Consider $|\psi\rangle = \sum_{m=-j}^{j} \tilde{\psi}_m |m\rangle$

$$e^{-iJ_z \frac{1}{\hbar}\phi} |\psi\rangle = \sum_{m=-j}^{j} \tilde{\psi}_m e^{-im\phi} |m\rangle$$

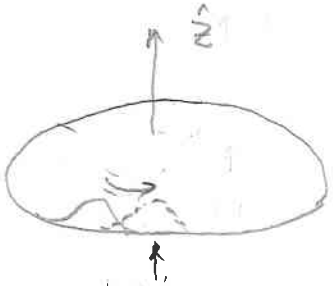
If we associate the function

$\psi(\theta) = \sum_{m=-j}^{j} \frac{e^{im\theta}}{\sqrt{2\pi}} \tilde{\psi}_m$ with $|\psi\rangle$

then $e^{-iJ_z\phi/\hbar}|\psi\rangle$ will be associated

with $\sum_{m=-j}^{j} \frac{e^{-im\theta}}{\sqrt{2\pi}} e^{im\phi} \tilde{\psi}_m = \psi(\theta-\phi)$

and we have rotated the function $\psi(\theta)$ through ϕ! [value of $\psi(\theta)$ has been replaced by value at $\theta-\phi$.]



$\theta-\phi$ moved to θ

Calculate $\int_0^{2\pi} |\psi(\theta)|^2 d\theta$

$$= \sum_{m',m} \psi^*_{m'} \psi_m \int_0^{2\pi} \frac{e^{i(m-m')\theta}}{2\pi} d\theta$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(m-m')\theta} d\theta$$

if $m' \neq m$ $= \frac{1}{2\pi} \frac{e^{i(m-m')2\pi} - 1}{i(m-m')} = 0$

if $m' = m$ $= \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1$

View $\psi(\theta)$ as a <u>wave function</u> associated with $|\psi\rangle = \sum_{m=-j}^{j} \tilde{\psi}_m |m\rangle$

$\ni \int_0^{2\pi} |\psi(\theta)|^2 d\theta = 1$

Not much choice of wave function since we can only vary the $2j+1$ amplitudes $\tilde{\psi}_m$. Examine the special state $\tilde{\psi}_m = \frac{1}{\sqrt{2j+1}} e^{i\phi m}$

$$\psi(\theta) = \sum_{m=-j}^{j} \tilde{\psi}_m \frac{e^{-im\theta}}{\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi(2j+1)}} \sum_{m=-j}^{j} e^{im(\theta-\phi)}$$

3/30/21

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Sum is a standard partial geometric series:

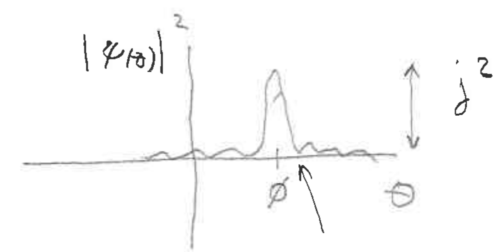
$$\psi(\theta) = \frac{1}{\sqrt{(2\pi)(2j+1)}}\, e^{-j(\theta-\phi)} \sum_{k=0}^{2j} e^{ik(\theta-\phi)}$$

$$= \frac{1}{\sqrt{2\pi(2j+1)}}\, e^{-j(\theta-\phi)}\, \frac{1-e^{i(2j+1)(\theta-\phi)}}{1-e^{i(\theta-\phi)}}$$

[recall $(1+z+z^2+\dots z^n)(1-z) = 1-z^{n+1}$]

$$= \frac{1}{\sqrt{2\pi(2j+1)}}\, \frac{e^{-(j+\frac{1}{2})(\theta-\phi)} - e^{i(j+\frac{1}{2})(\theta-\phi)}}{e^{-i(\theta-\phi)/2} - e^{+i(\theta-\phi)/2}}$$

$$= \frac{1}{\sqrt{2\pi(2j+1)}}\, \frac{\sin(j+\frac{1}{2})(\theta-\phi)}{\sin((\theta-\phi/2)}$$



$\theta - \phi = \dfrac{\pi}{j+\frac{1}{2}}$

area under curve $\sim \underbrace{j^2}_{\text{height}} \times \underbrace{\frac{1}{j}}_{\text{width}} \dfrac{1}{(\sqrt{2j+1})^2} \sim 1$ ✓

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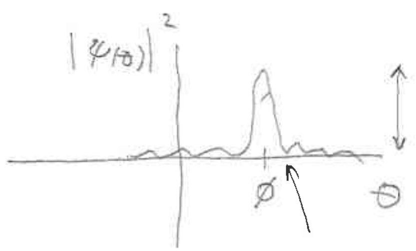

$\theta-\phi = \frac{\pi}{j+\frac{1}{2}}$

area under curve

$\sim j^2 \times \frac{1}{j}\,\frac{1}{(\sqrt{2j+1})^2} \sim 1$ ✓

height

width

Among the possible states $\psi(\theta)$ this is the most localized. However, we can't put our bead at a definite place unless we take $j \to \infty$ limit: which we should do!

- Ojects with fixed j can not have precise rotational locations
- However, for larger j these locations become more precise and the system more familiar
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Note: given $\psi(\theta)$ we can we can extract $\tilde{\psi}_m$:

$$\tilde{\psi}_m = \int_0^{2\pi} \frac{e^{-im\theta}}{\sqrt{2\pi}} \psi(\theta)\, d\theta$$

$$= \int_0^{2\pi} \frac{e^{-im\theta}}{\sqrt{2\pi}} \left[\sum_{m=-\infty}^{\infty} \frac{e^{im'\theta}}{\sqrt{2\pi}} \tilde{\psi}_{m'} \right] d\theta = \tilde{\psi}_m$$

$$\psi(\theta) \sim \tilde{\psi}_m \sim |\psi\rangle = \sum_{m=-\infty}^{\infty} \tilde{\psi}_m |m\rangle$$

What is $(J_z \psi)(\theta)$?

$$(J_z \psi)(\theta) \sim J_z \sum_{m=-\infty}^{\infty} \tilde{\psi}_m |m\rangle$$

$$\sim \sum_{m=-\infty}^{\infty} \underbrace{\hbar m \tilde{\psi}_m}\, |m\rangle$$

$$(J_z \psi)(\theta) = \sum_{m=-\infty}^{\infty} \frac{e^{im\theta}}{\sqrt{2\pi}} (\hbar m \tilde{\psi}_m)$$

$$= -i\hbar \frac{\partial}{\partial\theta} \sum_{m=-\infty}^{\infty} \frac{e^{im\theta}}{\sqrt{2\pi}} \tilde{\psi}_m = -i\hbar \frac{\partial}{\partial\theta} \psi(\theta)$$

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Thus $J_z = -i\hbar \frac{\partial}{\partial\theta}$ and

we have developed the quantum theory for a point particle moving on a ring:

① Two complementary descriptions:

a) $|\psi\rangle = \sum_{m=-\infty}^{\infty} \tilde{\psi}_m |m\rangle$, $\sum_{m=-\infty}^{\infty} |\tilde{\psi}_m|^2 = 1$

$J_z |m\rangle = \hbar m |m\rangle$

$|\tilde{\psi}_m|^2 =$ probability of finding $\hbar m$ when measuring J_z

b) $\psi(\theta) = \sum_{m=-\infty}^{\infty} \frac{e^{im\theta}}{\sqrt{2\pi}} \tilde{\psi}_m$ } Theory of Fourier series

$|\psi(\theta)|^2 \Delta\theta =$ prob. of finding particle in the interval $[\theta, \theta+\Delta\theta]$

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The connection between these two descritions can be neatly described if we view

$|\psi\rangle = \sum_{m=-\infty}^{\infty} \tilde{\psi}_m |m\rangle$ as an abstract Hilbert space expression

and the association

$|\psi\rangle \to \psi(\theta)$ & $|m\rangle \to \frac{e^{im\theta}}{\sqrt{2\pi}}$

as a specific realization of that abstract description using a complex vector space of functions

$\langle \psi' | \psi \rangle \longrightarrow \int_0^{2\pi} \psi'^*(\theta)\, \psi(\theta)\, d\theta$

$\langle m' | m \rangle \longrightarrow \int_0^{2\pi} \frac{e^{-im'\theta}}{\sqrt{2\pi}} \frac{e^{+im\theta}}{\sqrt{2\pi}} d\theta = \delta_{mm'}$

$|\psi\rangle = \sum_m \tilde{\psi}_m |m\rangle \longrightarrow \psi(\theta) = \sum_m \tilde{\psi}_m \frac{e^{im\theta}}{\sqrt{2\pi}}$

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$$\tilde{\psi}_m = \langle m|\psi\rangle \to \tilde{\psi}_m = \int_0^{2\pi} d\theta\, \frac{e^{-im\theta}}{\sqrt{2\pi}}\,\psi(\theta)$$

$$J_z\,|\psi\rangle \to -i\hbar\frac{\partial}{\partial\theta}\,\psi(\theta)$$

$$J_z|\psi_m\rangle = \hbar m|\psi_m\rangle \to -i\hbar\frac{\partial}{\partial\theta}\frac{e^{im\theta}}{\sqrt{2\pi}} = \hbar m\,\frac{e^{im\theta}}{\sqrt{2\pi}}$$

Replace J_z & θ by p and x

unwrap the circle

$$\psi_{pos}(x) = \sqrt{\frac{1}{R}}\,\psi_{ang}\left(\theta = \frac{x}{R}\right) \qquad p = \frac{1}{R}\left(-i\hbar\frac{\partial}{\partial\theta}\right)$$

$$= -i\hbar\frac{\partial}{\partial(R\theta)}$$

$$\int_0^{2\pi R} \psi'^*_{pos}(x)\,\psi_{pos}(x)\,dx \qquad = -i\hbar\frac{\partial}{\partial x}$$

$$= \int_0^{2\pi R} \frac{1}{R}\,\psi'_{ang}\left(\frac{x}{R}\right)^*\psi_{ang}\left(\frac{x}{R}\right)dx \qquad \frac{dx}{R} = d\theta$$

$$= \int_0^{2\pi} \psi'_{ang}(\theta)^*\,\psi_{ang}(\theta)$$

$$\therefore\ |p_m\rangle \to \frac{e^{im\frac{x}{R}}}{\sqrt{2\pi R}} = \frac{e^{i\frac{1}{\hbar}p_m x}}{\sqrt{L}}$$

$$p_m = \frac{\hbar m}{R}$$ } only discrete quantized momenta are allowed

$J_z\,|\psi\rangle \;\rightarrow\; -i\hbar \frac{\partial}{\partial\theta}\,\psi(\theta)$

$J_z\,|\psi_m\rangle = \hbar m\,|\psi_m\rangle \;\rightarrow\; -i\hbar\frac{\partial}{\partial\theta}\frac{e^{im\theta}}{\sqrt{2\pi}} = \hbar m\,\frac{e^{im\theta}}{\sqrt{2\pi}}$

Replace J_z & θ by p and x

$0 \quad 2\pi R = L$, x — unwrap the circle

$\psi_{pos}(x) = \sqrt{\frac{1}{R}}\,\psi_{ang}\left(\theta = \frac{x}{R}\right)$

$p = \frac{1}{R}\left(-i\hbar\frac{\partial}{\partial\theta}\right) = -i\hbar\frac{\partial}{\partial(R\theta)} = -i\hbar\frac{\partial}{\partial x}$

$\int_0^{2\pi R} \psi'_{pos}(x)^*\,\psi_{pos}(x)\,dx$

$= \int_0^{2\pi R} \frac{1}{R}\,\psi'_{ang}\left(\frac{x}{R}\right)^* \psi_{ang}\left(\frac{x}{R}\right) dx \qquad \frac{dx}{R} = d\theta$

$= \int_0^{2\pi} \psi'_{ang}(\theta)^*\,\psi_{ang}(\theta)$

$\therefore\; |p_m\rangle \rightarrow \frac{e^{im\frac{x}{R}}}{\sqrt{2\pi R}} = \frac{e^{i\frac{1}{\hbar}p_m x}}{\sqrt{L}}$

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We have now developed a quantum theory of a particle in a "periodic box" $\quad 0 \le x \le L$

- states described by wave functions

$|\psi\rangle \rightarrow \psi(x) \qquad \langle\psi'|\psi\rangle = \int_0^L \psi'^*(x)\,\psi(x)\,dx$

$\psi(0) = \psi(L), \quad \frac{d\psi}{dx}(0) = \frac{d\psi}{dx}(L)$

$p_x = -i\hbar\frac{\partial}{\partial x} \qquad |p_m\rangle \rightarrow \frac{e^{ip_m x/\hbar}}{\sqrt{L}}$

$p_m = \frac{2\pi m\hbar}{L} \qquad P\,|p_m\rangle = p_m\,|p_m\rangle$

$|\langle p_m|\psi\rangle|^2$ = prob of finding momentum $p = p_m$

$|\psi(x)|^2\Delta x$ = prob of finding position between x & $x+\Delta x$

To complete this development we must take $L \rightarrow \infty$

Use $-\frac{L}{2} \le x \le \frac{L}{2}$ so boundary disappears

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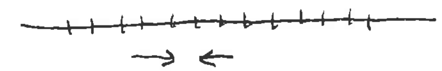
problem with

$$\tilde{\psi}_m = \int_{-L}^{L} \frac{e^{-i p_m x/\hbar}}{\sqrt{L}}\,\psi(x)\,dx \xrightarrow[L\to\infty]{} 0$$

& $p_m = \frac{2\pi\hbar}{L}$ become increasingly dense

allowed p_m



$\frac{2\pi\hbar}{L}$

Probability of finding any one value of $p_m \sim \frac{1}{L} \to 0$ but number in an interval between p & $p + \Delta p$ $\sim \Delta p / \left(\frac{2\pi\hbar}{L}\right) \sim L \to \infty$

Probability of finding momenta between p & $p + \Delta p$

$$= \sum_{\substack{m \\ p \le p_m \le p+\Delta p}} |\tilde{\psi}_m|^2 \approx |\tilde{\psi}_m|^2 \Delta p \frac{L}{2\pi\hbar}$$

$$= \frac{\Delta p\, L}{2\pi\hbar} \left| \int_{-L/2}^{L/2} \frac{e^{-i p_m x/\hbar}}{\sqrt{L}}\,\psi(x)\,dx \right|^2$$

$$= \Delta p\, |\tilde{\psi}(p)|^2 \quad \text{if} \quad \underbrace{\tilde{\psi}(p)}_{\sqrt{\frac{L}{2\pi\hbar}}\,\tilde{\psi}_m} = \int_{-\infty}^{\infty} \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\,\psi(x)\,dx$$

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$\rightarrow \leftarrow$ $\frac{2\pi\hbar}{L}$

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Probability of finding momenta between p & $p+\Delta p$

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$$= \Delta p\, |\tilde{\psi}(p)|^2 \quad \text{if} \quad \tilde{\psi}(p) = \int_{-\infty}^{\infty} \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \psi(x)\, dx$$

$\sqrt{\frac{L}{2\pi\hbar}}\, \tilde{\psi}_m$

The reverse also works

$$\psi(x) = \sum_{m=-\infty}^{\infty} \frac{e^{ip_m x/\hbar}}{\sqrt{L}} \tilde{\psi}_m$$

$\tilde{\psi}(p)\sqrt{\frac{2\pi\hbar}{L}}$

$$= \underbrace{\sum_{m=-\infty}^{\infty} \underbrace{\frac{2\pi\hbar}{L}}_{\Delta p}}_{\int_{-\infty}^{\infty} dp} \frac{e^{ip_m x/\hbar}}{\sqrt{2\pi\hbar}} \tilde{\psi}(p)$$

Riemann sum

Thus,

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} dx\, \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \psi(x)$$

$$\psi(x) = \int_{-\infty}^{\infty} dp\, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \tilde{\psi}(p)$$

Theory of Fourier transforms

$x_{op}|\psi\rangle \rightarrow x\,\psi(x)$ $\qquad p_{op}|\psi\rangle = -i\hbar \frac{\partial}{\partial x}\psi(x)$

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{\psi}(p)|^2 dp = 1$$

Note

$$\langle x \rangle = \langle \psi | x_p | \psi \rangle = \int_{-\infty}^{\infty} x\, |\psi(x)|^2 dx$$