The connection between these two descritions can be neatly described if we view

$$|\psi\rangle = \sum_{m=-\infty}^{\infty} \tilde{\psi}_m |m\rangle$$ as an abstract Hilbert space expression

and the association

$$|\psi\rangle \to \psi(\theta) \quad \& \quad |m\rangle \to \frac{e^{im\theta}}{\sqrt{2\pi}}$$

as a specific realization of that abstract description using a complex vectorspace of functions

$$\langle\psi'|\psi\rangle \to \int_0^{2\pi} \psi'^*(\theta)\, \psi(\theta)\, d\theta$$

$$\langle m'|m\rangle \to \int_0^{2\pi} \frac{e^{-im'\theta}}{\sqrt{2\pi}} \frac{e^{+im\theta}}{\sqrt{2\pi}}\, d\theta = \delta_{mm'}$$

$$|\psi\rangle = \sum_m \tilde{\psi}_m |m\rangle \to \psi(\theta) = \sum_m \tilde{\psi}_m \frac{e^{im\theta}}{\sqrt{2\pi}}$$

$$\tilde{\psi}_m = \langle m|\psi\rangle \to \tilde{\psi}_m = \int_0^{2\pi} d\theta\, \frac{e^{-im\theta}}{\sqrt{2\pi}} \psi(\theta)$$

$$J_z |\psi\rangle \to -i\hbar \frac{\partial}{\partial\theta} \psi(\theta)$$

$$J_z|\psi_m\rangle = \hbar m|\psi_m\rangle \to -i\hbar \frac{\partial}{\partial\theta} \frac{e^{im\theta}}{\sqrt{2\pi}} = \hbar m \frac{e^{im\theta}}{\sqrt{2\pi}}$$

Replace J_z & θ by p and x

unwrap the circle

$$\psi_{pos}(x) = \sqrt{\frac{1}{R}}\, \psi_{ang}\left(\theta = \frac{x}{R}\right) \qquad p = \frac{1}{R}\left(-i\hbar \frac{\partial}{\partial\theta}\right) = -i\hbar \frac{\partial}{\partial(R\theta)} = -i\hbar \frac{\partial}{\partial x}$$

$$\int_0^{2\pi R} \psi'_{pos}(x)^*\, \psi_{pos}(x)\, dx$$

$$= \int_0^{2\pi R} \frac{1}{R} \psi'_{ang}\left(\frac{x}{R}\right)^* \psi_{ang}\left(\frac{x}{R}\right) dx \qquad \frac{dx}{R} = d\theta$$

$$= \int_0^{2\pi} \psi'_{ang}(\theta)^*\, \psi_{ang}(\theta)$$

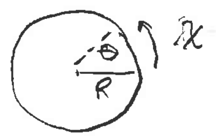
$$\therefore \quad |p_m\rangle \to \frac{e^{im\frac{x}{R}}}{\sqrt{2\pi R}} = \frac{e^{i\frac{1}{\hbar}p_m x}}{\sqrt{L}}$$

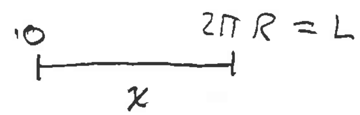
$$p_m = \frac{\hbar m}{R}$$ } only discrete quantized momenta are allowed

$J_z\,|\psi\rangle \rightarrow -i\hbar\frac{\partial}{\partial\theta}\psi(\theta)$

$J_z|\psi_m\rangle = \hbar m|\psi_m\rangle \rightarrow -i\hbar\frac{\partial}{\partial\theta}\frac{e^{im\theta}}{\sqrt{2\pi}} = \hbar m\frac{e^{im\theta}}{\sqrt{2\pi}}$

Replace J_z & θ by p and x





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$\psi_{pos}(x) = \sqrt{\frac{1}{R}}\,\psi_{ang}\left(\theta = \frac{x}{R}\right)$

$p = \frac{1}{R}\left(-i\hbar\frac{\partial}{\partial\theta}\right)$
$= -i\hbar\frac{\partial}{\partial(R\theta)}$
$= -i\hbar\frac{\partial}{\partial x}$

$\int_0^{2\pi R}\psi'_{pos}(x)^*\,\psi_{pos}(x)\,dx$

$= \int_0^{2\pi R}\frac{1}{R}\,\psi'_{ang}\left(\frac{x}{R}\right)^*\psi_{ang}\left(\frac{x}{R}\right)dx \qquad \frac{dx}{R} = d\theta$

$= \int_0^{2\pi}\psi'_{ang}(\theta)^*\,\psi_{ang}(\theta)$

$\therefore\ |p_m\rangle \rightarrow \frac{e^{im\frac{x}{R}}}{\sqrt{2\pi R}} = \frac{e^{i\frac{1}{\hbar}p_m x}}{\sqrt{L}}$

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We have now developed a quantum theory of a particle in a "periodic box" $0 \le x \le L$

- states described by wave functions

$|\psi\rangle \rightarrow \psi(x) \qquad \langle\psi'|\psi\rangle = \int_0^L \psi'^*(x)\,\psi(x)\,dx$

$\psi(0) = \psi(L), \quad \frac{d\psi}{dx}(0) = \frac{d\psi}{dx}(L)$

$p_x = -i\hbar\frac{\partial}{\partial x} \qquad |p_m\rangle \rightarrow \frac{e^{ip_m x/\hbar}}{\sqrt{L}}$

$p_m = \frac{2\pi m\hbar}{L} \qquad P|p_m\rangle = p_m|p_m\rangle$

$|\langle p_m|\psi\rangle|^2$ = prob of finding momentum $p = p_m$

$|\psi(x)|^2\Delta x$ = prob of finding position between x & $x+\Delta x$

To complete this development we must take $L \rightarrow \infty$

Use $-\frac{L}{2} \le x \le \frac{L}{2}$ so boundary disappears

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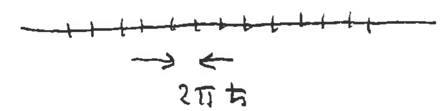
problem with

$$\tilde{\psi}_m = \int_{-L}^{L} \frac{e^{-ip_m x/\hbar}}{\sqrt{L}}\,\psi(x)\,dx \xrightarrow[L\to\infty]{} 0$$

& $p_m = \frac{2\pi\hbar}{L}$ become increasingly dense

allowed p_m



Probability of finding any one value of $p_m \sim \frac{1}{L} \to 0$ but number in an interval between p & $p + \Delta p$ $\sim \Delta p / \left(\frac{2\pi\hbar}{L}\right) \sim L \to \infty$

Probability of finding momenta between p & $p + \Delta p$

$$= \sum_{\substack{m \\ p \le p_m \le p+\Delta p}} |\tilde{\psi}_m|^2 \approx |\tilde{\psi}_m|^2 \,\Delta p \frac{L}{2\pi\hbar}$$

$$= \frac{\Delta p\, L}{2\pi\hbar} \left| \int_{-L/2}^{L/2} \frac{e^{-ip_m x/\hbar}}{\sqrt{L}}\,\psi(x)\,dx \right|^2$$

$$= \Delta p\, |\tilde{\psi}(p)|^2 \quad \text{if} \quad \tilde{\psi}(p) = \int_{-\infty}^{\infty} \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\,\psi(x)\,dx$$

$$\sqrt{\frac{L}{2\pi\hbar}}\,\tilde{\psi}_m$$

problem with

$$\tilde{\psi}_m = \int_{-L}^{L} \frac{e^{-ip_m x/\hbar}}{\sqrt{L}} \psi(x)\, dx \underset{L\to\infty}{\longrightarrow} 0$$

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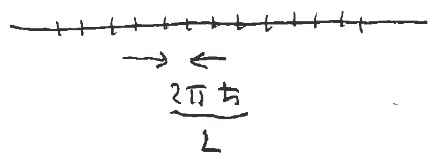

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$\sqrt{\frac{L}{2\pi\hbar}}\, \tilde{\psi}_m$

The reverse also works

$$\psi(x) = \sum_{m=-\infty}^{\infty} \frac{e^{ip_m x/\hbar}}{\sqrt{L}} \tilde{\psi}_m$$

$\tilde{\psi}(p) \sqrt{\frac{2\pi\hbar}{L}}$

$$= \underbrace{\sum_{m=-\infty}^{\infty} \underbrace{\frac{2\pi\hbar}{L}}_{\Delta p}}_{\int_{-\infty}^{\infty} dp} \frac{e^{ip_m x/\hbar}}{\sqrt{2\pi\hbar}} \tilde{\psi}(p)$$

Riemann sum

Thus,

$$\tilde{\psi}(p) = \int_{-\infty}^{\infty} dx\, \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}} \psi(x)$$

$$\psi(x) = \int_{-\infty}^{\infty} dp\, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}} \tilde{\psi}(p)$$

Theory of Fourier transforms

$x_{op}|\psi\rangle \to x\,\psi(x)$ $\quad p_{op}|\psi\rangle = -i\hbar \frac{\partial}{\partial x}\psi(x)$

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{\psi}(p)|^2 dp = 1$$

Note

$$\langle x \rangle = \langle \psi | x_p | \psi \rangle = \int_{-\infty}^{\infty} x\, |\psi(x)|^2 dx$$

The reverse also works

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Thus,

$$\left.\begin{aligned} \tilde{\psi}(p) &= \int_{-\infty}^{\infty} dx\, \frac{e^{-ipx/\hbar}}{\sqrt{2\pi\hbar}}\, \psi(x) \\ \psi(x) &= \int_{-\infty}^{\infty} dp\, \frac{e^{ipx/\hbar}}{\sqrt{2\pi\hbar}}\, \tilde{\psi}(p) \end{aligned}\right\} \text{Theory of Fourier transforms}$$

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2. Uncertainty principle:

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}$$

Average values of x & p are easy to express for a state $|\psi\rangle$

$$\langle x \rangle = \langle \psi | x^{op} | \psi \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 x\, dx \equiv \bar{x}$$

$$\langle p \rangle = \langle \psi | p^{op} | \psi \rangle = \int_{-\infty}^{\infty} \psi^*(x)\left(-i\hbar \frac{d}{dx}\right)\psi(x)\, dx \equiv \bar{p}$$

$$= \int_{-\infty}^{\infty} |\tilde{\psi}(p)|^2 p\, dp$$

What about the fluctuation of results for x about $\langle x \rangle$? Use

$$(\Delta x)^2 \equiv \langle \psi | (x^{op} - \bar{x})^2 | \psi \rangle = \int_{-\infty}^{\infty} (x - \bar{x})^2 |\psi(x)|^2\, dx$$

&

$$(\Delta p)^2 \equiv \langle \psi | (p^{op} - \bar{p})^2 | \psi \rangle = \int_{-\infty}^{\infty} (p - \bar{p})^2 |\tilde{\psi}(p)|^2\, dp$$

need square so quantity does not average to zero!

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Since since results for x and p are both determined by the same wave function they are NOT independent. Heisenberg's uncertainty relation $\Delta x \, \Delta p \geq \frac{\hbar}{2}$ states this and can be prove now that Δx & Δp are defined. critical property is

$$[x^{op}, p^{op}] = x^{op} p^{op} - p^{op} x^{op}$$
$$= x\left(-i\hbar \frac{\partial}{\partial x}\right) - \left(-i\hbar \frac{\partial}{\partial x}\right) x$$
$$= i\hbar$$

measurements of x & p interfere with each other as did J_y & J_z

Define $X^{op} = x^{op} - \bar{x}$, $P^{op} = p^{op} - \bar{p}$ and examine

$$|\Phi\rangle = (X^{op} - i\alpha P^{op}) |\psi\rangle$$

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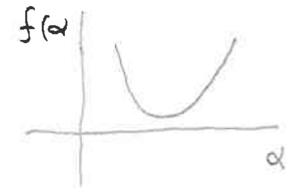
Work out

$$0 \leq \langle \Phi | \Phi \rangle = \left([X^{op} - i\alpha P^{op}]|\psi\rangle, [X^{op} - i\alpha P^{op}]|\psi\rangle\right)$$

$$0 \leq \left(|\psi\rangle, [X^{op} + i\alpha P^{op}][X^{op} - i\alpha P^{op}]|\psi\rangle\right)$$

$$0 \leq \langle \psi | \{(X^{op})^2 - i\alpha [X^{op}, P^{op}] + \alpha^2 (P^{op})^2\} | \psi \rangle$$

$$0 \leq (\Delta x)^2 + \alpha\hbar + \alpha^2 (\Delta p)^2 = f(\alpha)$$



No real roots

$\alpha = (-\hbar \pm \sqrt{\hbar^2 - 4\Delta x \Delta p})/2(\Delta p)^2$

then $b^2 \leq 4ac$

$\hbar^2 \leq 4(\Delta x)^2(\Delta y)^2$

or $\Delta x \Delta p \geq \frac{\hbar}{2}$

It is impossible to prepare a state where a subsequent measurement of momentum or instead position will each have uncertainties with

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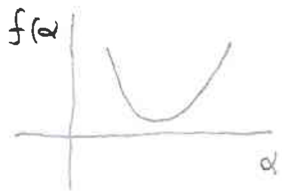

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Note $\Delta x \Delta p = 0$ is possible

$\langle \Psi | \Psi \rangle = 0$ or $(X^{op} - i\alpha P^{op})|\psi\rangle = 0$

assume $\bar{x} = \bar{p} = 0$, then

$$\left\{ x - i\alpha\left(-i\hbar \frac{d}{dx}\right) \right\} \psi(x) = 0$$

$$\frac{d}{dx}\psi(x) = \frac{\alpha}{\hbar} x \psi(x)$$

and $\psi(x) = e^{\frac{1}{2}\frac{\alpha}{\hbar}x^2}$ choose $\alpha < 0$

a Gaussian wave function and an instructive example:

$\psi(x) = N e^{-\frac{x^2}{4D^2}}$ find N

$$1 = \int_{-\infty}^{\infty} N^2 e^{-\frac{x^2}{2D^2}} dx = N^2 D\sqrt{2} \underbrace{\int_{-\infty}^{\infty} e^{-y^2} dy}_{\sqrt{\pi}}$$

$\frac{x}{\sqrt{2}D}$

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$$\Delta x^2 = \int x^2 |\psi(x)|^2 dx$$

$$= \frac{1}{D\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2D^2}} dx = \frac{2D^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2} dy \qquad \frac{x}{\sqrt{2}D}$$

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$$= D^2$$

Finally evaluate

$$\tilde{\psi}(p) = \frac{1}{\sqrt{D\sqrt{2\pi}}} \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} e^{-\frac{x^2}{4D^2}} dx$$

$$= \frac{1}{\sqrt{D\sqrt{2\pi}}} \frac{1}{\sqrt{2\pi\hbar}} \underbrace{\int_{-\infty}^{\infty} e^{-(x + 2ipD^2/\hbar)^2 \frac{1}{4D^2}} dx}_{2D\sqrt{\pi} = \sqrt{2}D\sqrt{2\pi}} \; e^{-\frac{p^2 D^2}{\hbar^2}}$$

$$= \frac{1}{\sqrt{\underbrace{\frac{\hbar}{2D}}_{\Delta p^2}\sqrt{\pi}}} e^{-\frac{p^2}{(\hbar/D)^2}} \qquad 4\Delta p^2 \qquad \Delta p = \frac{1}{2}\frac{\hbar}{D}$$

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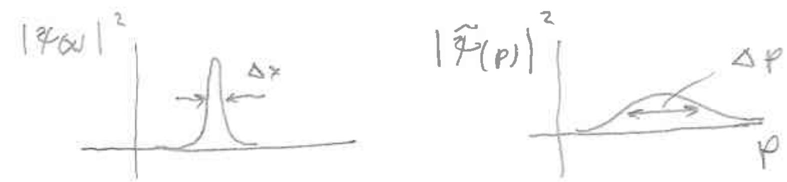
$$\Delta x \, \Delta p = \frac{\hbar}{2}$$

Perfect example of uncertainty principle:

$$\psi(x) = \frac{1}{\sqrt{\Delta x \sqrt{2\pi}}} e^{-\frac{x^2}{4\Delta x^2}} \qquad \tilde{\psi}(p) = \frac{1}{\sqrt{\Delta p\sqrt{2\pi}}} e^{-\frac{p^2}{4\Delta p^2}}$$

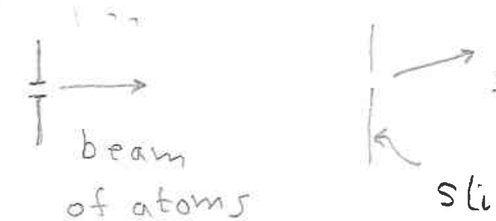


$$\Delta x \Delta p = \frac{\hbar}{2}$$

Gedanken experiments

(does $\Delta x \Delta p \geq \frac{\hbar}{2}$ make sense?)

(a)



Measure momentum transferred to slit to predict momentum of atom after it passes through the slit.

Determine Δx & Δp to high precision!?

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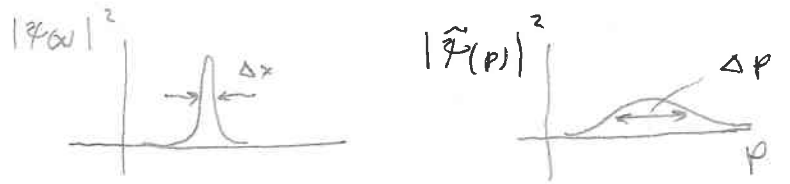

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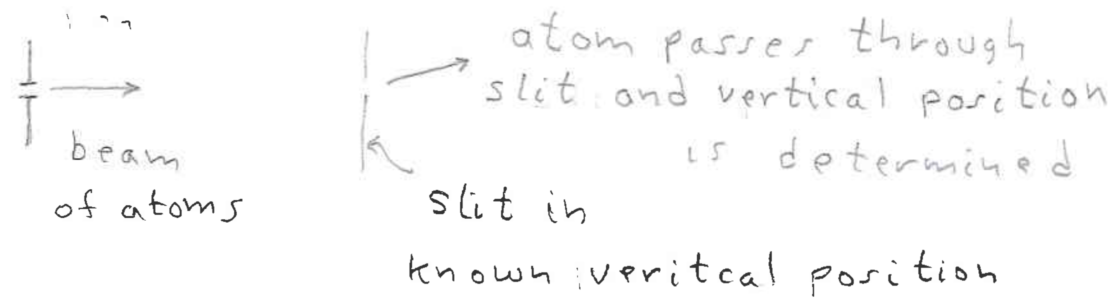

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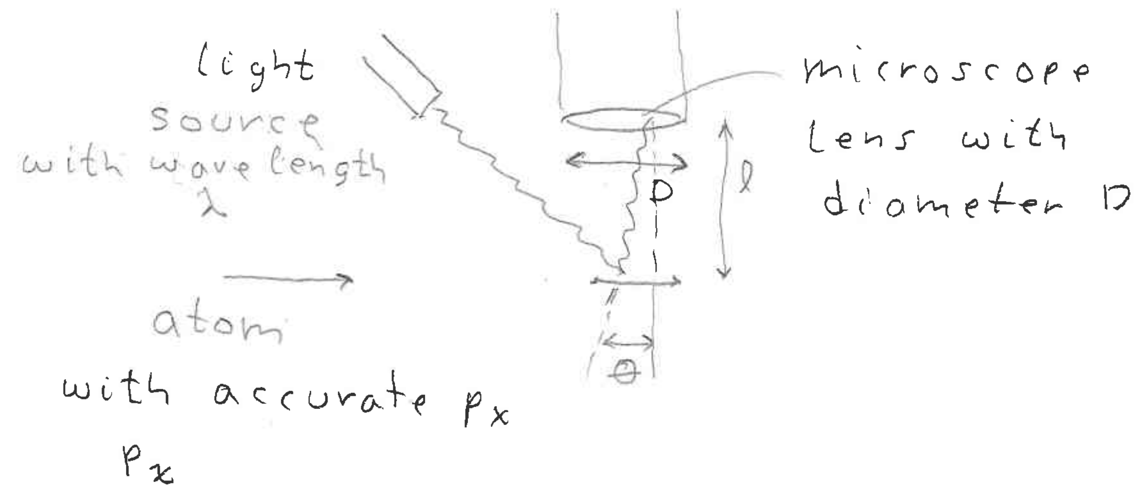
Not correct If slit also obeys uncertainty princple and we know it position accurately (small Δx_{slit}) the its initial momentum p_x^{slit} is not accurate,

$$\Delta p_x^{slit} \geq \frac{\hbar}{2\Delta x_{slit}}$$

so accurate measurement of final momentum of slit does not determime momentum tranrferred to atom

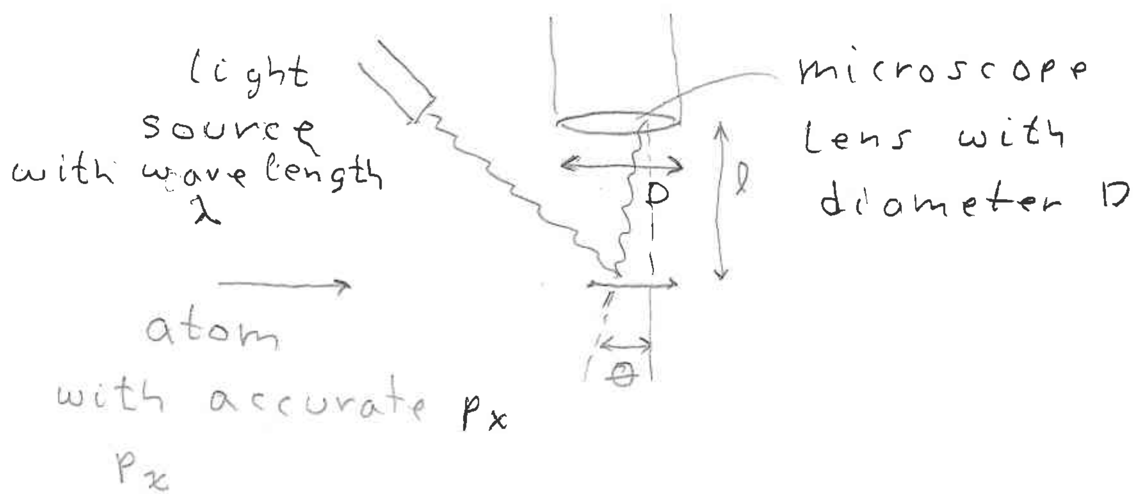
(b) Heisenberg's microscope



Not correct   If slit also obeys uncertainty princple and we know it position accurately (small Δx_{slit}) the its initial momentum p_x^{slit} is not accurate,

$$\Delta p_x^{slit} \geq \frac{\hbar}{2\Delta x_{slit}}$$ so accurate measurement of final momentum of slit does not determime momentum tranrferred to atom

(b) Heisenberg's microscope



- microscope measures angular position of atom with resolving power $\Delta\theta = \frac{\lambda}{D}$ (Rayleigh criterion)

  $\therefore \Delta x = \Delta\theta\, l = \frac{\lambda l}{D}$

- Use weak pulse of light with energy E & momentum E/c
- Direction of scattered light is uncertain by $\Delta\phi = \frac{D}{l}$
- Therefore momentum transferred to atom when light scatters into microscope is also uncertain

  $\Delta p_x = E/c \times \Delta\phi = \frac{E}{c} \cdot \frac{D}{l}$

- $\frac{\hbar}{2} \leq \Delta x\, \Delta p = \frac{\lambda l}{D} \cdot \frac{E}{c}\frac{D}{l} = \frac{\lambda}{c} E$

energy of light beam must obey

$E \geq \hbar \frac{c}{\lambda} = \hbar\nu$ Plank's photon's

must quantize light to for consistency