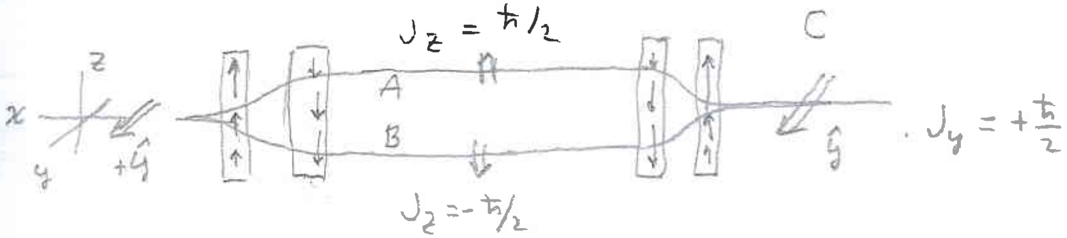
7. The quantum mechanics of measurements:

Consider two pairs of Stern-Gerlach magnets:



If we add sensitive photographic emulsion to leg B that is exposed if particle has $J_z = \hbar/2$, then polarization on the right disappears: With the emulsion we measure J_z and so J_z is known when it reaches region C so J_y must be zero.

We must be able to derive this result from quantum theory if we include the emulsion in our quantum state.

We want to combine the state of our Cesium atom $|\pm 1/2\rangle_z$ with the quantum state of the emulsion which we might also represent with two states:

$|yes\rangle$ (exposed) & $|no\rangle$ (not exposed)

What does linear algebra offer?

Start with two vector spaces:

D with N_D basis vectors $|d_i\rangle$, $1 \le i \le N_D$

E with N_E basis vectors $|e_i\rangle$, $1 \le i \le N_E$

I Their cartesian product $D \oplus E$ with $N_D + N_E$ basis vectors $|d_1\rangle \dots |d_{N_D}\rangle$ and $|e_1\rangle \dots |e_{N_E}\rangle$: $|\psi\rangle = \sum_{i=1}^{N_D} D_i |d_i\rangle + \sum_{j=1}^{N_E} E_i |e_i\rangle$

II Their tensor product $D \otimes E$ with $N_D \times N_E$ basis vectors $|d_i\rangle \otimes |e_j\rangle$

and $|\psi\rangle = \sum_{i=1}^{N_D} \sum_{j=1}^{N_E} C_{ij} |d_i\rangle \otimes |e_j\rangle$

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I Cartesian product is good to describe a classical 2-particle system

$$\vec{v} = r_x^{(1)} \hat{e}_x^{(1)} + r_y^{(1)} \hat{e}_y^{(1)} + r_z^{(1)} \hat{e}_z^{(1)} + r_x^{(2)} \hat{e}_x^{(2)} + r_y^{(2)} \hat{e}_y^{(2)} + r_z^{(2)} \hat{e}_z^{(2)}$$

six basis vector $\hat{e}_x^{(1)}, \hat{e}_y^{(1)}, \ldots \hat{e}_z^{(2)}$

$(r_x^{(1)}, r_y^{(1)}, r_z^{(1)})$ & $(r_x^{(2)}, r_y^{(2)}, r_z^{(2)})$ locate particle 1 & particle 2

II Tensor product is perfect for quantum mechanics. Consider deuterium atom:

$\leftarrow j = 1/2$ electron

$j = 1$ nucleus

Combine $m_z = \pm 1/2$ electron states with $m_z = 1, 0, -1$ deuteron states

$$|\psi\rangle = \sum_{m_z^D=-1}^{1} \sum_{m_z^e=-1/2}^{+1/2} \psi_{m_z^D, m_z^e} |m_z^D\rangle^D \otimes |m_z^e\rangle^e$$

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$$|\psi\rangle = \psi_{-1\,-1/2} |-1\rangle^D \otimes |-1/2\rangle^e + \psi_{-1\,+1/2} |-1\rangle^D \otimes |+1/2\rangle^e$$
$$+ \psi_{0\,-1/2} |0\rangle^D \otimes |-1/2\rangle^e + \psi_{0\,+1/2} |0\rangle^D \otimes |+1/2\rangle^e$$
$$+ \psi_{+1\,-1/2} |+1\rangle^D \otimes |-1/2\rangle^e + \psi_{+1,+1/2} |+1\rangle^D \otimes |+1/2\rangle^e$$

6-dimensional

$\left|\psi_{m_z^D\, m_z^e}\right|^2$ = prob to find m_z^D for deuteron and m_z^e for electron when independent measurements of deuteron and electrons J_z are made

$$J_i^{total} = J_i^D \otimes J_i^e \qquad i = x, y, z$$

acts only on deuteron — acts only on electron

J_i^D and J_j^e are independent

$$[J_i^D, J_j^e] = 0$$

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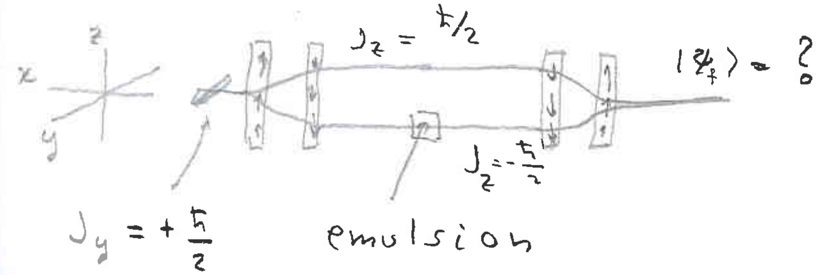
Now we can follow motion of Cesium atoms including the quantum mechanical measurement of each atom's value of J_z:



state of emulsion "no" atom has passed

Initial wave function

$$|\psi_i\rangle = \frac{1}{\sqrt{2}}\left\{|1/2\rangle_{m_z} + i|-1/2\rangle_{m_z}\right\} \otimes |no\rangle$$

Final wave function

$$|\psi_f\rangle = \frac{1}{\sqrt{2}}|1/2\rangle_{m_z} \otimes |no\rangle + \frac{i}{\sqrt{2}}|-1/2\rangle \otimes |yes\rangle$$

atom and emulsion states are "entangled"

$$\langle J_z\rangle = \langle\psi_f|\, J_z\, |\psi_f\rangle$$
$$= \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\langle 1/2|_z \otimes \langle no| - \frac{i}{\sqrt{2}}\langle -1/2|_z \otimes \langle yes|\right]$$
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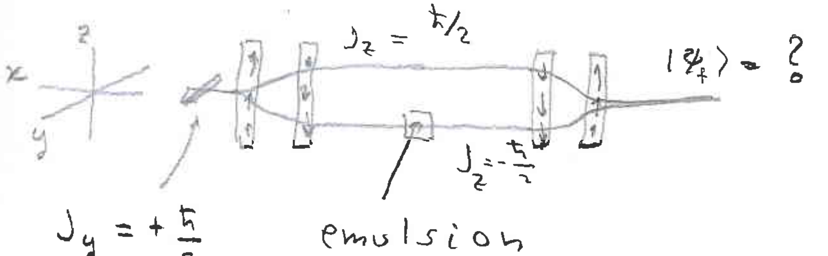

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$$\langle J_y\rangle = \langle\psi_f|\, J_y\, |\psi_f\rangle$$

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$$\times\ \underbrace{\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\left[\frac{1}{\sqrt{2}}|1/2\rangle_z \otimes |no\rangle + \frac{i}{\sqrt{2}}|-1/2\rangle_z \otimes |yes\rangle\right]}_{\left[\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}|-1/2\rangle_z \otimes |no\rangle + \frac{i}{\sqrt{2}}|+1/2\rangle_z \otimes |yes\rangle\right]}$$

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Relation between operators and observables?

For a finite (D) dimensional quantum vector space each hermitian operator corresponds to a quantity that might be measured. If $O = O^\dagger$

① There exists D states $\{|\lambda_n\rangle\}_{1 \le n \le D}$ which are eigenstates of O

$O|\lambda_n\rangle = \lambda_n |\lambda_n\rangle$ (homework problem)

② λ_n are real:

$$\lambda_n = (|\lambda_n\rangle, O|\lambda_n\rangle) = (O|\lambda_n\rangle, |\lambda_n\rangle)$$
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③ $\langle \lambda_{n'} | \lambda_n \rangle = 0$ if $\lambda_{n'} \neq \lambda_n$:

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$$\Rightarrow \quad 0 = \underbrace{(\lambda_{n'} - \lambda_n)}_{\text{if } \neq 0 \;\Rightarrow} \langle \lambda_{n'} | \lambda_n \rangle$$

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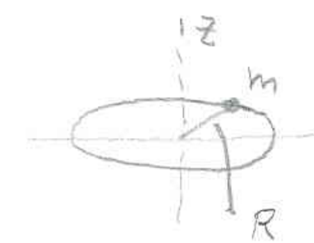
C Next introduce a quantum mechanical description of a particle's position.

1. Start with angular momentum and try to identify the position of the object we are rotating

Imagine a bead on a circular wire with $\hat{z}$ the axis of symmetry



If this system has angular momentum j we have $2j+1$ states $|m\rangle$ $-j \le m \le j$

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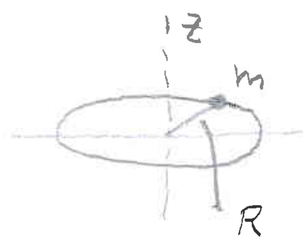

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Consider $|\psi\rangle = \sum_{m=-j}^{j} \tilde{\psi}_m |m\rangle$

$$e^{-iJ_z \frac{1}{\hbar}\phi}|\psi\rangle = \sum_{m=-j}^{j} \tilde{\psi}_m e^{-im\phi}|m\rangle$$

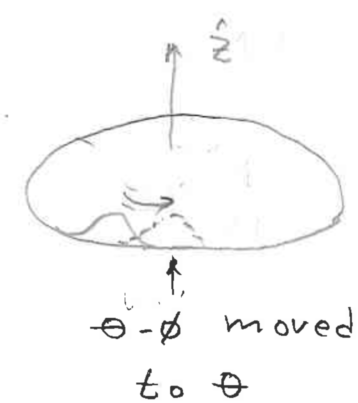
If we associate the function

$$\psi(\theta) = \sum_{m=-j}^{j} \frac{e^{im\theta}}{\sqrt{2\pi}} \tilde{\psi}_m \quad \text{with } |\psi\rangle$$

then $e^{-iJ_z\phi/\hbar}|\psi\rangle$ will be associated

$$\text{with } \sum_{m=-j}^{j} \frac{e^{-im\theta}}{\sqrt{2\pi}} e^{im\phi}\tilde{\psi}_m = \psi(\theta-\phi)$$

and we have rotated the function $\psi(\theta)$ through ϕ! [value of $\psi(\theta)$ has been replaced by value at $\theta-\phi$.]



Calculate $\int_0^{2\pi} |\psi(\theta)|^2 d\theta$

$$= \sum_{m',m} \psi^*_{m'} \psi_m \int_0^{2\pi} \frac{e^{i(m-m')\theta}}{2\pi} d\theta$$

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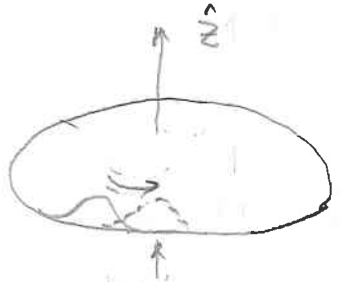

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$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-m')\theta} d\theta$$

if $m' \neq m$ $= \frac{1}{2\pi} \frac{e^{i(m-m')2\pi} - 1}{i(m-m')} = 0$

if $m' = m$ $= \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$

Thus, $\psi(\theta)$ is the wave function

associated with $|\psi\rangle = \sum_{m=-j}^{j} \tilde{\psi}_m |m\rangle$

& $\int_0^{2\pi} |\psi(\theta)|^2 d\theta = 1$

Not much choice of wave function

since we can only vary the $2j+1$

amplitudes $\tilde{\psi}_m$. Examine the

Special state $\tilde{\psi}_m = \frac{1}{\sqrt{2j+1}} e^{-i\phi m}$

$$\psi(\theta) = \sum_{m=-j}^{j} \tilde{\psi}_m \cdot \frac{e^{-im\theta}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi(2j+1)}} \sum_{m=-j}^{j} e^{im(\theta-\phi)}$$

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$$\psi(\theta) = \frac{1}{\sqrt{(2\pi)(2j+1)}} e^{-j(\theta-\phi)} \sum_{k=0}^{2j} e^{ik(\theta-\phi)}$$

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[recall $(1+z+z^2+\dots z^n)(1-z) = 1-z^{n+1}$]

$$= \frac{1}{\sqrt{2\pi(2j+1)}} \frac{e^{-(j+\frac{1}{2})(\theta-\phi)} - e^{i(j+\frac{1}{2})(\theta-\phi)}}{e^{-i(\theta-\phi)/2} - e^{+i(\theta-\phi)/2}}$$

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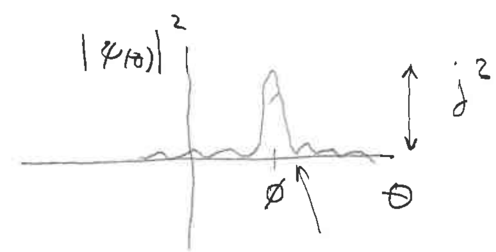

$\theta - \phi = \dfrac{\pi}{j+\frac{1}{2}}$

area under curve $\sim \underset{\text{height}}{j^2} \times \underset{\text{width}}{\dfrac{1}{j}} \dfrac{1}{(\sqrt{2j+1})^2} \sim 1$ ✓

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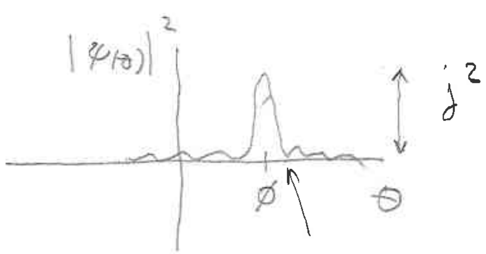

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Among the possible states $\psi(\theta)$ this is the most localized.

However, we can't put our bead at a definite place unless we take $j \to \infty$ limit: which we should do!

- Ojects with fixed j can not have precise rotational locations
- However, for larger j these locations become more precise and the system more physical
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What is $(J_z \psi)(\theta)$?

$$(J_z \psi)(\theta) \sim J_z \sum_{m=-\infty}^{\infty} \tilde{\psi}_m |m\rangle$$

$$\sim \sum_{m=-\infty}^{\infty} \underbrace{\hbar m \tilde{\psi}_m}\, |m\rangle$$

$$(J_z \psi)(\theta) = \sum_{m=-\infty}^{\infty} \frac{e^{im\theta}}{\sqrt{2\pi}} (\hbar m \tilde{\psi}_m)$$

$$= -i\hbar \frac{\partial}{\partial\theta} \sum_{m=-\infty}^{\infty} \frac{e^{im\theta}}{\sqrt{2\pi}} \tilde{\psi}_m = -i\hbar \frac{\partial}{\partial\theta} \psi(\theta)$$

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What is $(J_z \psi)(\theta)$?

$$(J_z \psi)(\theta) \sim J_z \sum_{m=-\infty}^{\infty} \tilde{\psi}_m |m\rangle$$

$$\sim \sum_{m=-\infty}^{\infty} \underbrace{\hbar m \tilde{\psi}_m} |m\rangle$$

$$(J_z \psi)(\theta) = \sum_{m=-\infty}^{\infty} \frac{e^{im\theta}}{\sqrt{2\pi}} (\hbar m \tilde{\psi}_m)$$

$$= -i\hbar \frac{\partial}{\partial\theta} \sum_{m=-\infty}^{\infty} \frac{e^{im\theta}}{\sqrt{2\pi}} \tilde{\psi}_m = -i\hbar \frac{\partial}{\partial\theta} \psi(\theta)$$

Thus $J_z = -i\hbar \frac{\partial}{\partial\theta}$ and
we have developed the quantum theory for a point particle moving on a ring:

1. Two complementary descriptions:

a) $|\psi\rangle = \sum_{m=-\infty}^{\infty} \tilde{\psi}_m |m\rangle$, $\sum_{m=-\infty}^{\infty} |\tilde{\psi}_m|^2 = 1$

$J_z |m\rangle = \hbar m |m\rangle$

$|\tilde{\psi}_m|^2 =$ probability of finding $\hbar m$ when measuring J_z

b) $\psi(\theta) = \sum_{m=-\infty}^{\infty} \frac{e^{im\theta}}{\sqrt{2\pi}} \tilde{\psi}_m$

$|\psi(\theta)|^2 \Delta\theta =$ prob. of finding particle in the interval $[\theta, \theta + \Delta\theta]$

$$\int_0^{2\pi} |\psi(\theta)|^2 d\theta = 1$$