Physics UN2802 — Professor Christ

due 04/14/2022

Assignment #21

Reading:

April 12	Quantum Notes pages 1-26
April 14	French and Taylor 4-3

Problems:

172. French and Taylor 3-5

173. A physical "observable" $\mathcal{O}$ is represented in quantum mechanics by an operator O which acts on the vectors of a complex vector space. The set of vectors $|\lambda_n\rangle_{0\le n\le\infty}$ forms a basis for this vector space. Each of these vectors corresponds to a definite value λ_n of the observable $\mathcal{O}$ and obeys:

$$O|\lambda_n\rangle = \lambda_n|\lambda_n\rangle.$$

These states are normalized to unit length and are mutually orthogonal:

$$\begin{aligned} \langle\lambda_n|\lambda_m\rangle &= 1 \quad \text{for} \quad n = m \\ \langle\lambda_n|\lambda_m\rangle &= 0 \quad \text{for} \quad n \neq m. \end{aligned}$$

(a) Consider the quantum state

$$|\psi\rangle = \frac{1}{2}|\lambda_1\rangle + \frac{\sqrt{3}}{2}|\lambda_2\rangle.$$

What is the probability that a measurement of $\mathcal{O}$ will return the value λ_1? The value λ_2? What is the average result that will be obtained if O is measured many times on identical copies of the above state $|\psi\rangle$?

(b) For the quantum state

$$|\psi\rangle = N\sum_{n=0}^{\infty}\frac{a^n}{\sqrt{n!}}|\lambda_n\rangle$$

find the value of the constant N which will insure that this new state $|\psi\rangle$ has unit length: $\langle\psi|\psi\rangle = 1$. Here a is a fixed complex number.
(Hint: Use the Taylor series for the exponential $e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$).

174. Consider a spin-1 particle, again using the basis of eigenstates of J_z now with eigenvalues $+\hbar$, 0 and $-\hbar$. Use the three, 3×3 angular momentum matrices given in problem 164. Assume the particle has magnetic moment $\vec{\mu} = \gamma \vec{J}$ and that a constant magnetic field B is applied in the $+z$ direction.

(a) Initially the particle is in the state with $J_y = +\hbar$. Find that state in the prescribed basis. Assume this is the state of the system at $t = 0$.

(b) Find the resulting state $|\psi(t)\rangle$ as a function of the time t.

(c) Find the average values of the three components of the angular momentum of the particle at the time t: $\langle\psi(t)|J_i|\psi(t)\rangle$, $1 \le i \le 3$.

175. If the quantum state of a particle is given by the wave function

$$\psi(x) = N \begin{cases} 0 & x \le -d \\ d + x & -d \le x \le 0 \\ d - x & 0 \le x \le d \\ 0 & d \le x \end{cases}$$

(a) Determine the constant N so the wave function $\psi(x)$ will obey the normalization condition

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 dx = 1$$

(b) What is the probability that the particle will be found in the interval $-d/2 \le x \le +d/2$?

176. Consider the tensor product of two complex vector spaces D and E with dimensions M and N respectively. Let $|d_i\rangle$, $1 \le i \le M$ and $|e_j\rangle$, $1 \le j \le N$ be orthonormal sets of basis vectors for each of these vector spaces. Thus, vectors u and v in D and E can be written:

$$u = \sum_{i=1}^{M} u_i |d_i\rangle$$
$$v = \sum_{j=1}^{N} v_j |e_j\rangle.$$

Define the $M \times N$ combinations $|d_i\rangle \otimes |e_j\rangle$, $1 \le i \le M$, $1 \le j \le N$ as an orthonormal basis in the tensor product $D \otimes E$ of the vector spaces

D and E. If the product $u \otimes v$ is defined as:

$$u \otimes v = \sum_{i=1}^{M} \sum_{j=1}^{N} u_i v_j |d_i\rangle \otimes |e_j\rangle, \tag{1}$$

show that if w and x are elements of D_N and E_N respectively then the inner product of $u \otimes v$, and $w \otimes x$ is given by:

$$(w \otimes x, u \otimes v) = (w, u) \cdot (x, v). \tag{2}$$

177. Consider a quantum system composed of **two** spin-1/2 particles, call them D and E. A general quantum state ψ can be written:

$$\psi = a_{11}|1\rangle^D \otimes |1\rangle^E + a_{12}|1\rangle^D \otimes |2\rangle^E + a_{21}|2\rangle^D \otimes |1\rangle^E + a_{22}|2\rangle^D \otimes |2\rangle^E \tag{3}$$

where the individual states $|1\rangle^D$, $|2\rangle^D$, $|1\rangle^E$, $|2\rangle^E$ represent particle D with $J_z = +\hbar/2$, particle D with $J_z = -\hbar/2$, particle E with $J_z = +\hbar/2$, particle E with $J_z = -\hbar/2$ respectively.

Next, define the two angular momentum operators $\vec{J}^D$ and $\vec{J}^E$ which affect only the states $|a\rangle^D$ and $|b\rangle^E$ respectively and can be written in terms of Pauli matrices in the usual way. Thus,

$$J_i^D \left(|a\rangle^D \otimes |b\rangle^E\right) = \frac{\hbar}{2} \sum_{k=1,2} \sigma^i_{k,a} (|k\rangle^D \otimes |b\rangle^E) \tag{4}$$

$$J_i^E \left(|a\rangle^D \otimes |b\rangle^E\right) = \frac{\hbar}{2} \sum_{k=1,2} \sigma^i_{k,b} (|a\rangle^D \otimes |k\rangle^E), \tag{5}$$

so that J_i^D affects only the $|a\rangle^D$ vectors and J_i^E affects only the $|b\rangle^E$ vectors. For example:

$$J_y^D \left(|2\rangle^D \otimes |1\rangle^E\right) = \frac{-i\hbar}{2} |1\rangle^D \otimes |1\rangle^E. \tag{6}$$

Finally define the total angular momentum operator $\vec{J}^{\text{tot}} = \vec{J}^D + \vec{J}^E$.

(a) Show that the state $|1\rangle_D \otimes |1\rangle_E$ is an eigenstate of J_z^{tot} with eigenvalue $J_z = \hbar$.

(b) Show by direct evaluation that this state is also an eigenvector of the operator $(\vec{J}^{\text{tot}})^2 = (J_x^{\text{tot}})^2 + (J_y^{\text{tot}})^2 + (J_z^{\text{tot}})^2$ with eigenvalue $2\hbar^2$.

(c) Show by direct evaluation that the state $|1\rangle_D \otimes |2\rangle_E - |2\rangle_D \otimes |1\rangle_E$ is also an eigenstate of the operators J_z^{tot} and $(\vec{J}^{\text{tot}})^2$ with eigenvalue 0 for each.