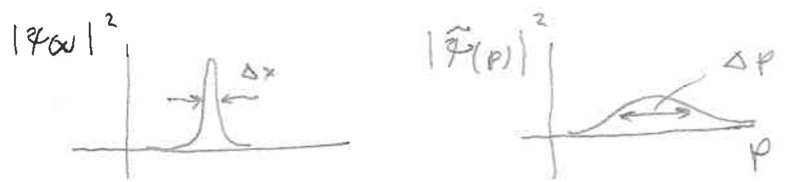
04/06/2021

(367)

Perfect example of uncertainty principle:

$$\psi(x) = \frac{1}{\sqrt{\Delta x \sqrt{2\pi}}} e^{-\frac{x^2}{4\Delta x^2}} \qquad \tilde{\psi}(p) = \frac{1}{\sqrt{\Delta p \sqrt{2\pi}}} e^{-\frac{p^2}{4\Delta p^2}}$$

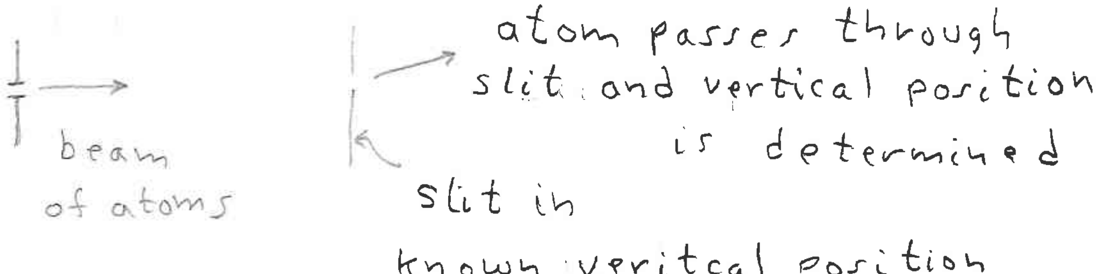


$$\Delta x \Delta p = \frac{\hbar}{2}$$

Gedanken experiments

(does $\Delta x \Delta p \geq \frac{\hbar}{2}$ make sense?)

(a)



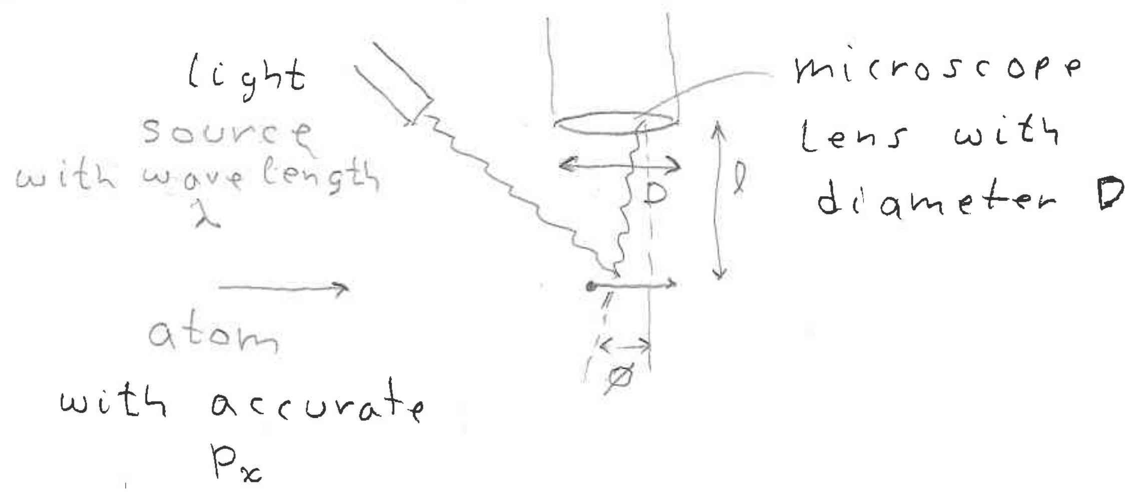
Measure momentum transferred to slit to predict momentum of atom after it passes through the slit.

Determine Δx & Δp to high precision!?

(368)

Not correct If slit also obeys uncertainty princple and we know its position accurately (small Δx_{slit}) then its initial momentum p_x^{slit} is not accurate, $\Delta p_x^{slit} \geq \frac{\hbar}{2\Delta x_{slit}}$ so accurate measurement of final momentum of slit does not determime momentum transferred to atom

(b) Heisenberg's microscope



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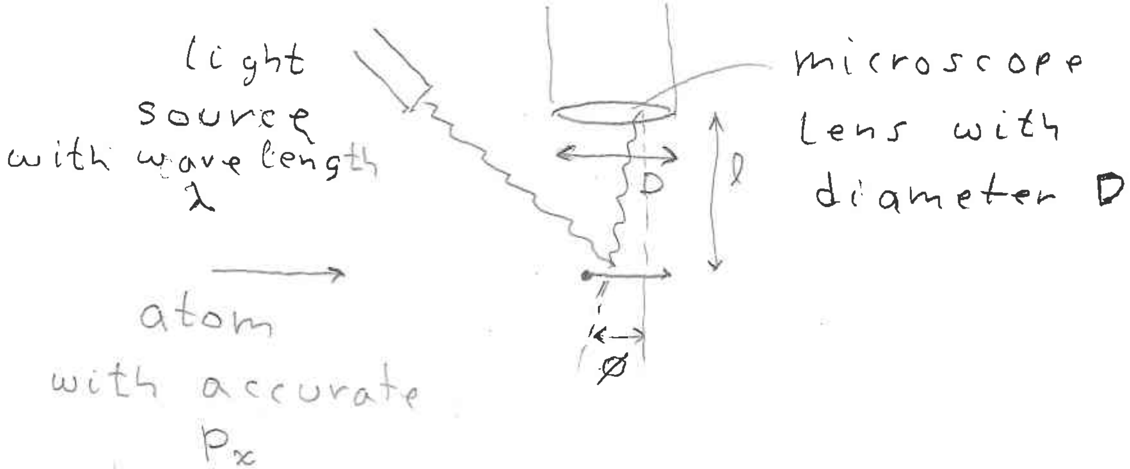

- microscope measures angular position of atom with resolving power $\Delta\theta = \frac{\lambda}{D}$ (Rayleigh criterion)

  $\therefore \Delta x = \Delta\theta\,\ell = \frac{\lambda \ell}{D}$

- Use weak pulse of light with energy E & momentum E/c
- Direction of scattered light is uncertain by $\Delta\phi = \frac{D}{\ell}$
- Therefore momentum transferred to atom when light scatters into microscope is also uncertain

  $\Delta p_x = E/c \times \Delta\phi = \frac{E}{c} \cdot \frac{D}{\ell}$

- $\frac{\hbar}{2} \leq \Delta x\, \Delta p_x = \frac{\lambda \ell}{D} \cdot \frac{E}{c}\frac{D}{\ell} = \frac{\lambda}{c} E$

energy of light beam must obey

$E \geq \hbar \frac{c}{\lambda} = \hbar\nu$ Plank's photon's!

Must quantize light too for consistency

- microscope measures angular position of atom with resolving power $\Delta\theta = \frac{\lambda}{D}$ (Rayleigh criterion)

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D. One dimensional Quantum Mechanics

1. Consider a quantum particle with mass m and position x moving in one dimension.

   Determine how a state $|\psi\rangle$ changes with time by solving

   $$i\hbar \frac{d}{dt} |\psi(t)\rangle = H_{op} |\psi(t)\rangle$$

   where the energy operator or Hamitonian is given by

   $$H_{op} = \frac{p_{op}^2}{2m} + V(x_{op}) \qquad (*)$$

   in analogy with $H = -\vec{\mu} \cdot \vec{B}$ for particle with magnentic moment $\vec{\mu} = \gamma \vec{J}$. $(*)$ most easily realized using wave functions

$|\psi\rangle \to \psi(x)$ $\quad x_{op}|\psi\rangle \to x\,\psi(x)$ $\quad p_{op}|\psi\rangle = -i\hbar \frac{d}{dx}\psi(x)$

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$|\psi\rangle \to \psi(x) \quad x_{op}|\psi\rangle \to x\,\psi(x) \quad P_{op}|\psi\rangle = -i\hbar\frac{d}{dx}\psi(x)$

so

$$H_{op}|\psi\rangle \to \left\{ \frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 + V(x) \right\}\psi(x)$$

We would like to use our exponential solution:

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \quad \& \quad |\psi(t=0)\rangle = |\psi(0)\rangle$$

solved by $|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$

But "How can we compute ?"

This is easy if $|\psi(0)\rangle = |E_n\rangle$, an eigenstate of H:

$$H|E_n\rangle = E_n|E_n\rangle$$

$$\& \quad |E_n(t)\rangle = e^{-i\frac{1}{\hbar}Ht}|E_n\rangle = e^{-i\frac{1}{\hbar}E_nt}|E_n\rangle$$

If $|\psi(0)\rangle = \sum_n c_n|E_n\rangle$, $|\psi(t)\rangle = \sum_n c_n e^{-i\frac{E_nt}{\hbar}}|E_n\rangle$ a general solution.

so

$$H_{op}|\psi\rangle \rightarrow \left\{ \frac{1}{2m}\left(-i\hbar \frac{d}{dx}\right)^2 + V(x) \right\} \psi(x)$$

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$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \left[\frac{P_{op}^2}{2m} + V(x_{op})\right]|\psi(t)\rangle$$

using solutions of the time-independent Schrödinger equation

$$\left[\frac{P_{op}^2}{2m} + V(x_{op})\right]|E_n\rangle = E_n|\psi_n\rangle$$

or

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} + V(x)\psi_n(x) = E_n\psi_n(x)$$

where $|E_n\rangle \rightarrow \psi_n(x)$ is an energy eigenstate of H with eigenvalue E_n

This is easy for this week's homework where $V(x) = 0$ so eigenstates of $H = \frac{P_{op}^2}{2m}$ come from eigenstates of P_{op}

$$|P_m\rangle \rightarrow \frac{e^{2\pi i m x/L}}{\sqrt{L}}$$

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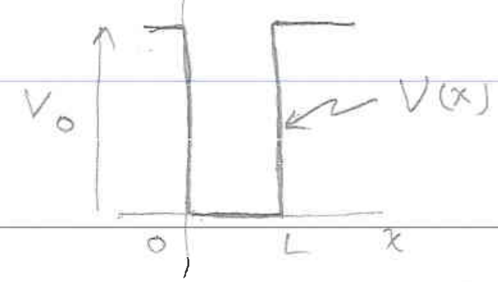
2. Work out 2 examples

a) $V(x)$ very large except when $0 \le x \le L$



Assume V_0 is so large that $\psi(0) = \psi(L) = 0$

solve $\frac{1}{2m}(-\hbar^2)\frac{d^2}{dx^2}\psi_n(x) = E_n\psi_n(x)$

while requiring $\psi(0) = \psi(L) = 0$

Just our old SHO equation with the general solution

$$\psi(x) = A\sin(kx+\phi)$$

with $\frac{1}{2m}\hbar^2k^2 = E$ or $k = \frac{1}{\hbar}\sqrt{2mE}$

$\psi(0) = 0 \Rightarrow A\sin\phi = 0 \Rightarrow \phi = 0$

$\psi(L) = 0 \Rightarrow A\sin kL = 0 \Rightarrow k_n = \frac{2\pi}{L}n$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{1}{2m}\Big(\underbrace{\frac{2\pi n}{L}\hbar}_{P_n}\Big)^2$$

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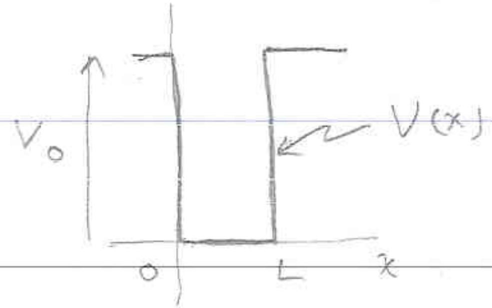

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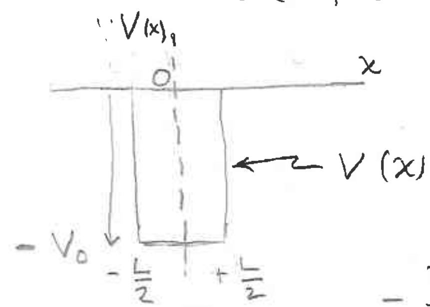
Momentum eigenstates don't work: $e^{\pm i p_n x/\hbar}$ -- the particle can't have only one sign of p_n but

$$\psi_n = \sqrt{\frac{2}{L}} \sin\left(\frac{p_n x}{\hbar}\right) = \sqrt{\frac{2}{L}} \frac{e^{+ip_n x/\hbar} - e^{-ip_n x/\hbar}}{2i}$$

with $p_n = \hbar k_n = \frac{2\pi n \hbar}{L}$ works!

b) Finite potential well



Find "bound" states with $E_n < 0$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E_n \psi(x)$$

$x < -\frac{L}{2}$ or $+\frac{L}{2} < x$

$$\frac{d^2\psi}{dx^2} = \underbrace{-E_n \frac{2m}{\hbar^2}}_{\kappa^2 > 0} \psi$$

$-\frac{L}{2} \le x \le \frac{L}{2}$

$$-\frac{d^2\psi}{dx^2} = \underbrace{(E_n + V_0)\frac{2m}{\hbar^2}}_{k^2 > 0} \psi$$

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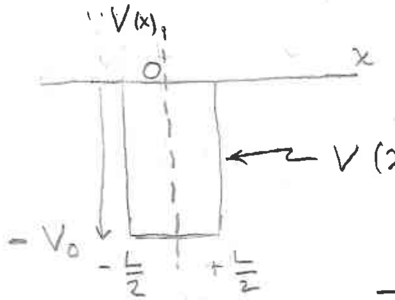

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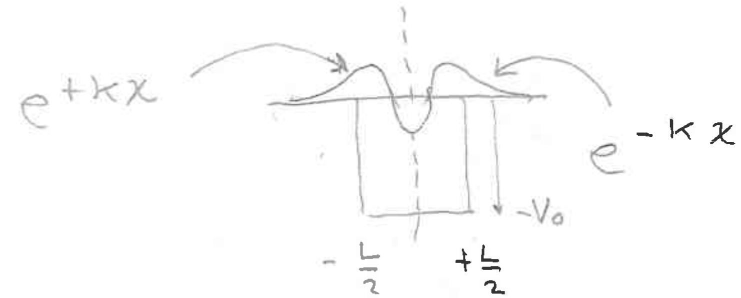


Can separate even & odd solutions

$$\psi(-x) = \pm\psi(+x)$$

Find even solutions

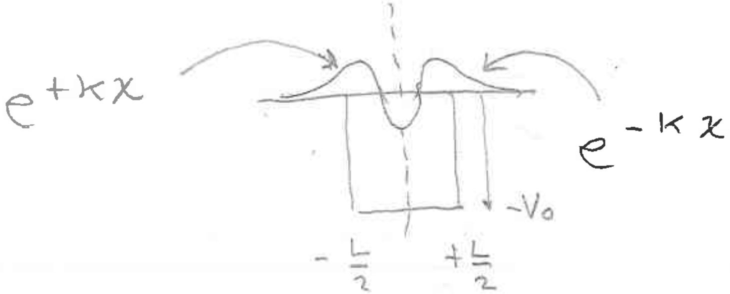
$$\psi(x) = \begin{cases} A\cos kx & x < \frac{L}{2} \\ B e^{-\kappa x} & \frac{L}{2} < x \end{cases}$$

require $\psi(x)$ and $\frac{d\psi}{dx}$ are continuous at $x = \frac{L}{2}$ so $\frac{d\psi}{dx}$ and $\frac{d^2\psi}{dx^2}$ are finite at $x = \frac{L}{2}$

$$\left.\begin{aligned} A\cos\frac{kL}{2} &= B e^{-\kappa L/2} \\ -kA\sin k\frac{L}{2} &= -\kappa B e^{-\kappa L/2} \end{aligned}\right\}\ \text{take ratio}$$

$$k\tan\frac{kL}{2} = \kappa$$

a trancendental equation, solve graphically



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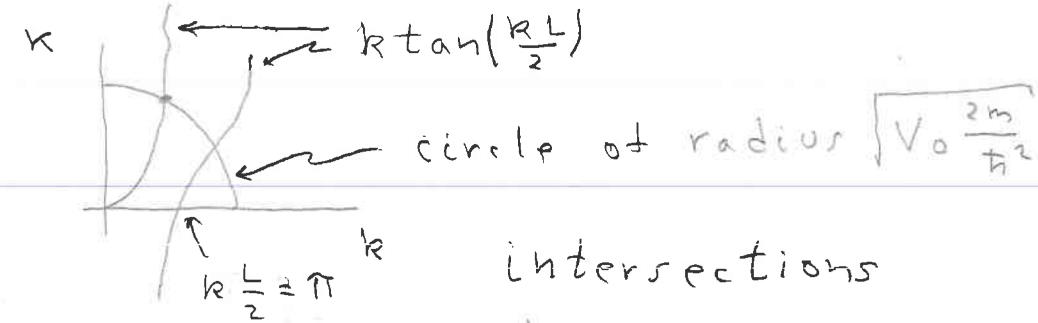
a trancendental equation, solve graphically

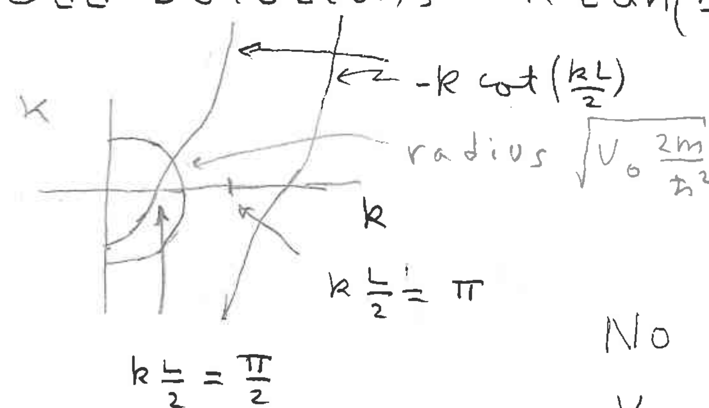
$$\kappa^2 = -E_n\frac{2m}{\hbar^2} \qquad k^2 = (V_0 + E_n)\frac{2m}{\hbar^2}$$

$$\therefore\ \kappa^2 + k^2 = V_0\frac{2m}{\hbar^2}$$



intersections determine allowed energies. At least one solution even as $V_0 \to 0$!

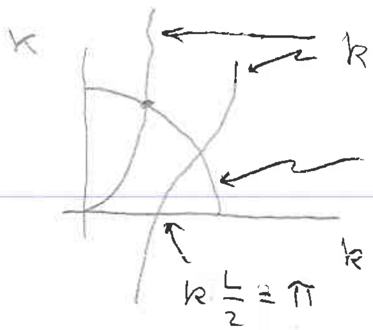
Odd solutions $\quad k\tan\left(\frac{kL}{2}\right) \to -k\cot\left(\frac{kL}{2}\right)$



No solution when V_0 becomes too small

$$K^2 = -E_n \frac{2m}{\hbar^2} \qquad k^2 = (V_0 + E_n)\frac{2m}{\hbar^2}$$

$$\therefore \quad K^2 + k^2 = V_0 \frac{2m}{\hbar^2}$$



$k\tan(\frac{kL}{2})$

circle of radius $\sqrt{V_0 \frac{2m}{\hbar^2}}$

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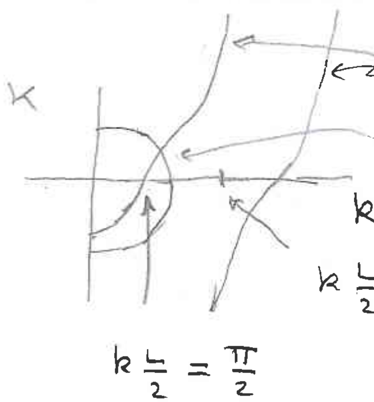

$-k\cot(\frac{kL}{2})$

radius $\sqrt{V_0 \frac{2m}{\hbar^2}}$

No solution when V_0 becomes too small

3. Ehrenfest relations

See a connection to classical physics: Examine

$$\bar{x}(t) = \langle \psi(t) | x_{op} | \psi(t) \rangle$$

$$\frac{d}{dt}\bar{x}(t) = \frac{d}{dt}\left(|\psi(t)\rangle, x_{op}|\psi(t)\rangle \right)$$

$$= \left(\frac{-i}{\hbar} H |\psi(t)\rangle, x_{op}|\psi(t)\rangle \right) + \left(|\psi(t)\rangle, x_{op}\frac{-i}{\hbar} H|\psi(t)\rangle \right)$$

$$= \frac{-i}{\hbar}\langle \psi(t) | x_{op} H - H x_{op} | \psi(t) \rangle$$

However, $\left[x_{op}, \frac{p_{op}^2}{2m} + V(x_{op}) \right] = i\hbar \frac{p_{op}}{m}$

since $x_{op} p_{op}^2 - p_{op}^2 x_o$

$$= \underbrace{(x_{op}p_{op} - p_{op}x_{op})}_{i\hbar} p_{op} + p_{op} \underbrace{(x_{op}p_{op} - p_{op}x_{op})}_{i\hbar}$$

$$\therefore \frac{d}{dt}\bar{x}(t) = \langle \psi(t) | \frac{p_{op}}{m} | \psi(t) \rangle$$

$$= \frac{\bar{p}(t)}{m}$$