

## Assignment #20

**Don't miss the problems on the second and third pages!**

Reading:

*April 5:* French and Taylor 4-1 and 4-2

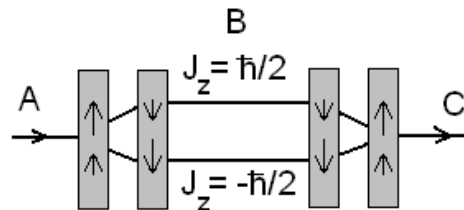
Problems:

166. Purcell 9.8

(Note, the center of the metal box lies on the line  $x = 0, y = 0$ .)

167. French and Taylor 2-5

168. The apparatus shown on the right uses a series of regions of inhomogeneous magnetic field to separate an incoming beam of spin-1/2 neutral atoms into their  $J_z = \pm\hbar/2$  components and then to recombine them back into a single beam. The magnetic



fields are arranged so that the only effect on the quantum state when passing from region A to region B and from region B to region C is the spatial separation shown in the figure. Assume that the atoms in the incoming beam are each in the quantum state with  $J_y = +\hbar/2$ .

- If an additional magnetic field in the  $+z$ -direction is imposed in region B on the atoms passing on the upper path ( $J_z = +\hbar/2$ ) which increases the phase of the states passing on the upper path by the angle  $\phi$ , what will be the resulting average value of  $\vec{J}$  found for the atoms in region C?
- Assume that the atoms have  $\vec{\mu} = \gamma\vec{J}$  with  $\gamma = e/mc$  where  $e$  and  $m$  are the electron's charge and mass. If the atoms are traveling at a speed of  $3 \cdot 10^5$  cm/sec and this additional magnetic field is present for 10 cm of the upper path in region B, use the time translation operator  $\exp(+i\vec{\mu} \cdot \vec{B}t/\hbar)$  to determine how strong an additional magnetic field (measured in Gauss) is needed in this region to produce a change in phase of  $\phi = +\pi/2$  radians for the states passing through the upper path?

169. A spin-1/2 particle with magnetic moment  $\vec{\mu} = \gamma \vec{J}$  is initially in the state

$$\left| +\frac{1}{2} \right\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Use the basis of states  $|\pm \frac{1}{2}\rangle_z$  with  $J_z = \pm \frac{\hbar}{2}$  and  $J_i = \frac{\hbar}{2} \sigma^i$ ,  $i \in \{x, y, z\}$  where  $\sigma^i$  are the three conventional Pauli matrices. A constant magnetic field  $\vec{B} = B\hat{y}$  is imposed in the  $y$  direction.

- (a) Find the resulting state  $|\psi(t)\rangle$  as a function of the time  $t$ .
- (b) Find the average values of the three components of the angular momentum of the particle at the time  $t$ :  $\langle \psi(t) | J_i | \psi(t) \rangle$ ,  $i \in \{x, y, z\}$ .

170. If  $O$  is a hermitian operator, show that for any state  $|\psi\rangle$  the matrix element  $\langle \psi | O^2 | \psi \rangle$  must be non-negative.

171. Consider a hermitian operator  $O$  acting on a complex vector space of finite dimension  $N$ . The following problem outlines a series of steps that can be used to prove that such an operator must possess  $N$  eigenstates which form a basis for that vector space. Thus, when solving this problem you should use only the fact that the operator is hermitian,  $O^\dagger = O$ , but no property stated in class regarding its complete basis of eigenstates.

Consider a “sphere” of states  $|\psi\rangle$  with unit norm,  $\| |\psi\rangle \|^2 = 1$ . Make the reasonable assumption that there should exist a vector  $|\psi_0\rangle$  which lies on this sphere for which the expression,

$$\left( |\psi_0\rangle, O|\psi_0\rangle \right) = \langle \psi_0 | O | \psi_0 \rangle \tag{1}$$

takes a value less than or equal to the value of  $\langle \psi | O | \psi \rangle$  found for any state  $|\psi\rangle$  on this sphere. (Prove the truth of this assumption if you wish.) Consider a second, unit vector  $|\delta\psi\rangle$  which lies in the  $N - 1$  dimensional space of vectors perpendicular to  $|\psi_0\rangle$ :

$$\left( |\delta\psi\rangle, |\psi_0\rangle \right) = \langle \delta\psi | \psi_0 \rangle = 0. \tag{2}$$

- (a) Define the vector

$$|\psi_\epsilon\rangle = \frac{1}{\sqrt{1 + |\epsilon|^2}} \left( |\psi_0\rangle + \epsilon |\delta\psi\rangle \right) \tag{3}$$

where  $\epsilon$  is a complex number. Show that the state  $|\psi_\epsilon\rangle$  also has unit length.

- (b) Since  $|\psi_0\rangle$  is a state of unit length which minimizes the expectation value of  $O$  among those states of unit length we can conclude that

$$\left( |\psi_\epsilon\rangle, O|\psi_\epsilon\rangle \right) \geq \left( |\psi_0\rangle, O|\psi_0\rangle \right). \tag{4}$$

Show that if evaluated to first order in the small complex number  $\epsilon$  this inequality implies that for all states  $|\delta\psi\rangle$

$$\left( |\delta\psi\rangle, O|\psi_0\rangle \right) = 0. \tag{5}$$

- (c) Explain how this result implies that  $O|\psi_0\rangle$  must be proportional to  $|\psi_0\rangle$  or

$$O|\psi_0\rangle = \lambda_0|\psi_0\rangle, \quad (6)$$

if the proportionality constant is labeled  $\lambda_0$ , proving  $O$  has at least one eigenvector.

- (d) Consider the  $N - 1$  dimensional space of vectors that are orthogonal to  $|\psi_0\rangle$  and repeat the above argument to find a second eigenvector  $|\psi_1\rangle$  of  $O$ .
- (e) Repeating the above procedure, deduce that any hermitian  $O$  acting on a finite dimensional vector space of dimension  $N$  can be diagonalized.