Physics UN2802 — Professor Christ
due 04/21/2022

Assignment #22

Don't miss the problem on page 2

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Reading:

April 19	French and Taylor 4-3, 8.1-8.6
April 21	French and Taylor 8.7-8.12

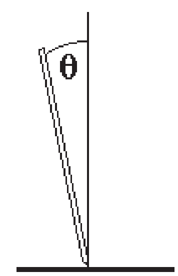
Problems:

178. Show that if the wave function $\psi(x)$ is real then its Fourier transform, $\widetilde{\psi}(p)$ obeys the condition $\widetilde{\psi}(p)^* = \widetilde{\psi}(-p)$. Recall that

$$\widetilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} e^{-ip\,x/\hbar}\, \psi(x)dx.$$

179. Compute the Fourier transform $\widetilde{\psi}(p)$ of the wave function $\psi(x)$ given in problem 175 from Assignment #21.

180. Combine uncertainty principle constraints on the initial conditions with a classical description of the resulting time-dependent motion to estimate the maximum length of time a pencil can be balanced on its point. Simplify the problem by treating the pencil's motion as restricted to a two-dimensional plane. Locate the pencil's position by the angle θ that it makes with the vertical direction.



(a) Make a rough numerical estimate in cgs units of the moment of inertia, I, of the pencil, rotating about its point in the 2-dimensional plane.

(b) Given an initial value of $\theta = \theta_0$ and the initial angular momentum L_0, find the resulting angular position, $\theta(t)$ at a later time t predicted by classical mechanics in the approximation that $\theta(t) \ll 1$. [Hint: this is similar to simple harmonic motion but now calculated about a point of unstable equilibrium, with sinh() and cosh() solutions.]

(c) Determine initial values θ_0 and L_0 which are consistent with the uncertainty relation $\Delta\theta\ \Delta L \geq \hbar$ and yet allow the pencil to remain vertical for as long a time as possible. [Hint: to discuss the long time behavior, the negative exponentials in the sinh() and cosh() solutions can be neglected.]

(d) Approximately, how long can the pencil remain with $\theta < 0.1$?

181. Consider a particle moving in a periodic box of length 20 and unit mass in an appropriate set of units in which $\hbar = 1$. Assume that initially the particle is in a localized state described by the wave function

$$\psi(x) = N \begin{cases} 0 & 0 \leq x \leq x_1 \\ 1 - \cos\left(2\pi \frac{x - x_1}{x_2 - x_1}\right) & x_1 \leq x \leq x_2 \\ 0 & x_2 \leq x \leq 20 \end{cases} ,$$

where $x_1 = 2$ and $x_2 = 4$. Use Python to answer the following questions. It may be convenient to build upon the two Fourier series examples linked to the web site: PythonPage.

(a) Determine the normalization constant N accurate to four decimal places. [Here using the Python routine numpy.quad, introduced in the second Fourier series Python example, to numerically evaluate an integral that you could compute analytically may be the easiest approach.]

(b) Express the wave function $\psi(x)$ as a Fourier series using the quantized momenta $p_m = 2\pi m/L$ where $-100 \leq m \leq 100$. Submit as your answer to this part of the problem a graph of the difference between the exact function $\psi(x)$ and the function obtained from this truncated Fourier series approximation. [As in part (a), the recommended approach is to use numpy.quad to simplify the evaluation of the Fourier coefficients – let the computer do the work for you.]

(c) Introduce the energy operator $H = p^2/2$ and use the time development

$$|\psi(t)\rangle = e^{-i\frac{p_{\text{op}}^2}{2}t}|\psi(t=0)\rangle$$

implied by the Schrodinger equation:

$$i\frac{\partial}{\partial t}\psi(x,t) = \frac{p_{\text{op}}^2}{2}\psi(x,t)$$

to determine $\psi(x,t)$ at the later times $t = 0.2n$ where the integer n varies between 0 and 6. (Recall that the action of p_{op} that appears in the energy operator H is very simple when acting on each term in the Fourier series expansion worked out part (b).) Submit a composite graph plotting $\psi(x)$ for each of these times as your answer to this part of this problem.

(d) Extend the length of the periodic box to 80 and repeat your answer to part (c) but now use the initial wave function

$$\psi_{\text{new}}(x) = e^{ip_0 x}\psi(x),$$

where the constant momentum $p_0 = 60$. [Here we are using the operator $\exp(ip_0 x_{\text{op}}/\hbar)$ to "translate" the average momentum of the state $|\psi\rangle$ from zero to p_0.]