+\tfrac{1}{2}\rangle_z$ $-90°$ about x axis

$$e^{-i\frac{J_x}{\hbar}\left(-\frac{\pi}{2}\right)}|\tfrac{1}{2}\rangle_z = e^{+i\sigma_x\frac{\pi}{4}}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\pi}{4} & +i\sin\pi/4 \\ +i\sin\pi/4 & \cos\frac{\pi}{4}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}\frac{1}{\sqrt{2}} & +i\frac{1}{\sqrt{2}} \\ +i\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}\\ i\frac{1}{\sqrt{2}}\end{pmatrix} \checkmark$$

Second consistency check:

recall $|\psi(\theta,\phi)\rangle = \begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{+i\phi/2}\end{pmatrix}$

has an average spin

$$\langle\psi(\theta,\phi)|\vec{J}|\psi(\theta,\phi)\rangle = \frac{\hbar}{2}\hat{n}(\theta,\phi)$$

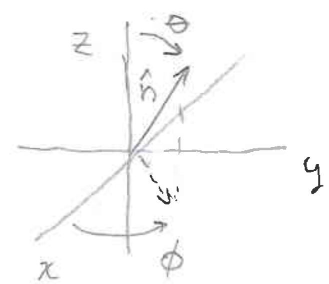
$$e^{-i\frac{1}{\hbar}J_z\vartheta}|\psi(\theta,\phi)\rangle = e^{-i\sigma_z\frac{\vartheta}{2}}\begin{pmatrix}\cos\frac{\theta}{2}e^{-i\phi/2}\\ \sin\frac{\theta}{2}e^{+i\phi/2}\end{pmatrix}$$

$\frac{\hbar}{2}\sigma_z$

$$= \begin{pmatrix}e^{-i\frac{\vartheta}{2}} & 0 \\ 0 & e^{+i\frac{\vartheta}{2}}\end{pmatrix}\begin{pmatrix}\cos\theta/2\, e^{-i\phi/2}\\ \sin\theta/2\, e^{+i\phi/2}\end{pmatrix}$$

$$= \begin{pmatrix}\cos\frac{\theta}{2}e^{-i(\phi+\vartheta)/2}\\ \sin\frac{\theta}{2}e^{+i(\phi+\vartheta)/2}\end{pmatrix}$$

The expected increase in ϕ by ϑ



Second consistency check:

recall $|\psi(\theta,\phi)\rangle = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{+i\phi/2} \end{pmatrix}$

has an average spin

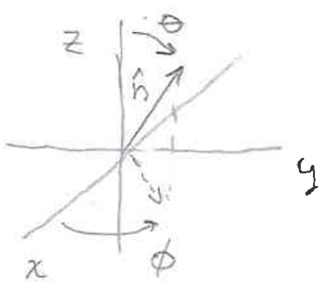
$$\langle \psi(\theta,\phi) | \vec{J} | \psi(\theta,\phi)\rangle = \frac{\hbar}{2} \hat{n}(\theta,\phi)$$

$$e^{-i\frac{1}{\hbar} \underset{\frac{\hbar}{2}\sigma_z}{J_z} \vartheta} |\psi(\theta,\phi)\rangle = e^{-i\sigma_z \frac{\vartheta}{2}} \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{+i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-i\frac{\vartheta}{2}} & 0 \\ 0 & e^{+i\frac{\vartheta}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta/2\, e^{-i\phi/2} \\ \sin\theta/2\, e^{+i\phi/2} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\frac{\theta}{2} e^{-i(\phi+\vartheta)/2} \\ \sin\frac{\theta}{2} e^{+i(\phi+\vartheta)/2} \end{pmatrix}$$

The expected increase in ϕ by ϑ



Next introduce dynamics and time dependence.

- If forces act on our quantum state $|\psi\rangle$ we expect it will change with time; $|\psi\rangle \to |\psi(t)\rangle$
- Assume like rotations, evolution in time is describe by a linear operator:
  $$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$
- $U(t)$ should not change the inner product, ie $U(t)^{-1} = U(-t) = U(t)^{\dagger}$.
- Should require $U(t_2)U(t_1) = U(t_2+t_1)$

For small Δt expand

$$U(\Delta t) = I - \frac{i}{\hbar}\Delta t \underbrace{H}_{\text{units of energy}} + O(\Delta t^2)$$

Just as for rotations $\quad [H = H^{\dagger}]$

$$R(x, \Delta\theta) = I - \frac{i}{\hbar} J_x \Delta\theta + O(\Delta\theta^2)$$

Next introduce dynamics and time dependence.

- If forces act on our quantum state $|\psi\rangle$ we expect it will change with time: $|\psi\rangle \rightarrow |\psi(t)\rangle$
- Assume like rotations, evolution in time is describe by a linear operator: $|\psi(t)\rangle = U(t)|\psi(0)\rangle$
- $U(t)$ should not change the inner product, ie $U(t)^{-1} = U(-t) = U(t)^{\dagger}$.
- Should require $U(t_2)\,U(t_1) = U(t_2+t_1)$

For small Δt expand

units of energy

$$U(\Delta t) = I - \frac{i}{\hbar}\Delta t\, H + O(\Delta t^2)$$

Just as for rotations $[H = H^{\dagger}]$

$$R(x, \Delta\theta) = I - \frac{i}{\hbar} J_x \Delta\theta + O(\Delta\theta^2)$$

Just as for rotations

$$U(t+\Delta t) = U(\Delta t)\,U(t)$$

$$U(t) + \Delta t \frac{dU}{dt}(t) = \left[I - \frac{i\Delta t}{\hbar} H\right] U(t)$$

$$\& \quad \frac{dU}{dt}(t) = -\frac{i}{\hbar} H\, U(t) \quad , \quad U(0) = I$$

$$\Rightarrow \quad U(t) = e^{-\frac{i}{\hbar}Ht}$$

Thus, $|\psi(t)\rangle = U(t)|\psi(0)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle$

$$\& \quad i\hbar \frac{d}{dt}|\psi(t)\rangle = i\hbar \underbrace{\frac{d}{dt}U(t)}_{-\frac{i}{\hbar}H\,U(t)}|\psi(0)\rangle$$

$$= H\, U(t)\,|\psi(0)\rangle$$

or

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = H\,|\psi(t)\rangle$$

The Schrodinger equation. From classical Hamiltonian mechanics guess that H is the energy operator.

Just as for rotations

$$U(t+\Delta t) = U(\Delta t)\, U(t)$$

$$U(t) + \Delta t \frac{dU}{dt}(t) = \left[I - i\frac{\Delta t}{\hbar} H \right] U(t)$$

$$\& \quad \frac{dU}{dt}(t) = -\frac{i}{\hbar} H\, U(t) \quad , \quad U(0) = I$$

$$\Rightarrow \quad U(t) = e^{-\frac{i}{\hbar} H t}$$

Thus, $|\psi(t)\rangle = U(t)|\psi(0)\rangle = e^{-\frac{i}{\hbar} H t} |\psi(0)\rangle$

$$\& \quad i\hbar \frac{d}{dt}|\psi(t)\rangle = i\hbar \underbrace{\frac{d}{dt} U(t)}_{-\frac{i}{\hbar} H\, U(t)} |\psi(0)\rangle$$

$$= H\, U(t)\, |\psi(0)\rangle$$

or

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H\, |\psi(t)\rangle$$

The Schrodinger equation. From classical Hamiltonian mechanics guess that H is the energy operator.

Try out with quantum angular momentum. If there is a magnetic moment $\vec{\mu}$ associated with our system we expect $\vec{\mu} = \gamma \vec{J}$ and $H = -\vec{\mu}\cdot\vec{B}$ if we impose a magnetic field $\vec{B}$. Expect

$$i\hbar\, |\psi(t)\rangle = H\, |\psi(t)\rangle = -\gamma \vec{J}\cdot B\, |\psi(t)\rangle.$$

Solved by exponentiation

$$|\psi(t)\rangle = e^{+i\frac{\gamma}{\hbar}\vec{J}\cdot\vec{B}\, t}\, |\psi(0)\rangle$$

$$= e^{-i\frac{1}{\hbar}\hat{n}\cdot\vec{J}\,\theta(t)}\, |\psi(0)\rangle$$

where $\hat{n} = \hat{B}$, $\theta = -\gamma B t$

our familiar Larmor precession

$$\vec{\tau} = \vec{\mu}\times\vec{B}, \quad \frac{d\vec{L}}{dt} = \vec{\mu}\times\vec{B} = \gamma\vec{L}\times\vec{B}$$

$$\text{or} \quad \frac{d}{dt}\vec{L} = \underbrace{-\gamma\vec{B}}_{\vec{\omega}}\times\vec{L} \quad \Big\}\ \text{rotation with } \vec{\omega} = -\gamma\vec{B}$$

Try out with quantum angular momentum.

If there is a magnetic moment $\vec{\mu}$ associated with our system we expect $\vec{\mu} = \gamma \vec{J}$ and $H = -\vec{\mu} \cdot \vec{B}$ if we impose a magnetic field $\vec{B}$.

Expect

$$i\hbar \,| \psi(t) \rangle = H \,| \psi(t) \rangle = -\gamma \vec{J} \cdot B \,| \psi(t) \rangle .$$

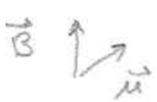
Solved by exponentiation

$$| \psi(t) \rangle = e^{+i \frac{\gamma}{\hbar} \vec{J} \cdot \vec{B} t} \,| \psi(0) \rangle$$

$$= e^{-i \frac{1}{\hbar} \hat{n} \cdot \vec{J} \, \theta(t)} \,| \psi(0) \rangle$$

where $\hat{n} = -\hat{B}$, $\theta = -\gamma B t$

our familiar Larmor precession

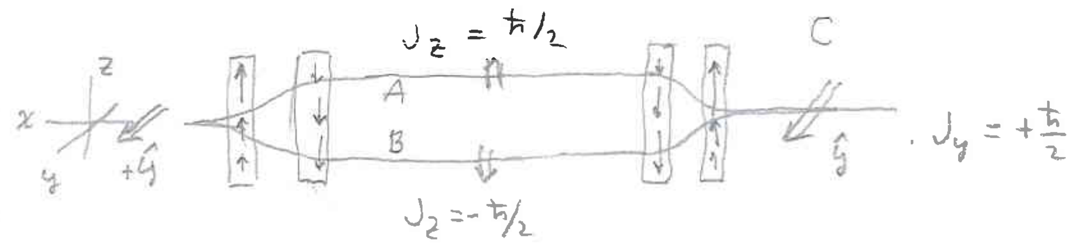


$$\vec{\tau} = \vec{\mu} \times \vec{B}, \quad \frac{d\vec{L}}{dt} = \vec{\mu} \times \vec{B} = \gamma \vec{L} \times \vec{B}$$

or $\frac{d}{dt} \vec{L} = \underbrace{-\gamma \vec{B}}_{\vec{\omega}} \times \vec{L}$ } rotation with $\vec{\omega} = -\gamma \vec{B}$

7. The quantum mechanics of measurements:

Consider two pairs of Stern-Gerlach magnets:

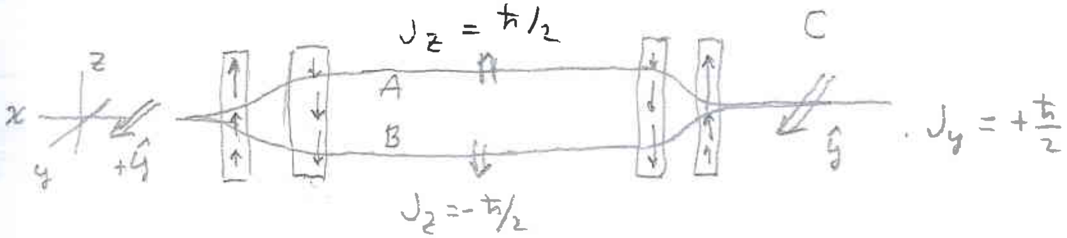


If we add sensitive photographic emulsion to leg B that is exposed if particle has $J_z = \hbar/2$, then polarization on the right disappears: With the emultion we measure J_z and so J_z is known when it reaches region C so J_y must be zero.

We must be able to derive this result from quantum theory if we include the emulsion in our quantum state.

7. The quantum mechanics of measurements:

Consider two pairs of Stern-Gerlach magnets:



If we add sensitive photographic emulsion to leg B that is exposed if particle has $J_z = \hbar/2$, then polarization on the right disappears: With the emultion we measure J_z and so J_z is known when it reaches region C so J_y must be zero.

We must be able to derive this result from quantum theory if we include the emulsion in our quantum state.

We want to combine the state of our Cesium atom $|\pm 1/2\rangle_z$ with the quantum state of the emulsion which we might also represent with two states:

$|yes\rangle$ (exposed) & $|no\rangle$ (not exposed)

What does linear algebra offer?

Start with two vector spaces:

D with N_D basis vectors $|d_i\rangle$, $1 \le i \le N_D$

E with N_E basis vectors $|e_i\rangle$, $1 \le i \le N_E$

I Their cartesian product $D \oplus E$ with $N_D + N_E$ basis vectors $|d_1\rangle \ldots |d_{N_D}\rangle$ and $|e_1\rangle \ldots |e_{N_E}\rangle$: $|\psi\rangle = \sum_{i=1}^{N_D} D_i |d_i\rangle + \sum_{j=1}^{N_E} E_i |e_i\rangle$

II Their tensor product $D \otimes E$ with $N_D \times N_E$ basis vectors $|d_i\rangle \otimes |e_j\rangle$

and $|\psi\rangle = \sum_{i=1}^{N_D} \sum_{j=1}^{N_E} C_{ij} |d_i\rangle \otimes |e_j\rangle$

We want to combine the state of our Cesium atom $|\pm 1/2\rangle_z$ with the quantum state of the emulsion which we might also represent with two states:

$|yes\rangle$ (exposed) & $|no\rangle$ (not exposed)

What does linear algebra offer?

Start with two vector spaces:

D with N_D basis vectors $|d_i\rangle$, $1 \le i \le N_D$

E with N_E basis vectors $|e_i\rangle$, $1 \le i \le N_E$

I Their cartesian product $D \oplus E$ with $N_D + N_E$ basis vectors $|d_1\rangle \ldots |d_{N_D}\rangle$ and $|e_1\rangle \ldots |e_{N_E}\rangle$: $|\psi\rangle = \sum_{i=1}^{N_D} D_i |d_i\rangle + \sum_{j=1}^{N_E} E_i |e_i\rangle$

II Their tensor product $D \otimes E$ with $N_D \times N_E$ basis vectors $|d_i\rangle \otimes |e_j\rangle$ and $|\psi\rangle = \sum_{i=1}^{N_D} \sum_{j=1}^{N_E} C_{ij} |d_i\rangle \otimes |e_j\rangle$

I Cartesian product is good to describe a classical 2-particle system

$$\vec{v} = r_x^{(1)} \hat{e}_x^{(1)} + r_y^{(1)} \hat{e}_y^{(1)} + r_z^{(1)} \hat{e}_z^{(1)} + r_x^{(2)} \hat{e}_x^{(2)} + r_y^{(2)} \hat{e}_y^{(2)} + r_z^{(2)} \hat{e}_z^{(2)}$$

six basis vector $\hat{e}_x^{(1)}, \hat{e}_y^{(1)}, \ldots \hat{e}_z^{(2)}$

$(r_x^{(1)}, r_y^{(1)}, r_z^{(1)})$ & $(r_x^{(2)}, r_y^{(2)}, r_z^{(2)})$ locate particle 1 & particle 2

II Tensor product is perfect for quantum mechanics. Consider deuterium atom:

Combine $m_z = \pm 1/2$ electron states with $m_z = 1, 0, -1$ deuteron states

$$|\psi\rangle = \sum_{m_z^D=-1}^{1} \sum_{m_z^e=-1/2}^{+1/2} \psi_{m_z^D, m_z^e} |m_z^D\rangle^D \otimes |m_z^e\rangle^e$$

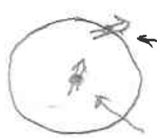
I Cartesian product is good to describe a classical 2-particle system

$$\vec{r} = r_x^{(1)}\hat{e}_x^{(1)} + r_y^{(1)}\hat{e}_y^{(1)} + r_z^{(1)}\hat{e}_z^{(1)} + r_x^{(2)}\hat{e}_x^{(2)} + r_y^{(2)}\hat{e}_y^{(2)} + r_z^{(2)}\hat{e}_z^{(2)}$$

six basis vector $\hat{e}_x^{(1)}, \hat{e}_y^{(1)}, \ldots \hat{e}_z^{(2)}$

$(r_x^{(1)}, r_y^{(1)}, r_z^{(1)})$ & $(r_x^{(2)}, r_y^{(2)}, r_z^{(2)})$ locate partice 1 & partice 2

II Tensor product is perfect for quantum mechanics. Consider deuterium atom:



← $j=1/2$ electron

$j=1$ nucleus

Combine $m_z = \pm 1/2$ electron states with $m_z = 1, 0, -1$ deuteron states

$$|\psi\rangle = \sum_{m_z^D=-1}^{1} \sum_{m_z^e=-1/2}^{+1/2} \psi_{m_z^D, m_z^e} |m_z^D\rangle^D \otimes |m_z^e\rangle^e$$

$$|\psi\rangle = \psi_{-1\,-1/2}|-1\rangle^D \otimes |-1/2\rangle^e + \psi_{-1\,+1/2}|-1\rangle^D \otimes |+1/2\rangle^e$$
$$+ \psi_{0\,-1/2}|0\rangle^D \otimes |-1/2\rangle^e + \psi_{0\,+1/2}|0\rangle^D \otimes |+1/2\rangle^e$$
$$+ \psi_{+1\,-1/2}|+1\rangle^D \otimes |-1/2\rangle^e + \psi_{+1,+1/2}|+1\rangle^D \otimes |+1/2\rangle^e$$

6-dimensional

$|\psi_{m_z^D\, m_z^e}|^2$ = prob to find m_z^D for deuteron and m_z^e for electron when independent measurements of deuteron and electrons J_z are made

$$J_i^{total} = J_i^D \otimes J_i^e \qquad i = x, y, z$$

acts only on deuteron

acts only on electron

J_i^D and J_j^e are independent

$$[J_i^D, J_j^e] = 0$$