$$|\psi_4\rangle = \underbrace{(-1)^{f(0)}}_{\substack{\text{determines} \\ f(0)}} \, |\underbrace{f(0) \oplus f(1)}_{\substack{\text{determines if} \\ f(0) = f(1) \text{ or } f(0) = \overline{f(1)}}}\rangle \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

One appication of $U_f$ is enough!

4. Deutsch-Josza algorithm:

Consider an n-bit binary number $x$ and a binary valued function $f(x)$. We must evaluate $f(x)$ $2^n$ times to fully determine it. Consider a restricted set of function $f(x)$ obey one of 3 conditions

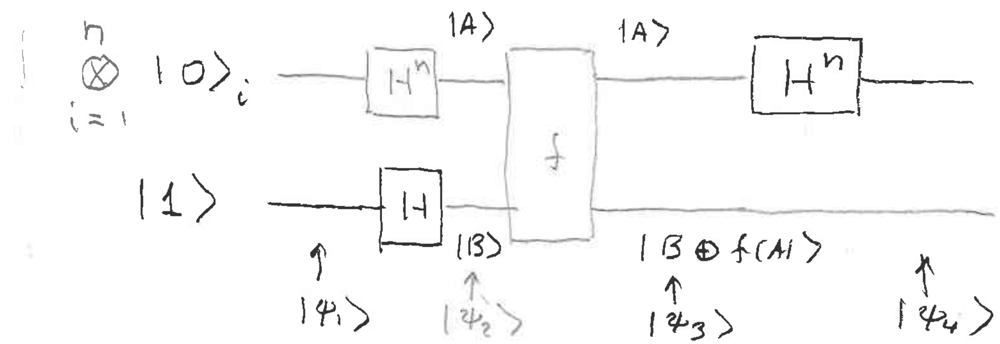
a) $f(x) = 0$ for all $x$

b) $f(x) = 1$ for all $x$

c) $f(x) = 0$ for $1/2$ of the $2^n$ inputs and
$f(x) = 1$ for the other half



Find a quantum circuit that will tell to which class $f$ belongs in one appication:




$$|\psi_1\rangle = \left[\bigotimes_{i=1}^{n} |0\rangle_i\right] \otimes |1\rangle$$

$$|\psi_2\rangle = \left[\bigotimes_{i=1}^{n} \frac{|0\rangle_i + |1\rangle_i}{\sqrt{2}}\right] \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

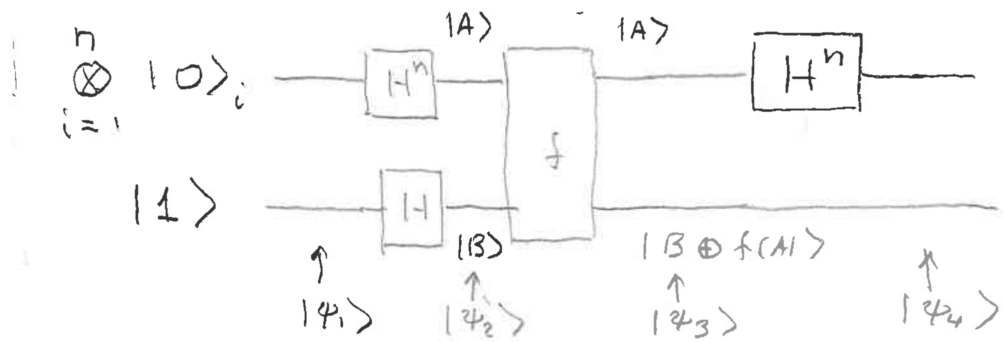
$$= \frac{1}{\sqrt{2^n}} \left\{\sum_x \bigotimes_{i=1}^{n} |x_i\rangle_i\right\} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \left\{\sum_x \bigotimes_{i=1}^{n} |x_i\rangle_i (-1)^{f(x)}\right\} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2^n}} \left\{\sum_x \bigotimes_{i=1}^{n} \frac{|0\rangle_i + (-1)^{x_i} |1\rangle_i}{\sqrt{2}} (-1)^{f(x)}\right\}$$

$$\times \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

Find a quantum circuit that will tell to which class f belongs in one appication:




$$|\psi_1\rangle = \left[\bigotimes_{i=1}^{n} |0\rangle_i\right] \otimes |1\rangle$$

$$|\psi_2\rangle = \left[\bigotimes_{i=1}^{n} \frac{|0\rangle_i + |1\rangle_i}{\sqrt{2}}\right] \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2^n}} \left\{ \sum_x \bigotimes_{i=1}^{n} |x_i\rangle_i \right\} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2^n}} \left\{ \sum_x \bigotimes_{i=1}^{n} |x_i\rangle_i (-1)^{f(x)} \right\} \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

$$|\psi_4\rangle = \frac{1}{\sqrt{2^n}} \left\{ \sum_x \bigotimes_{i=1}^{n} \frac{|0\rangle_i + (-1)^{x_i} |1\rangle_i}{\sqrt{2}} (-1)^{f(x)} \right\}$$

$$\times \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$



Finally project onto the state $|\phi\rangle = \bigotimes_{i=1}^{n} |0\rangle_i \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

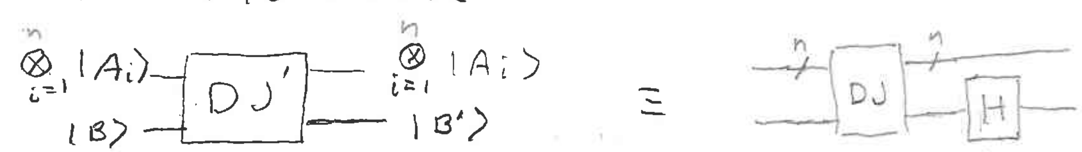
$$\langle \phi | \psi_4 \rangle = \frac{1}{2^n} \sum_x (-1)^{f(x)}$$

$$= \begin{cases} 1 & \text{(a) } f(x) = 0 \text{ for all } x \\ -1 & \text{(b) } f(x) = 1 \text{ for all } x \\ 0 & \text{(c) } f(x) = 0 \text{ for half the inputs } f(x) = 1 \text{ for the other half} \end{cases}$$

---

How can we measure the $-1$ when the phase of a quantum state usually has no meaning?

1st introduce





$$\begin{cases} |\Theta\rangle & \text{(a)} \\ \perp |\Theta\rangle & \text{(b)} \\ -|\Theta\rangle & \text{(c)} \end{cases}$$

$$|\Theta\rangle = \bigotimes_{i=1}^{n} |0\rangle_i \otimes |1\rangle$$

4/15/21



Finally project onto the state $|\phi\rangle = \bigotimes_{i=1}^{n} |0\rangle_i \otimes \frac{|0\rangle - |1\rangle}{\sqrt{2}}$

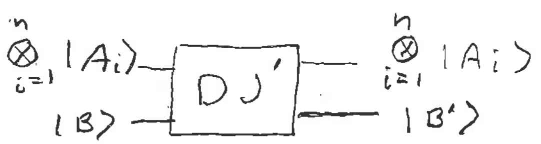
$$\langle \phi | \psi_4 \rangle = \frac{1}{2^n} \sum_x (-1)^{f(x)}$$

$$= \begin{cases} 1 & \text{(a) } f(x) = 0 \text{ for all } x \\ -1 & \text{(b) } f(x) = 1 \text{ for all } x \\ 0 & \text{(c) } f(x) = 0 \text{ for half the inputs } f(x) = 1 \text{ for the other half} \end{cases}$$

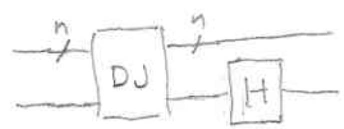
---

How can we measure the $-1$ when the phase of a quantum state usually has no meaning?

$1^{st}$ introduce




$\equiv$



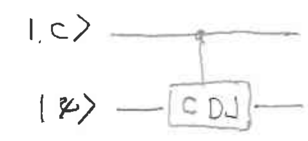


$\begin{cases} |\theta\rangle & \text{(a)} \\ \perp |\theta\rangle & \text{(b)} \\ -|\theta\rangle & \text{(c)} \end{cases}$

$|\theta\rangle = \bigotimes_{i=1}^{n} |0\rangle_i \otimes |1\rangle$

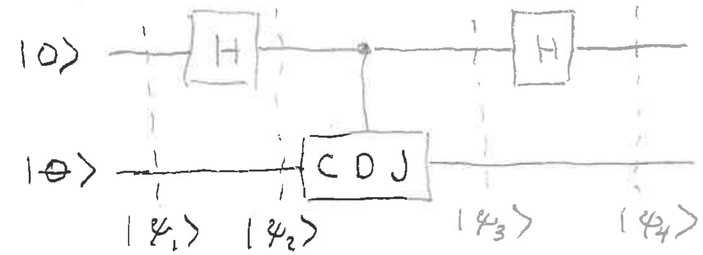


Next define "controlled DJ'"




$\begin{cases} |\psi\rangle & c = 1 \\ [DJ']|\psi\rangle & c = 1 \end{cases}$

Finally use




$$|\psi_1\rangle = |0\rangle \otimes |\theta\rangle$$

$$|\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |\theta\rangle$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}} \left\{ |0\rangle \otimes |\theta\rangle + |1\rangle \otimes [DJ']|\theta\rangle \right\}$$

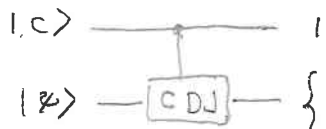
$$|\psi_4\rangle = \frac{1}{2} \left\{ (|0\rangle + |1\rangle) \otimes |\theta\rangle + (|0\rangle - |1\rangle) \otimes [DJ']|\theta\rangle \right\}$$

$$= \left\{ |0\rangle \otimes \left[ \frac{|\theta\rangle + [DJ']|\theta\rangle}{2} \right] + |1\rangle \otimes \left[ \frac{|\theta\rangle - [DJ']|\theta\rangle}{2} \right] \right.$$

(a): $|0\rangle \otimes |\theta\rangle$ (b): $|1\rangle \otimes |\theta\rangle$ (c): $|A\rangle \otimes |\theta\rangle^{\perp}$

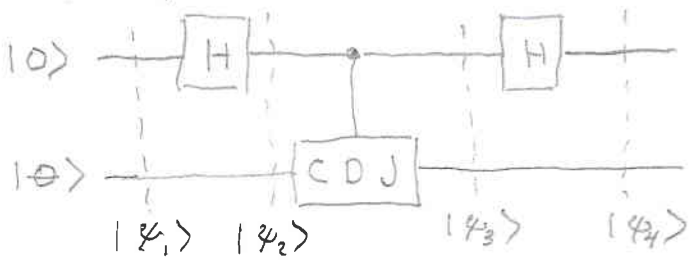
Dim 1 1 $2 \times (2^{n+1} - 1)$

Next define "controlled DJ′"




$$\begin{cases} |\psi\rangle & c=1 \\ [DJ']|\psi\rangle & c=1 \end{cases}$$

Finally use




$$|\psi_1\rangle = |0\rangle \otimes |\theta\rangle$$

$$|\psi_2\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes |\theta\rangle$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}\Big\{|0\rangle \otimes |\theta\rangle + |1\rangle \otimes [DJ']|\theta\rangle\Big\}$$

$$|\psi_4\rangle = \frac{1}{2}\Big\{(|0\rangle + |1\rangle) \otimes |\theta\rangle + (|0\rangle - |1\rangle) \otimes [DJ']|\theta\rangle\Big\}$$

$$= \Big\{ |0\rangle \otimes \Big[\frac{|\theta\rangle + [DJ']|\theta\rangle}{2}\Big] + |1\rangle \otimes \Big[\frac{|\theta\rangle - [DJ']|\theta\rangle}{2}\Big]$$

(a): $|0\rangle \otimes |\theta\rangle$ (b): $|1\rangle \otimes |\theta\rangle$ (c): $|A\rangle \otimes |\theta\rangle^{\perp}$

Dim 1 1 $2 \times (2^{n+1} - 1)$



# 5. Teleportation

## a) Einstein-Podolsky-Rosen "paradox"

Two factors in an entanged state can fly far apart and then be measured by space-like separated observers:

$$\text{Lab A} \quad \xleftarrow{e^-} \pi^0 \longrightarrow e^+ \quad \text{Lab B}$$

$$|1/2\rangle_{e^-} \otimes |-1/2\rangle_{e^+} - |-1/2\rangle_{e^-} \otimes |+1/2\rangle_{e^+}$$

$$\downarrow$$

$$|1/2\rangle^{A}_{e^-} \otimes |-1/2\rangle^{B}_{e^+} - |-1/2\rangle^{A}_{e^-} \otimes |+1/2\rangle^{B}_{e^+}$$

If observer A measures $J_z = +\hbar/2$ she knows when B measures $J_z$ he will find $J_z = -\hbar/2$ !

Strange but not wrong. Not a paradox and verified by experiment.

# 5. Teleportation

## a) Einstein-Podolsky-Rosen "paradox"

Two factors in an entanged state can fly far apart and then be measured by space-like separated observers:

$$\text{Lab A} \quad \overset{e^-}{\longleftarrow} \ \pi^0 \longrightarrow e^+ \quad \text{Lab B}$$

$$|\tfrac{1}{2}\rangle_{e^-} \otimes |-\tfrac{1}{2}\rangle_{e^+} - |-\tfrac{1}{2}\rangle_{e^-} \otimes |+\tfrac{1}{2}\rangle_{e^+}$$

$$\downarrow$$

$$|\tfrac{1}{2}\rangle^{A}_{e^-} \otimes |-\tfrac{1}{2}\rangle^{B}_{e^+} - |-\tfrac{1}{2}\rangle^{A}_{e^-} \otimes |+\tfrac{1}{2}\rangle^{B}_{e^+}$$

If observer A measures $J_z = +\hbar/2$ she knows when B measures $J_z$ he will find $J_z = -\hbar/2$ !

Strange but not wrong. Not a paradox and verified by experiment.



b) Such an EPR state can be used to achieve a very strange transfer of quantum information:

(i) Assume labs A + B share an EPR state $\left[ |0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B \right] \frac{1}{\sqrt{2}}$

in Lab A

in Lab B

(ii) Further assume that there is a third Qbit $|Q\rangle = \frac{1}{\sqrt{|\alpha|^2 + |\beta|^2}} \left( \alpha|0\rangle + \beta|1\rangle \right)$ in Lab A so the complete quantum state lies in an 8-dim Hilbert space and is

$$|Q\rangle \otimes \left\{ |0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B \right\}$$

(iii) The state $|\psi\rangle$ can be created in B if A sends B a 2-bit number!

b) Such an EPR state can be used to achieve a very strange transfer of quantum information:

(i) Assume labs A + B share an EPR state $\left[|0\rangle_A |1\rangle_B + |1\rangle_A |0\rangle_B\right]\frac{1}{\sqrt{2}}$

in Lab A

in Lab B

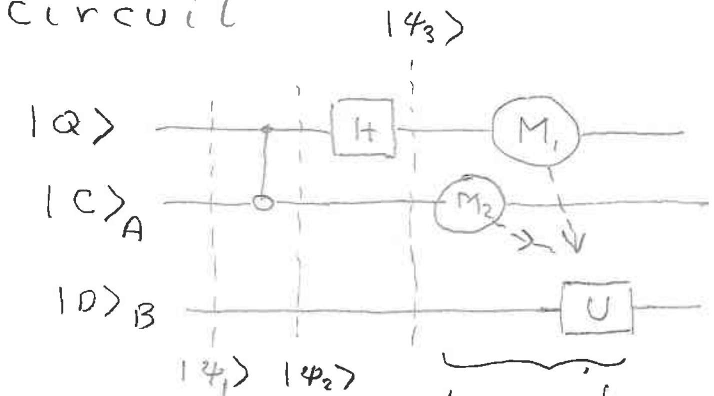
(ii) Further assume that there is a third Qbit $|Q\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}(\alpha|0\rangle + \beta|1\rangle)$ in Lab A so the complete quantum state lies in an 8-dim Hilbert space and is

$$|Q\rangle \otimes \left\{|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\right\}$$

(iii) The state $|\psi\rangle$ can be created in B if A sends B a 2-bit number!



This can be accomphished with the following quantum circuit



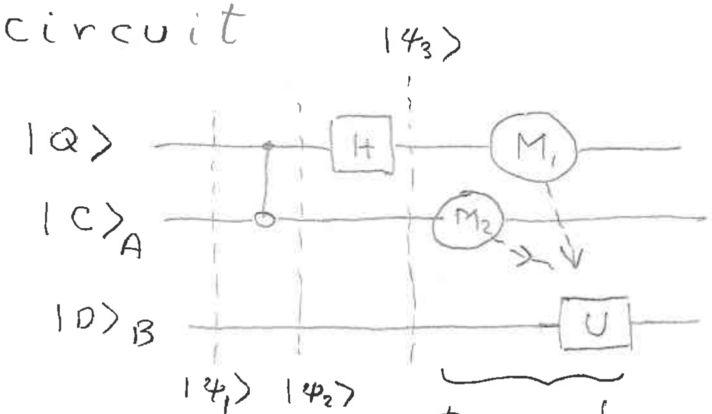

transfer 2-bit instruction to Lab B

$$|\psi_1\rangle = \frac{1}{\sqrt{2}\sqrt{|\alpha|^2+|\beta|^2}}\left(\alpha|0\rangle + \beta|1\rangle\right)\otimes\left(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\right)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}\sqrt{2}}\left\{\alpha|0\rangle \otimes \left(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\right) + \beta|1\rangle \otimes \left(|1\rangle_A \otimes |1\rangle_B + |0\rangle_A \otimes |0\rangle_B\right)\right\}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\left\{\alpha\left(\frac{|0\rangle + |1\rangle}{2}\right)\otimes\left(|0\rangle_A \otimes |1\rangle_B + |1\rangle_A \otimes |0\rangle_B\right) + \beta\left(\frac{|0\rangle - |1\rangle}{2}\right)\otimes\left(|1\rangle_A \otimes |1\rangle_B + |0\rangle_A \otimes |0\rangle_B\right)\right\}$$

This can be accomplished with the following quantum circuit




transfer 2-bit instruction to Lab B

$$|\psi_1\rangle = \frac{1}{\sqrt{2}\sqrt{|\alpha|^2+|\beta|^2}}\left(\alpha|0\rangle + \beta|1\rangle\right)\otimes\left(|0\rangle_A\otimes|1\rangle_B + |1\rangle_A\otimes|0\rangle_B\right)$$

$$|\psi_2\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}\sqrt{2}}\Big\{\alpha|0\rangle\otimes\left(|0\rangle_A\otimes|1\rangle_B + |1\rangle_A\otimes|0\rangle_B\right)$$
$$+\beta|1\rangle\otimes\left(|1\rangle_A\otimes|1\rangle_B + |0\rangle_A\otimes|0\rangle_B\right)\Big\}$$

$$|\psi_3\rangle = \frac{1}{\sqrt{|\alpha|^2+|\beta|^2}}\Big\{\alpha\left(\frac{|0\rangle+|1\rangle}{2}\right)\otimes\big(|0\rangle_A\otimes|1\rangle_B$$
$$+|1\rangle_A\otimes|0\rangle_B\big)$$
$$+\beta\left(\frac{|0\rangle-|1\rangle}{2}\right)\otimes\left(|1\rangle_A\otimes|1\rangle_B + |0\rangle_A\otimes|0\rangle_B\right)\Big\}$$



Now collect terms

$$|\psi_3\rangle = \frac{1}{2}\Big\{|0\rangle\otimes|0\rangle_A\otimes\left[\frac{\alpha|1\rangle_B+\beta|0\rangle_B}{\sqrt{|\alpha|^2+|\beta|^2}}\right]$$
$$+|0\rangle\otimes|1\rangle_A\otimes\left[\frac{\alpha|0\rangle_B+\beta|1\rangle_B}{\sqrt{|\alpha|^2+|\beta|^2}}\right]$$
$$+|1\rangle\otimes|0\rangle_A\otimes\left[\frac{\alpha|1\rangle_B-\beta|0\rangle_B}{\sqrt{|\alpha|^2+|\beta|^2}}\right]$$
$$+|1\rangle\otimes|1\rangle_A\otimes\left[\frac{\alpha|0\rangle_B-\beta|1\rangle_B}{\sqrt{|\alpha|^2+|\beta|^2}}\right]\Big\}$$

Finally the first two factors are measured in Lab A and the results transmitted to Lab B and used to determine the transformation U to create $|Q\rangle$ in Lab B:

$0,0:\quad U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad 1,0:\quad U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$0,1:\quad U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\qquad 1,1:\quad U = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

Now collect terms

$$|\psi_3\rangle = \frac{1}{2}\Big\{ |0\rangle \otimes |0\rangle_A \otimes \left[ \frac{\alpha|1\rangle_B + \beta|0\rangle_B}{\sqrt{|\alpha|^2+|\beta|^2}} \right]$$

$$+ |0\rangle \otimes |1\rangle_A \otimes \left[ \frac{\alpha|0\rangle_B + \beta|1\rangle_B}{\sqrt{|\alpha|^2+|\beta|^2}} \right]$$

$$+ |1\rangle \otimes |0\rangle_A \otimes \left[ \frac{\alpha|1\rangle_B - \beta|0\rangle_B}{\sqrt{|\alpha|^2+|\beta|^2}} \right]$$

$$+ |1\rangle \otimes |1\rangle_A \otimes \left[ \frac{\alpha|0\rangle_B - \beta|1\rangle_B}{\sqrt{|\alpha|^2+|\beta|^2}} \right] \Big\}$$

Finally the first two factors are measured in Lab A and the results transmitted to Lab B and used to determine the transformation $U$ to create $|Q\rangle$ in Lab B:

$0,0$: $U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad$ $1,0$: $U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$0,1$: $U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad$ $1,1$: $U = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$



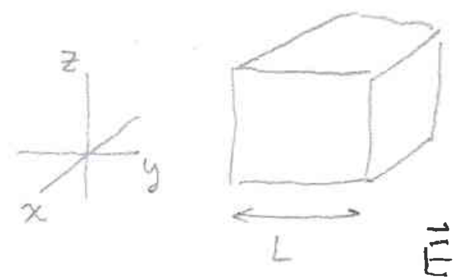
# F Quantum field theory and the emergence of particles

Treat Maxwell's equations using quantum mechanics

$$\vec{\nabla}\cdot\vec{E} = 0 \qquad \vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \vec{\nabla}\times\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t}$$

Use a conducting box of side $L$




Consider a single "mode" with

$$\vec{E}(x,y,z) = \mathcal{E}(t)\,\hat{x}\,\sin\!\left(\frac{\pi}{L}y\right)\sin\!\left(\frac{\pi}{L}z\right)$$

$\perp$ to both $y$-$z$ faces ($x = 0$ & $L$)

$\parallel$ to but vanishes on the other four faces.

$$\frac{1}{c}\frac{\partial\vec{B}}{\partial t} = -\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial y}E_x\,\hat{z} + \frac{\partial}{\partial z}E_x\,\hat{y}$$

$$= \mathcal{E}(t)\frac{\pi}{L}\left[\cos\frac{\pi}{L}y\,\sin\frac{\pi}{L}z\,\hat{z} - \sin\frac{\pi}{L}y\,\cos\frac{\pi}{L}z\,\hat{y}\right]$$

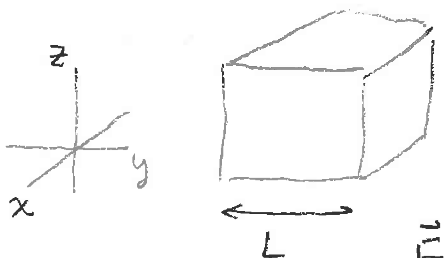
# F Quantum field theory and the emergence of particles

Treat Maxwell's equations using quantum mechanics

$$\vec\nabla\cdot\vec E = 0 \qquad \vec\nabla\cdot\vec B = 0$$

$$\vec\nabla\times\vec E = -\frac{1}{c}\frac{\partial B}{\partial t} \qquad \vec\nabla\times\vec B = \frac{1}{c}\frac{\partial \vec E}{\partial t}$$

Use a conducting box of side L



Consider a single "mode" with

$$\vec E(x,y,z) = \mathcal{E}(t)\,\hat x\,\sin\!\left(\frac{\pi}{L}y\right)\sin\!\left(\frac{\pi}{L}z\right)$$

$\perp$ to both y-z faces ($x = 0$ & $L$)

// to but vanishes on the other four faces.

$$\frac{1}{c}\frac{\partial\vec B}{\partial t} = -\vec\nabla\times\vec E = -\frac{\partial}{\partial y}E_x\,\hat z + \frac{\partial}{\partial z}E_x\,\hat y$$

$$= \mathcal{E}(t)\frac{\pi}{L}\left[\cos\frac{\pi}{L}y\,\sin\frac{\pi}{L}z\,\hat z - \sin\frac{\pi}{L}y\cos\frac{\pi}{L}z\,\hat y\right]$$



$\Rightarrow$

$$\vec B(t) = \mathcal{B}(t)\left[\cos\frac{\pi}{L}y\,\sin\frac{\pi}{L}z\,\hat z - \sin\frac{\pi}{L}y\,\cos\frac{\pi}{L}z\,\hat y\right]$$

and $\frac{1}{c}\frac{d\mathcal{B}(t)}{dt} = \frac{\pi}{L}\mathcal{E}(t)$

$$\text{Energy} = \int_0^L dx\int_0^L dy\int_0^L dz\,\frac{1}{8\pi}\left\{\vec E^2 + \vec B^2\right\}$$

$$= \frac{1}{8\pi}\int_{L^3} dx\,dy\,dz\left\{\mathcal{E}^2(t)\sin^2\frac{\pi y}{L}\sin^2\frac{\pi z}{L} + \mathcal{B}^2(t)\left[\cos^2\frac{\pi y}{L}\sin^2\frac{\pi z}{L} + \sin^2\frac{\pi y}{L}\cos^2\frac{\pi z}{L}\right]\right\}$$

recall $\int_0^L \sin^2\frac{\pi x}{L}\,dx = \int_0^L \cos^2\frac{\pi x}{L}\,dx = \frac{1}{2}L$

$\Rightarrow$

$$= \frac{1}{8\pi}L^3\frac{1}{4}\left\{\mathcal{E}^2(t) + 2\,\mathcal{B}^2(t)\right\}$$

$\frac{L}{\pi c}\frac{d\mathcal{B}}{dt}$

$$= \frac{L^3}{8\pi}\frac{1}{4}\left\{\left(\frac{L}{\pi c}\right)^2\left(\frac{d\mathcal{B}}{dt}\right)^2 + 2\,\mathcal{B}^2\right\}$$

$$= \frac{1}{2}m\left(\frac{d\mathcal{B}}{dt}\right)^2 + \frac{1}{2}m\,\omega^2\mathcal{B}^2(t)$$

$$p = m\frac{d\mathcal{B}}{dt} \qquad m = \frac{L^5}{16\pi^3c^2}$$

$$\omega = \sqrt{2}\,\frac{\pi c}{L}$$

$\Rightarrow$

$$\vec{B}(t) = \mathcal{B}(t)\left[\cos\frac{\pi}{L}y \, \sin\frac{\pi}{L}z \, \hat{z} - \sin\frac{\pi}{L}y \, \cos\frac{\pi}{L}z \, \hat{y}\right]$$

and $\frac{1}{c}\frac{d\mathcal{B}(t)}{dt} = \frac{\pi}{L}\mathcal{E}(t)$

$$\text{Energy} = \int_0^L dx \int_0^L dy \int_0^L dz \, \frac{1}{8\pi}\left\{\vec{E}^2 + \vec{B}^2\right\}$$

$$= \frac{1}{8\pi}\int_{L^3} dx\,dy\,dz \left\{ \mathcal{E}^2(t) \sin^2\frac{\pi y}{L} \sin^2\frac{\pi z}{L} + \mathcal{B}^2(t)\left[\cos^2\frac{\pi y}{L}\sin^2\frac{\pi z}{L} + \sin^2\frac{\pi y}{L}\cos^2\frac{\pi z}{L}\right]\right\}$$

recall $\int_0^L \sin^2\frac{\pi x}{L}\,dx = \int_0^L \cos^2\frac{\pi x}{L}\,dx = \frac{1}{2}L$

$\Rightarrow$

$$= \frac{1}{8\pi} L^3 \frac{1}{4}\left\{\mathcal{E}^2(t) + 2\mathcal{B}^2(t)\right\}$$

$\frac{L}{\pi c}\frac{d\mathcal{B}}{dt}$

$$= \frac{L^3}{8\pi}\frac{1}{4}\left\{\left(\frac{L}{\pi c}\right)^2\left(\frac{d\mathcal{B}}{dt}\right)^2 + 2\mathcal{B}^2\right\}$$

$$= \frac{1}{2}m\left(\frac{d\mathcal{B}}{dt}\right)^2 + \frac{1}{2}m\omega^2\mathcal{B}^2(t)$$

$p = m\frac{d\mathcal{B}}{dt}$ $\qquad m = \frac{L^5}{16\pi^3 c^2}$

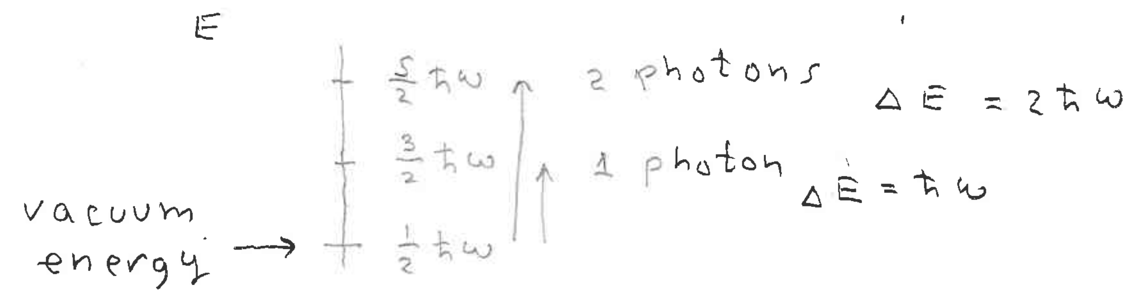
$\omega = \sqrt{2}\,\frac{\pi c}{L}$



A simple harmonic oscillator that is easy to make quantum mechanical

$$a = \sqrt{\frac{m\omega}{2\hbar}}\,\mathcal{B}_{op} + i\frac{1}{\sqrt{2m\omega\hbar}}\,P_{op}$$

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$




each photon carries energy

$$E = \hbar\omega = \frac{h}{2\pi}2\pi\nu = h\nu$$

Planck's energy quantization!