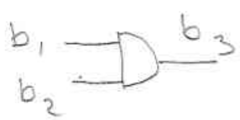
E Quantum computing

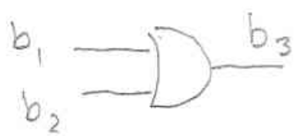
(taken from M. Nielsen & I. Chuang
Quantum Computation and Quantum
Information)

1. Digital computing (classical)

- Work with information encoded as a sequence of one's and zeros
  "1" ≡ on ≡ true , 0 = off = false
  use $(b_1, b_2, b_3, \ldots b_n)$, b_i = "0" or "1"
- Combine using simple Boolean gates

AND 

b_1	b_2	b_3
0	0	0
1	0	0
0	1	0
1	1	1

OR 

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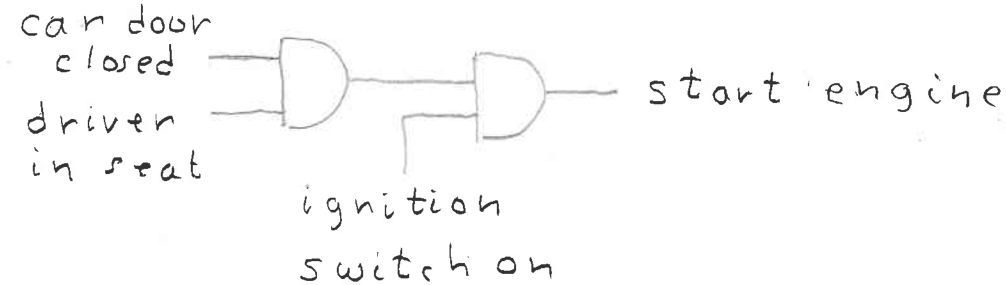
Example



individual signals are voltages

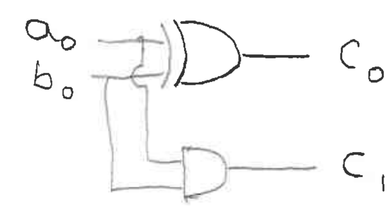
$0 \le V \le 0.25$ Volt ≡ '0'

$0.75 \le V \le 1$ Volt ≡ "1"

Number also represented by n bits

$$(b_n, b_{n-1}, b_{n-2} \ldots b_0) \equiv x = \sum_{i=0}^{n} b_i 2^i$$

$$(a_0) + (b_0) = (c_1, c_0)$$



or
$$(b_{31}, b_{30} \ldots b_0) = (-1)^{b_{31}} \times 2^{(-1)^{b_{30}} \times \sum_{i=24}^{29} b_i 2^{i-24}} \cdot \left(1 + \sum_{i=0}^{23} 2^{i-24} b_i\right)$$

32-bit floating point

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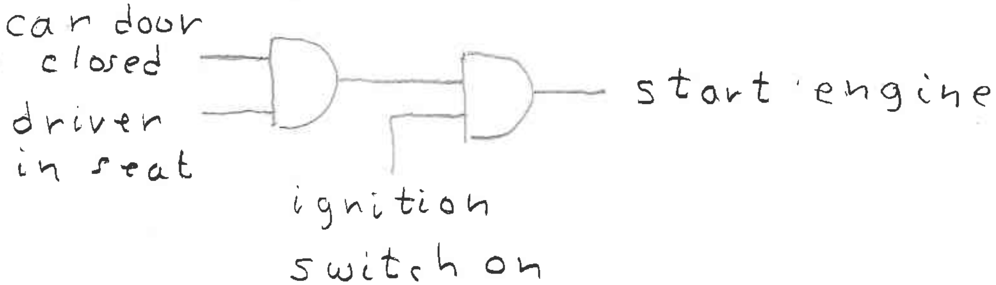

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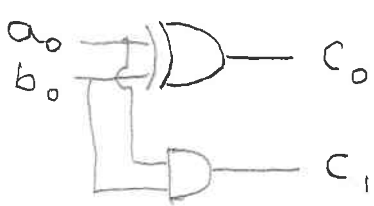

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32-bit floating point $\cdot \left(1 + \sum_{i=0}^{23} 2^{i-24} b_i\right)$

2. A quantum digital computer replaces a binary bit $b =$ '0' or '1' by a quantum state vector in a 2-dimensional Hilbert space

$$b \rightsquigarrow |Q\rangle = a|0\rangle + b|1\rangle.$$

A variable composed of n Qbits is formed from the tensor product of n single Qbits:

$$|q\rangle_n = \sum_{q_1=0}^{1} \sum_{q_2=0}^{1} \cdots \sum_{q_n=0}^{1} C_{q_1 q_2 \ldots q_n} \times |q_1\rangle \otimes |q_2\rangle \otimes \cdots \otimes |q_n\rangle$$

n qbits correspond to a 2^n dimensional complex vector!

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As in digital computers act on one, two & sometimes three Qbit states with simple "gates" which are unitary transformations acting on 2; 4- or 8-dim Hilbert space

Examples

ⓐ Not

$$|A\rangle \rightarrow \boxed{X} \rightarrow |B\rangle \qquad |B\rangle = \sigma_x |A\rangle \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|0\rangle \rightarrow |1\rangle , \quad |1\rangle \rightarrow |0\rangle$$

actually $a_0|0\rangle + a_1|1\rangle \rightarrow a_0|1\rangle + a_1|0\rangle$

ⓑ Hadamard gate

$$A \rightarrow \boxed{H} \rightarrow B \qquad |B\rangle = \frac{\sigma_z + \sigma_x}{\sqrt{2}} |A\rangle \qquad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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ⓒ Controlled not

$\underbrace{|B \oplus A\rangle}_{\text{exclusive or}}$

$A = 0, 1$
$B = 0, 1$

A	B	A⊕B
0	0	0
1	0	1
0	1	1
1	1	0

A is controling whether or not B is inverted

This also is a unitary operator

$$\underbrace{\begin{array}{c|c} \text{A B} & \begin{array}{cccc} (0,0) & (1,0) & (0,1) & (1,1) \end{array} \\ \hline \begin{array}{c} (0,0) \\ (1,0) \\ (0,1) \\ (1,1) \end{array} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}}_{H}$$

H simply shuffles the states and hence is unitary.

3. Now examine a more complex quantum circuit

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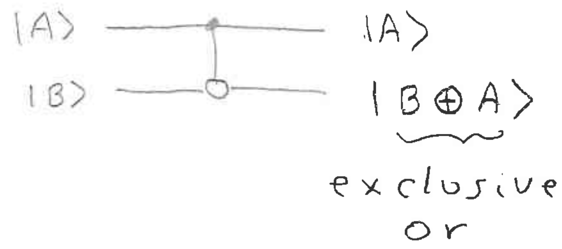

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The Deutsch algorithm

introduce a Boolean function $f(b)$, specified by two values

$f(0) = 0$ or 1 and $f(1) = 0$ or 1

Four such functions are possible

Consider a new quantum gate

$\equiv U_f$

This will be unitary since again shuffles the basis states.

How many times do we need to apply U_f to determine f?

Can be done in 2:

$|A=0\rangle \otimes |B=0\rangle \rightarrow |A=0\rangle \otimes |f(0)\rangle$

$|A=1\rangle \otimes |B=0\rangle \rightarrow |A=1\rangle \otimes |f(1)\rangle$

learn $f(0)$ & $f(1)$

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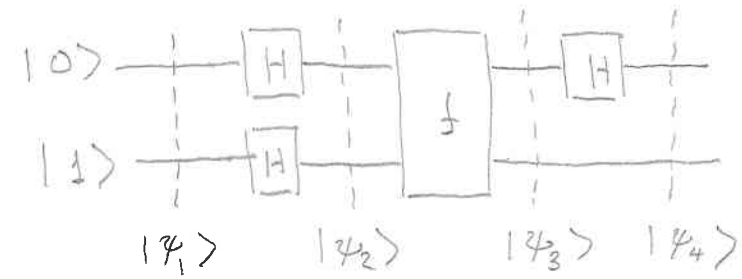
Do it in one application of U_f:



Use $|\psi\rangle = |A\rangle \times |B\rangle$

$$|\psi_1\rangle = |0\rangle \otimes |1\rangle$$

$$|\psi_2\rangle = \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right] \times \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

$$|\psi_3\rangle = \frac{1}{2}|0\rangle \otimes \left[\,|f(0)\rangle - |1 \oplus f(0)\rangle\right] + \frac{1}{2}|1\rangle \otimes \left[\,|f(1)\rangle - |1 \oplus f(1)\rangle\right]$$

$$= \frac{1}{2}\left\{ |0\rangle \otimes \left[|0\rangle - |1\rangle\right](-1)^{f(0)} + |1\rangle \otimes \left[|0\rangle - |1\rangle\right](-1)^{f(1)} \right\}$$

$$|\psi_4\rangle = \frac{1}{2}\left[\frac{|0\rangle + |1\rangle}{\sqrt{2}}(-1)^{f(0)} + \frac{|0\rangle - |1\rangle}{\sqrt{2}}(-1)^{f(1)}\right] \times \left[|0\rangle - |1\rangle\right]$$

$$= \left\{ |0\rangle \frac{\left[(-1)^{f(0)} + (-1)^{f(1)}\right]}{2} + |1\rangle \frac{\left[(-1)^{f(0)} - (-1)^{f(1)}\right]}{2} \right\} \otimes \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$$

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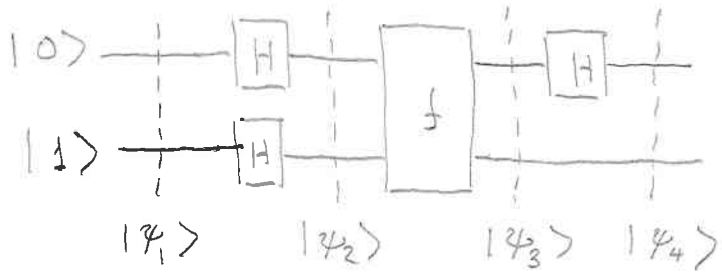

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One appication of U_f is enough!

4. Deutsch-Josza algorithm:

Consider an n-bit binary number x and a binary valued function $f(x)$. We must evaluate $f(x)$ 2^n times to fully determine it. Consider a restricted set of function $f(x)$ obey one of 3 conditions

a) $f(x) = 0$ for all x

b) $f(x) = 1$ for all x

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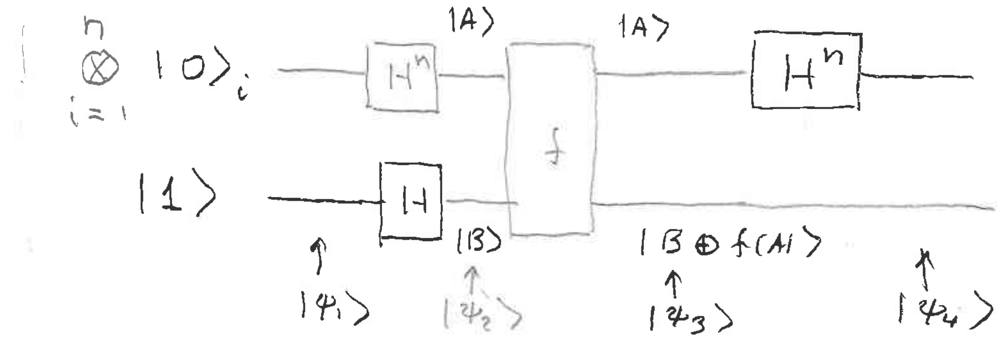
Find a quantum circuit that will tell to which class f belongs in one appication:



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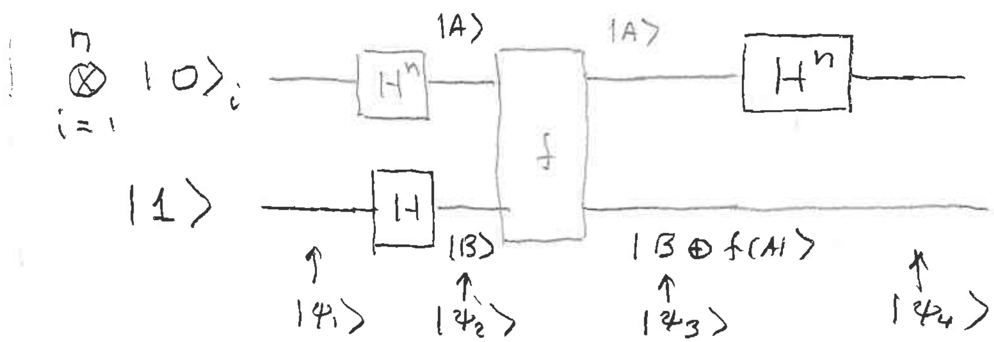

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Finally project onto the state $|\phi\rangle = \bigotimes_{i=1}^{n} |0\rangle_i \dfrac{|0\rangle - |1\rangle}{\sqrt{2}}$

$$\langle \phi | \psi_4 \rangle = \frac{1}{2^n} \sum_x (-1)^{f(x)}$$

$$= \begin{cases} 1 & \text{(a) } f(x) = 0 \text{ for all } x \\ -1 & \text{(b) } f(x) = 1 \text{ for all } x \\ 0 & \text{(c) } f(x) = 0 \text{ for half the inputs } f(x) = 1 \text{ for the other half} \end{cases}$$