4/19/21

# Review Session

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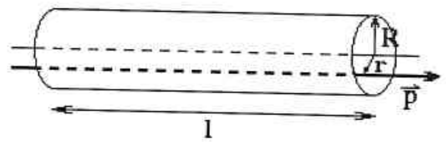
1. A cylinder of length $l$ and radius $R$ carries a uniform current density $j$.

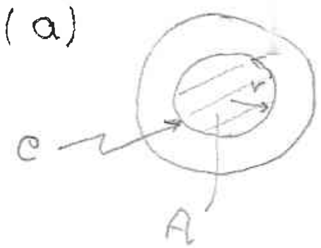
   (a) Find the magnetic field for both the regions $r > R$ and $r < R$. [10 points]




   (b) A particle with momentum $\vec{p}$ and charge $q$ moves parallel to and within the cylinder, displaced a distance $r$ from the cylinder's axis. Calculate the additional momentum $\vec{\Delta p}$ given the particle as it passes through the length $l$ of the cylinder. (You should assume that $\Delta p/p \ll 1$ so that the particle's spatial deflection by the magnetic forces can be neglected.) [10 points]

   (c) Including the effects of this $\vec{\Delta p}$ on the subsequent motion of the particle, show that such particles for all values of $r$, $0 \le r \le R$, will cross the axis of the cylinder at the same distance $D$ from the cylinder. Find this focal length $D$. [5 points]

(a)




Current flowing out of the page

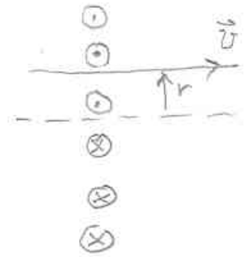
$$\oint_{C=\partial A} B \cdot dr = \frac{4\pi}{c} \int_A \vec{j} \cdot \hat{n}\, ds$$

$$2\pi r B(r) = \frac{4\pi}{c} \pi r^2 j$$

$$\Rightarrow \quad B(r) = \begin{cases} \frac{2\pi}{c} r j & r \le R \\ \frac{2I}{c} I & r \geq R \end{cases} \qquad I = \pi r^2 j$$

2

(b)



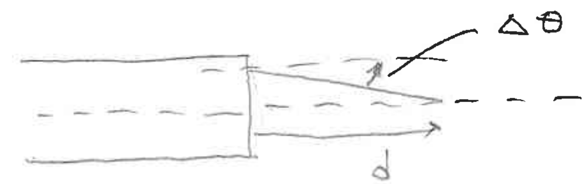

$$\vec{F} = q \frac{\vec{v}}{c} \times \vec{B} = -\hat{r}\, \frac{q v B}{c}$$

Particle a distance $r$ from the center will be deflected downward by

$$\Delta p = F T = \left[ q \frac{v}{c} \frac{2\pi j r}{c} \right] \frac{l}{v}$$

$$\Delta p = \frac{2\pi q j r l}{c^2}$$

(c)

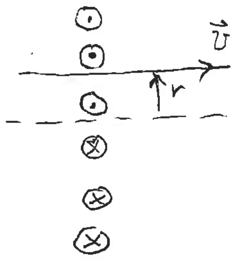



$$\Delta\theta\, d = r$$

$$d = \frac{r}{\Delta\theta} = \frac{r p}{\Delta p} = r p \frac{c^2}{2\pi q j r l}$$

$$= \frac{p c^2}{2\pi q j l} \sim \frac{E\, D/T}{q^2 \frac{1}{D^2 T} D} \sim \frac{D/T}{\frac{1}{D} P/T} \sim D \checkmark$$

(b)

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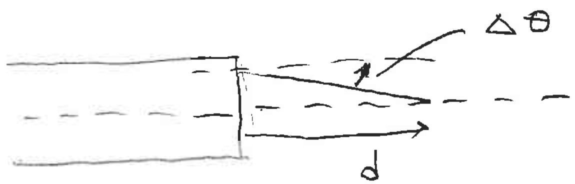
(c)

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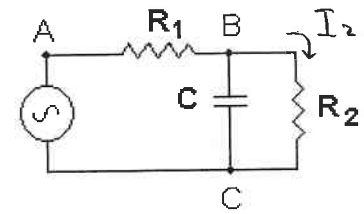
2. Consider the combination of two resistors, $R_1$ and $R_2$ and a capacitor $C$ driven by an applied voltage $V(t) = V_0 \cos(\omega t)$ configured as shown at the right.




(a) Find the complex impedance Z seen between the points A and C. [9 points]

(b) What is the actual current flowing through $R_1$ in both amplitude and phase? [8 points]

(c) What is the average power dissipated in the resistor $R_2$? [8 points]

(a)
$$Z = R_1 + \frac{1}{i\omega C + \frac{1}{R_2}} = R_1 + \frac{R_2}{1 + i\omega C R_2}$$

$$= \frac{R_1 + R_2 + i\omega C R_1 R_2}{1 + i\omega C R_2}$$

(b)
$$I = \frac{V(t)}{Z} = V_0 e^{i\omega t} \frac{1 + i\omega C R_2}{R_1 + R_2 + i\omega C R_2 R_1}$$

$$|I| = V_0 \left\{ \frac{1 + \omega^2 C^2 R_2}{(R_1 + R_2)^2 + (\omega C R_1 R_2)^2} \right\}^{1/2}$$

To calculate phase write

$$I = V_0 e^{i\omega t} \frac{(1 + i\omega C R_2)(R_1 + R_2 - i\omega C R_1 R_2)}{(R_1 + R_2)^2 + (\omega C R_2 R_1)^2}$$

$$I_r = |I| \cos(\omega t + \phi) \qquad \tan\phi = \frac{\omega C R_2 (R_1 \cancel{+ R_2}) \cancel{- \omega C R_1 R_2}}{R_1 + R_2 + \omega^2 C^2 R_1 R_2^2}$$

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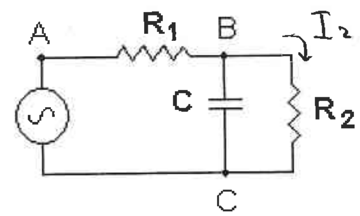


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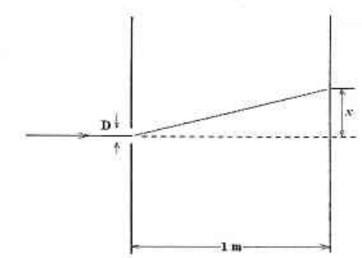
(c)
$$V_{BC} = V - I R_1$$
$$= V\left[1 - R_1 \frac{1 + i\omega C R_2}{R_1 + R_2 + i\omega C R_2 R_1}\right]$$
$$= V \frac{R_2}{R_1 + R_2 + i\omega C R_1 R_2}$$

$$I_2 = V_{BC}/R_2 \Rightarrow re(I_2) = re(V_{BC})/R_2$$
$$P = re(I_2)\, re(V_{BC}) = \langle \{ re(V_{BC}) \}^2 \rangle \frac{1}{R_2}$$
$$= \underbrace{\langle (\cos(\omega t + \phi))^2 \rangle}_{1/2} \frac{R_2 V_0^2}{[(R_1 + R_2)^2 + \omega^2 C^2 R_1 R_2]}$$

3. (a) A mono-energetic, horizontal beam of electrons, each with an energy of 1 electron Volt, is incident on a vertical screen with a horizontal slit of width $D = 10^{-5}$ cm. Make a rough numerical estimate of the vertical width of the pattern of electrons that strike a vertical screen a distance of 1 meter from the screen. Recall that an electron has mass $m = 0.911\ 10^{-27}$ gram, $\hbar = 1.05\ 10^{-27}$ erg sec and 1 eV $= 1.602\ 10^{-12}$ erg. [9 points]




$$\Delta X = D, \qquad \Delta p = \frac{\hbar}{\Delta X} = \frac{\hbar}{D}$$
$$x = 1\,m \times \Delta\theta = 1\,m\, \frac{\Delta p}{p} \approx 1\,m\, \frac{\hbar}{D p}$$

(c) $V_{BC} = V - IR_1$

$$= V\left[1 - R_1 \frac{1 + i\omega c R_2}{R_1 + R_2 + i\omega c R_2 R_1}\right]$$

$$= V \frac{R_2}{R_1 + R_2 + i\omega c R_1 R_2}$$

$$I_2 = V_{BC}/R_2 \Rightarrow re(I_2) = re(V_{BC})/R_2$$

$$P = re(I_2)\, re(V_{BC}) = \langle \{ re(V_{BC}) \}^2 \rangle \frac{1}{R_2}$$

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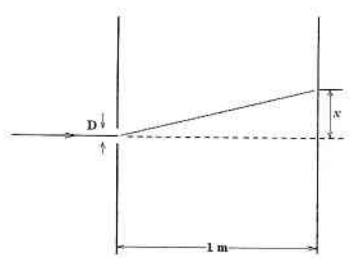


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(a) substitute

$$x = 100\,cm \frac{\hbar}{D\sqrt{2mE}}$$

$$x = 100\,cm \frac{1.05 \times 10^{-27}\ \text{erg sec}}{10^{-5}\,cm \times [2 \times 0.911 \times 10^{-27}\,gm \times 1.602 \times 10^{-12}\,erg]^{1/2}}$$

$$= 0.186\ cm$$

(b) Use the uncertainty principle, $\Delta r\ \Delta p \geq \hbar/2$ to show that the orbit of an electron bound to a proton must have a minimum size, recognizing that the orbiting electron will have energy equal to $p^2/2m - e^2/r$. Estimate the size of the hydrogen atom in terms of the electron's mass ($m$), charge ($e$) and Planck's constant $\hbar$ [8 points]

$$E = \frac{p^2}{2m} - \frac{e^2}{r} \qquad p \sim \frac{\hbar}{r}$$

$$\approx \frac{\hbar^2}{2m} \frac{1}{r^2} - \frac{e^2}{r}$$

minimum when

$$0 = \frac{dE}{dr} = -\frac{\hbar^2}{m} \frac{1}{r^3} + \frac{e^2}{r^2}$$

$$r = \frac{\hbar^2}{me^2} \checkmark$$

(a) substitute

$$x = 100\,\mathrm{cm}\,\frac{\hbar}{D\sqrt{2mE}}$$

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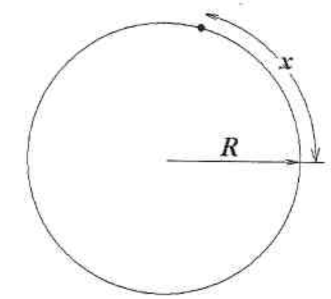
$$r = \frac{\hbar^2}{me^2} \checkmark$$



(c) A particle of mass $m$ and charge q is constrained to move in a circle of radius $R$. The particle is located by giving its position $x$ measured along the circumference. If a uniform magnetic field $B$ is imposed perpendicular to the plane of the circle, the energy operator can be written:




$$H_{\mathrm{op}} = \frac{(p_{\mathrm{op}} - eRB/c)^2}{2m}.$$

Find the allowed energies and the wave functions of the corresponding energy eigenstates for this system. [8 points]

Eigenfunctions of $p_{op}$ will be eigenfunctions of H

$$\psi_n(x) = e^{\frac{2\pi i}{L}nx}\frac{1}{\sqrt{L}} \quad \text{obey}$$

$$p_{op}|\psi_n\rangle \rightarrow -i\hbar\frac{d}{dx}\left[e^{\frac{2\pi i}{L}nx}\right]$$

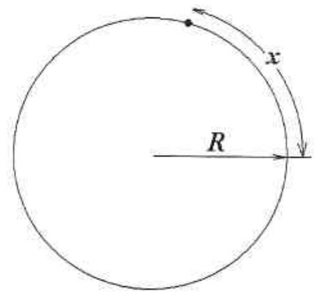
$$= \frac{2\pi}{L}n\hbar\, e^{\frac{2\pi i}{L}nx}$$

and $\psi_n(0) = \psi_n(L)$, $\frac{d\psi_n}{dx}(0) = \frac{d\psi_n}{dx}(L)$

$$E_n = \frac{\left(\frac{2\pi}{L}n\hbar - eRB/c^2\right)^2}{2m}$$

are the allowed energies

(c) A particle of mass $m$ and charge q is constrained to move in a circle of radius $R$. The particle is located by giving its position $x$ measured along the circumference. If a uniform magnetic field $B$ is imposed perpendicular to the plane of the circle, the energy operator can be written:




$$H_{\rm op} = \frac{(p_{\rm op} - eRB/c)^2}{2m}.$$

Find the allowed energies and the wave functions of the corresponding energy eigenstates for this system. [8 points]

Eigenfunctions of $P_{op}$ will be eigenfunctions of H

$\psi_n(x) = e^{\frac{2\pi i}{L} n x} \frac{1}{\sqrt{L}}$ obey

$P_{op}|\psi_n\rangle \to -i\hbar \frac{d}{dx}\left[e^{\frac{2\pi i}{L} n x}\right]$

$= \frac{2\pi}{L} n \hbar \, e^{\frac{2\pi i}{L} n x}$

and $\psi_n(0) = \psi_n(L)$, $\frac{d\psi_n}{dx}(0) = \frac{d\psi_n}{dx}(L)$

$$E_n = \frac{\left(\frac{2\pi}{L} n\hbar - eRB/c\right)^2}{2m}$$

are the allowed energies



4. A particle of mass $m$ moves within a very deep square well $V(x)$ given by

$$V(x) = \begin{cases} V_0 & x \leq 0 \text{ or } L \leq x \\ 0 & 0 \leq x \leq L \end{cases} \qquad (1)$$

where $V_0$ is a very large positive constant.

(a) Write down the energy operator (Hamiltonian) for this system and determine those allowed energies which are much smaller than $V_0$ and the corresponding eigenstates. [8 points]

$$H = \frac{P_{op}^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

Since $V(x)$ has a large discontinuity at $x = 0$ & $L$ so must $\frac{d^2\psi}{dx^2}$ so $\frac{d\psi}{dx}$ can be discontinuous $\lim_{\Delta x \to 0} \frac{\frac{d\psi}{dx}(+\Delta x) - \frac{d\psi}{dx}(-\Delta x)}{\Delta x} = \infty$

$\Rightarrow$ $\psi(x)$ must be continuous

$\Rightarrow$ $\psi(0) = \psi(L) = 0$

Use $\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi}{L} n x\right)$

$$E_n = \frac{(-\hbar^2)\left[-\left(\frac{\pi}{L} n\right)^2\right]}{2m} = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$$

(7)

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(8)

(b) Consider a state which at $t = 0$ has the wave function

$$\psi(x, 0) = N \sin\left(\frac{2\pi}{L}x\right) \cos\left(\frac{\pi}{L}x\right). \qquad (2)$$

Find the normalization constant $N$ needed so that the state $\psi(x, 0)$ has unit probability. [4 points]

(c) If the energy of this state is measured, what results are possible? With what probabilities will each of those results be found? [6 points]

(d) Find the state $\psi(x, t)$ into which the state above evolves at the time $t$. [4 points]

(e) Determine the smallest non-zero value of $t$ for which $\psi(x, t)$ is physically equivalent to $\psi(x, 0)$. [3 points]

(b) Easiest to first express $\psi(x, 0)$ in terms of eigenstates

Recall $\sin(A+B) = \sin A \cos B + \sin B \cos A$

$$\sin\left(\left(\frac{2\pi}{L} + \frac{\pi}{L}\right)x\right) = \sin\frac{2\pi}{L}x \cos\frac{\pi}{L}x + \sin\frac{\pi x}{L} \cos\frac{2\pi}{L}x$$

$$\sin\left(\left(\frac{2\pi}{L} - \frac{\pi}{L}\right)x\right) = \sin\frac{2\pi}{L}x \cos\frac{\pi}{L}x - \sin\frac{\pi x}{L} \cos\frac{2\pi}{L}x$$

$$\therefore \sin\left(\frac{2\pi}{L}x\right)\cos\left(\frac{\pi}{L}x\right) = \frac{1}{2}\sin\left(\frac{3\pi}{L}x\right) + \frac{1}{2}\sin\frac{\pi}{L}x$$

$$\psi(x, 0) = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{2}{L}}\sin\left(\frac{3\pi}{L}x\right) + \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right)\right\}$$

$$\& \quad N = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}} \times 2 = \frac{2}{\sqrt{L}}$$

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$$\& \quad N = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}} \times 2 = \frac{2}{\sqrt{L}}$$

(c) Will find $E_1 = \dfrac{\hbar^2\pi^2}{2mL^2}$ 50% of the time

$E_3 = \dfrac{\hbar^2\pi^2}{2mL^2}\cdot 9$ "

(d)
$$\psi(x,t) = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{2}{L}}\sin\left(\frac{\pi 3}{L}x\right)e^{-i\frac{\hbar\pi^2}{2mL^2}9t} + \sqrt{\frac{2}{L}}\sin\left(\frac{\pi}{L}x\right)e^{-i\frac{\hbar\pi^2}{2mL^2}t}\right.$$

e) Let $\omega = \dfrac{\hbar\pi^2}{2mL^2}$

$t=0$ state will reoccur, up to an overall phase when

$$e^{-i9\omega t} = e^{-i\omega t} \Rightarrow e^{-i8\omega t} = 1$$

or $8\omega t = 2\pi \Rightarrow t = \dfrac{\pi}{4}\dfrac{1}{\omega}$

$$= \frac{\pi}{4}\frac{2mL^2}{\hbar\pi^2} = \frac{mL^2}{2\hbar\pi}$$