3. Ehrenfest relations

see a connection to classical physics: Examine

$$\bar{x}(t) = \langle \psi(t) | x_{op} | \psi(t) \rangle$$

$$\frac{d}{dt}\bar{x}(t) = \frac{d}{dt}\left(|\psi(t)\rangle, x_{op} |\psi(t)\rangle \right)$$

$$= \left(\frac{-i}{\hbar} H |\psi(t)\rangle, x_{op} |\psi(t)\rangle \right)$$

$$+ \left(|\psi(t)\rangle, x_{op} \frac{-i}{\hbar} H |\psi(t)\rangle \right)$$

$$= \frac{-i}{\hbar} \langle \psi(t) | x_{op} H - H x_{op} | \psi(t) \rangle$$

However, $\left[x_{op}, \frac{p_{op}^2}{2m} + V(x_{op}) \right] = i\hbar \frac{p_{op}}{m}$

since $x_{op} p_{op}^2 - p_{op}^2 x_o$

$$= \underbrace{(x_{op} p_{op} - p_{op} x_{op})}_{i\hbar} p_{op} + p_{op} \underbrace{(x_{op} p_{op} - p_{op} x_{op})}_{i\hbar}$$

$$\therefore \frac{d}{dt}\bar{x}(t) = \langle \psi(t) | \frac{p_{op}}{m} | \psi(t) \rangle$$

$$= \frac{\bar{p}(t)}{m} \checkmark$$

Similarly

$$\frac{d}{dt}\bar{p}(t) = \frac{d}{dt}\left(|\psi(t)\rangle, p_{op} |\psi(t)\rangle \right)$$

$$= -\frac{i}{\hbar} \langle \psi(t) | [p_{op}, H] | \psi(t) \rangle$$

$$= -\frac{i}{\hbar} \langle \psi(t) | [p_{op}, V(x_{op})] | \psi(t) \rangle$$

Can be worked out if $V(x_{op})$ is a polynomial, e.g.

$$[p_{op}, x_{op}^3] = [p_{op}, x_{op}] x_{op}^2 + x_{op} [p_{op}, x_{op}] x_{op}^2$$

$$+ x_{op}^2 [p_{op}, x_{op}]$$

$$= -i\hbar\, 3 x_{op}^2$$

& in general $[p_{op}, V(x_{op})] = -i\hbar \frac{d}{dx} V(x_{op})$

and

$$\frac{d}{dt}\bar{p}(t) = \langle \psi(t) | \frac{d}{dx} V(x_{op}) | \psi(t) \rangle$$

$$\overset{?}{\approx} -\frac{dV}{dx}\left(\langle \psi(t) | V(x_{op}) | \psi(t) \rangle \right) = -\frac{dV}{dx}(\bar{x}(t))$$

must neglect fluctuation about $\bar{x}$

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$$\frac{d}{dt}\bar{p}(t) = \frac{d}{dt}\Big(|\psi(t)\rangle, p_{op}|\psi(t)\rangle \Big)$$

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$$[p_{op}, x^3_{op}] = [p_{op}, x_{op}]\,x^2_{op} + x_{op}[p_{op}, x_{op}]\,x_{op} + x^2_{op}[p_{op}, x_{op}]$$

$$= -i\hbar\, 3x^2_{op}$$

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and

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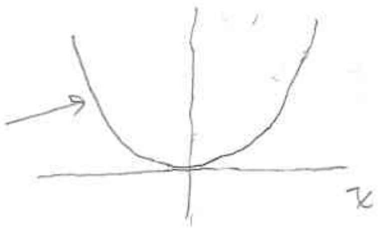
must neglect fluctuation about $\bar{x}$

4. Simple harmonic oscillator

$$H|\psi_n\rangle = \left[\frac{p^2_{op}}{2m} + \frac{1}{2}m x^2_{op}\right]|\psi_n\rangle = E_n|\psi_n\rangle$$

find E_n and $|\psi_n\rangle$



Easy to solve using raising and lowering operators.

Define a special operator

$$a = \sqrt{\frac{m\omega}{2\hbar}}\,x_{op} + i\frac{1}{\sqrt{2m\omega\hbar}}\,p_{op}$$

[classically:

$$a(t) = \sqrt{\frac{m\omega}{2\hbar}}\,A\cos\omega t + i\frac{1}{\sqrt{2m\omega\hbar}}\,m(-\omega\sin\omega t)$$

$$= \sqrt{\frac{m\omega}{2\hbar}}\,A\,e^{-i\omega t}$$]

Show $[H, a] = -\hbar\omega\, a$

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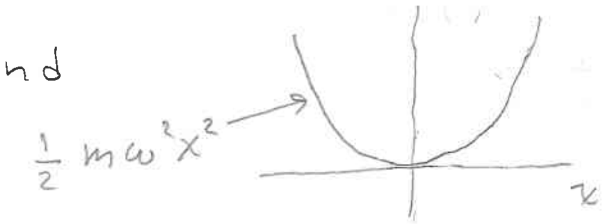

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show $[H, a] = -\hbar\omega\, a$

$$\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2,\ \sqrt{\frac{m\omega}{2\hbar}}\,x + i\frac{1}{\sqrt{2m\omega\hbar}}\,p\right]$$

$$= \frac{1}{2}\sqrt{\frac{\omega}{2m\hbar}}\,\underbrace{[p^2, x]}_{-i\hbar 2p} + i\frac{1}{2}\sqrt{\frac{m\omega}{2\hbar}}\,\omega\,\underbrace{[x^2, p]}_{+i\hbar 2x}$$

$$= \hbar\omega\frac{1}{\sqrt{2m\hbar\omega}}(-ip) - \hbar\omega\sqrt{\frac{m\omega}{2\hbar}}\,x$$

$$= -\hbar\omega\, a$$

This gives a an important property:

if $H|E_n\rangle = E_n|E_n\rangle$

then $H\,a|E_n\rangle = \{\underbrace{[H, a]}_{-\hbar\omega a} + a\underbrace{H}_{E_n}\}|E_n\rangle$

$$= (E_n - \hbar\omega)\, a\,|\psi_n\rangle$$

Thus $a|E_n\rangle$ is a new eigenvector of H with energy $E_n - \hbar\omega$

$$\left[\frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2,\ \sqrt{\frac{m\omega}{2\hbar}}\,x + i\frac{1}{\sqrt{2m\omega\hbar}}\,p\right]$$

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This implies

$$H\,a^n|E_n\rangle = (E_n - n\hbar\omega)|E_n\rangle$$

But $E_n = \langle E_n|\,\frac{p_{op}^2}{2m} + \frac{1}{2}m\omega^2x_{op}^2\,|E_n\rangle$

$$= \frac{1}{2m}\|\,p_{op}|E_n\rangle\|^2 + \frac{1}{2}m\omega^2\|x_{op}|E_n\rangle\|^2 \geq 0$$

Therefore, before $E_n - n\hbar\omega < 0$ we must have $a^n|E_n\rangle = 0$.

If n_0 is the smallest integer with $a^{n_0+1}|E_n\rangle = 0$, define

$$|0\rangle = N\,a^{n_0}|E_n\rangle$$

$|0\rangle$ obeys:

$\langle 0|0\rangle = 1$ (our choice for N)

$$\underbrace{a|0\rangle = 0}_{\text{we can solve this equation!}}$$

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we can solve this equation!

$$\left\{ \sqrt{\frac{m\omega}{2\hbar}}\, x + i \frac{1}{\sqrt{2m\omega\hbar}} \left(-i\hbar \frac{d}{dx}\right) \right\} \psi_0(x) = 0$$

$$\frac{d}{dx}\psi_0(x) = -\frac{m\omega}{\hbar} x\, \psi_0(x)$$

$$\Rightarrow \quad \psi_0(x) = N' e^{-\frac{m\omega}{2\hbar}x^2}$$

$$1 = N'^2 \int_{-\infty}^{\infty} e^{-\frac{m\omega}{\hbar}x^2} dx = N'^2 \sqrt{\frac{\hbar}{m\omega}}\sqrt{\pi}$$

$$\Rightarrow \quad N' = \sqrt[4]{\frac{m\omega}{\hbar\pi}}$$

$H|0\rangle = E_0|0\rangle$ What is E_0 ?

note $a^\dagger a = \left(\sqrt{\frac{m\omega}{2\hbar}}\, x - i\frac{1}{\sqrt{2m\omega\hbar}}\, p\right)\left(\sqrt{\frac{m\omega}{2\hbar}}\, x + i\frac{1}{\sqrt{2m\omega\hbar}}\, p\right)$

$$= \underbrace{\frac{m\omega}{2\hbar}x^2 + \frac{1}{2m\omega\hbar}p^2}_{\frac{1}{\hbar\omega}H} + \frac{i}{2}\underbrace{(xp - px)}_{i\hbar}$$

$$\Rightarrow \quad H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$\Rightarrow H|0\rangle = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)|0\rangle = \underbrace{\frac{1}{2}\hbar\omega}_{\text{ground state energy}}$$

$$\left\{\sqrt{\frac{m\omega}{2\hbar}}\,x + i\frac{1}{\sqrt{2m\omega\hbar}}\left(-i\hbar\frac{d}{dx}\right)\right\}\psi_0(x) = 0$$

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ground state energy

Useful to recognize

$$[a, a^\dagger] = \left[\sqrt{\frac{m\omega}{2\hbar}}x + i\frac{1}{\sqrt{2m\omega\hbar}}p,\; \sqrt{\frac{m\omega}{2\hbar}}x - i\frac{1}{\sqrt{2m\omega\hbar}}p\right]$$

$$= \frac{1}{2}\frac{-i}{\hbar}[x,p] + \frac{1}{2}\frac{+i}{\hbar}[p,x] = 1$$

Since $(Ha - aH) = -\hbar\omega a$
$a^\dagger H - Ha^\dagger = -\hbar\omega a^\dagger$ take "†"

note $(aH|\psi\rangle', |\psi\rangle) \equiv (|\psi\rangle', (aH)^\dagger|\psi\rangle)$

$(H|\psi'\rangle, a^\dagger|\psi\rangle) =$

$(|\psi'\rangle, Ha^\dagger|\psi\rangle) =$

so $(aH)^\dagger = Ha^\dagger$

Thus $[H, a^\dagger] = +\hbar\omega a^\dagger$ & $a^\dagger$ increases energy by $\hbar\omega$

$\therefore \; |n\rangle = N_n (a^\dagger)^n|0\rangle$

Is an energy eigenstate with

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

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$$[a, a^\dagger] = \left[\sqrt{\tfrac{m\omega}{2\hbar}}\,x + i\frac{1}{\sqrt{2m\omega\hbar}}\,p \,,\, \sqrt{\tfrac{m\omega}{2\hbar}}\,x - i\frac{1}{\sqrt{2m\omega\hbar}}\,p\right]$$

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$E_n = \hbar\omega(n+\frac{1}{2})$

If $\langle n|n\rangle = 1$

$$\|a^\dagger|n\rangle\|^2 = (a^\dagger|n\rangle, a^\dagger|n\rangle)$$

$$= (|n\rangle, \underbrace{a a^\dagger}_{\underbrace{[a^\dagger,a]}_{1} + \underbrace{a^\dagger a}_{n}} |n\rangle)$$

$$= n+1$$

$$\therefore\ |n+1\rangle = \frac{1}{\sqrt{n+1}} a^\dagger|n\rangle = \frac{1}{\sqrt{n+1}} a^\dagger\left(\frac{1}{\sqrt{n}} a|n-1\rangle\right)$$

$$= \frac{1}{\sqrt{(n+1)!}}(a^\dagger)^{n+1}|0\rangle$$

$$H|n\rangle = \hbar\omega(n+\tfrac{1}{2})|n\rangle$$

$$\& \ \psi_n(x) = \frac{1}{\sqrt{n!}}\underbrace{\left[\sqrt{\frac{m\omega}{2\hbar}}\,x - \sqrt{\frac{\hbar}{2m\omega}}\frac{d}{dx}\right]^n}_{\text{a polynomial in } x \text{ order } n} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \left[\sqrt{\frac{m\omega}{\pi\hbar}}\frac{1}{2^n n!}\right]^{1/2} \underbrace{h_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)}_{\text{Hermite polynomial}} e^{-\frac{m\omega}{2\hbar}x^2}$$

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Summary

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 = \hbar\omega (a^\dagger a + \tfrac{1}{2})$$

$$a = \sqrt{\frac{m\omega}{2\hbar}}\, x + i \frac{1}{\sqrt{2m\omega\hbar}}\, p$$

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger) \qquad p = \sqrt{\frac{m\omega\hbar}{2}} \frac{1}{i} (a - a^\dagger)$$

$$[a, a^\dagger] = 1$$

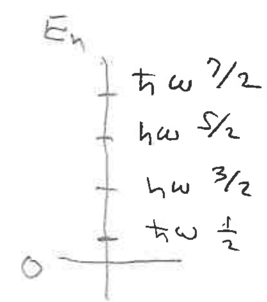
State $|n\rangle = \frac{1}{\sqrt{n!}} (a^\dagger)^n |0\rangle$ are eigenstates of H with evenly spaced eigenvalues

$$H |n\rangle = \hbar\omega (n + \tfrac{1}{2}) |n\rangle$$

$$a^\dagger |n\rangle = \sqrt{n+1}\, |n+1\rangle$$

$$a |n\rangle = \sqrt{n}\, |n-1\rangle$$

$$\langle x | n \rangle = \left[\frac{m\omega}{\hbar\pi} \right]^{1/4} e^{-\frac{m\omega}{2\hbar} x^2}$$



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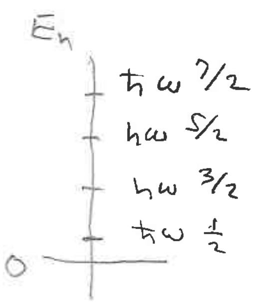
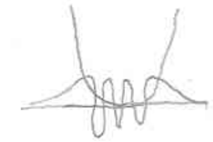


$$a^\dagger a\, |n\rangle = n\,|n\rangle$$

For large n



$h_n\left(\sqrt{\frac{m\omega}{\hbar}}\right) e^{-\frac{m\omega}{2\hbar}x^2}$ does not look classical!

Instead construct "coherent" states which show beautiful classical behavior:

Consider $|A\rangle$ which is an eigenstate of a: $a|A\rangle = A|A\rangle$.

Find $|A\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$

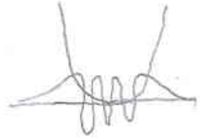
$$a|A\rangle = A|A\rangle$$

$$\sum_{n=0}^{\infty} c_n \underbrace{a|n\rangle}_{\sqrt{n}\,|n-1\rangle} = \sum_{n=0}^{\infty} c_n A|n\rangle$$

$$\sum_{n'} c_{n'+1}\sqrt{n'+1}\,|n\rangle = \sum_{n=0}^{\infty} c_n A|n\rangle$$

$$\Rightarrow \quad c_{n'+1}\sqrt{n'+1} = c_n A \quad \Rightarrow \quad c_n = \frac{1}{\sqrt{n!}} A^n N$$

For large n



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$$|A\rangle = N \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} A^n |n\rangle = N \sum_{n=0}^{\infty} \frac{1}{n!}(A a^\dagger)^n |0\rangle$$

$$1 = \langle A|A\rangle = N^2 \sum_n \left|\frac{1}{\sqrt{n!}} A^n\right|^2 = N^2 \sum_{n=0}^{\infty} \frac{1}{n!}(|A|^2)^n$$

$$= N^2 e^{+|A|^2} \Rightarrow N = e^{-\frac{1}{2}|A|^2}$$

$$\langle A|x|A\rangle = \sqrt{\frac{\hbar}{2m\omega}}(A + A^*) = \sqrt{\frac{2\hbar}{m\omega}}\, re(A)$$

$$\langle A|p|A\rangle = \sqrt{\frac{\hbar m\omega}{2}}\frac{1}{i}(A - A^*) = \sqrt{\frac{\hbar m\omega}{2}}\, im(A)$$

and $$e^{-iHt/\hbar}|A\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} A^n \underbrace{e^{-iHt/\hbar}}_{e^{-i\hbar\omega(n+\frac{1}{2})t/\hbar}} |n\rangle N$$

$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}(A e^{-i\omega t})^n |n\rangle e^{-i\frac{\omega t}{2}} N$$

$$= e^{-i\omega t/2}\,|A e^{-i\omega t}\rangle$$

and $$\langle x\rangle = \sqrt{\frac{2\hbar}{m\omega}}\, re(A e^{-i\omega t})$$

$$= \sqrt{\frac{2\hbar}{m\omega}}\,|A| \cos(\phi + \omega t)$$

$$\langle p\rangle = -\sqrt{\frac{2m\omega}{\hbar}}\,|A| \sin(\phi + \omega t)$$

Perfect classical motion! $\quad A = |A| e^{-i\phi}$

$$|A\rangle = N \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} A^n |n\rangle = N \sum_{n=0}^{\infty} \frac{1}{n!} (A a^\dagger)^n |0\rangle$$

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$$= \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} (A e^{-i\omega t})^n |n\rangle e^{-i\frac{\omega t}{2}} N$$

$$= e^{-i\omega t/2} |A e^{-i\omega t}\rangle$$

and $$\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}}\, \mathrm{re}(A e^{-i\omega t})$$

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Note $|A\rangle \to \psi_A(x)$ solves a simple equation:

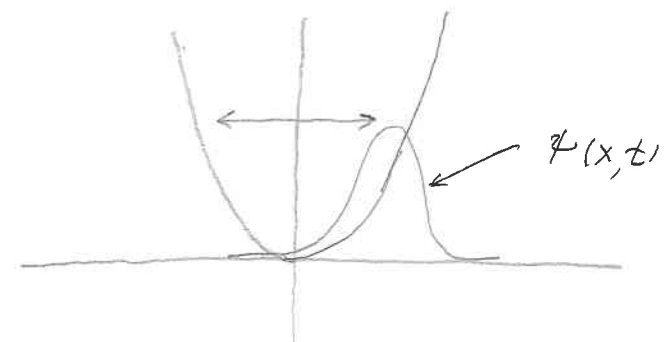
$$a|A\rangle = A|A\rangle$$

$$\left(\sqrt{\frac{m\omega}{2\hbar}}\, x + \sqrt{\frac{\hbar}{2m\omega}} \frac{d}{dx} \right) \psi_A(x) = A\, \psi_A(x)$$

$$\frac{d}{dx} \psi_A(x) = -\frac{m\omega}{\hbar} \left(x - \sqrt{\frac{2\hbar}{m\omega}}\, A \right) \psi_A(x)$$

$$\psi_A(x) = N e^{-\frac{m\omega}{2\hbar}\left(x - \sqrt{\frac{2\hbar}{m\omega}}\, \underbrace{A}_{|A|e^{-i\phi}e^{-i\omega t}}\right)^2}$$

simply a shifted Gaussian



oscillates back and forth classically without dispersion