Physics UN2802 Professor Christ
due 04/28/2022

Assignment #23

Reading:

April 26 Suggested: Nielsen & Chuang 1.1-1.4

Problems:

182. A particle of mass m moves in one-dimension under the influence of the finite square-well potential

$$V(x) = \begin{cases} V_0 & x \le -\frac{L}{2} \\ 0 & -\frac{L}{2} \le x \le \frac{L}{2} \\ V_0 & \frac{L}{2} \le x \end{cases} .$$

Because this potential is unchanged when x is replaced by $-x$, the energy eigenstates $\psi_n(x)$ obeying

$$\left\{ \frac{1}{2m}\left(-i\hbar\frac{d}{dx}\right)^2 + V(x) \right\} \psi_n(x) = E_n \psi_n(x) \tag{1}$$

can be chosen to be even or odd: $\psi_n(x) = \pm\psi_n(-x)$. We worked out in class a graphical quantization condition that determined the bound energies $E_n < V_0$ for the case that the wave function is even, $\psi_n(x) = \psi(-x)$. Find the analogous conditions for the odd wave functions, $\psi_n(x) = -\psi(-x)$. Make a sketch of the graphical condition that would apply in this case. Is there always at least one odd bound state as V_0 is made arbitrarily small?

183. Consider a wave function for a quantum state defined in three dimensions which depends on the three variable x, y and z: $\psi(x, y, z)$. Define the three "orbital" angular momentum operators by:

$$\begin{aligned} L_x &= y(-i\hbar\frac{\partial}{\partial z}) - z(-i\hbar\frac{\partial}{\partial y}) \\ L_y &= z(-i\hbar\frac{\partial}{\partial x}) - x(-i\hbar\frac{\partial}{\partial z}) \\ L_z &= x(-i\hbar\frac{\partial}{\partial y}) - y(-i\hbar\frac{\partial}{\partial x}) \end{aligned}$$

   (a) Show that $[L_x, L_y] = i\hbar L_z$

   (b) Show that the quantum state $\psi(x, y, z) = N \exp\{-(x^2 + y^2 + z^2)/4D^2\}$ is a zero eigenvector of L_i for each i:

$$L_i\psi(x, y, z) = 0.$$

   (c) Find the eigenvalues for the operator L_z for each of the states below:

$$\begin{aligned} \psi_A(x, y, z) &= N_A(x + i\,y)e^{-(x^2+y^2+z^2)/4D^2} \\ \psi_B(x, y, z) &= N_B(z)e^{-(x^2+y^2+z^2)/4D^2} \\ \psi_C(x, y, z) &= N_C(x - i\,y)e^{-(x^2+y^2+z^2)/4D^2}. \end{aligned}$$

   where the normalization factors N_A, N_B and N_C are constants.

184. Consider the quantum mechanical simple harmonic oscillator problem with coordinate x, momentum p and Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2. \tag{2}$$

Define the conventional lowering operator:

$$a = \sqrt{\frac{m\omega}{2\hbar}}x + i\frac{1}{\sqrt{2m\omega\hbar}}p. \tag{3}$$

Consider what is known as a coherent state defined in terms of a complex amplitude A:

$$|A\rangle = N \sum_{n=0}^{\infty} \frac{1}{n!}(Aa^\dagger)^n |0\rangle. \tag{4}$$

where $|0\rangle$ is the ground state. Since the state is completely determined by the complex constant A, it is reasonable to use the label 'A' inside the ket vector to identify the state.

(a) Find the normalization factor N needed if $\langle A|A\rangle = 1$.

(b) Show that

$$e^{-i\frac{Ht}{\hbar}}|A\rangle = |Ae^{-i\omega t}\rangle e^{-i\frac{1}{2}\omega t} \tag{5}$$

making the time dependence of this state easy to describe.

(c) Show that $a|A\rangle = A|A\rangle$.

(d) Use the expression for the operator x written in terms of the raising lowering operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger) \tag{6}$$

to evaluate $\langle A|x|A\rangle$.

(e) Beginning with the initial state $|A\rangle$ find the average value of position $\overline{x}(t)$ as a function of time. Show that it undergoes simple harmonic motion and find its amplitude and phase as a function of the complex number A.

185. Consider a particle of charge q and mass m whose one-dimensional motion is determined by the simple harmonic oscillator Hamiltonian:

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2.$$

(a) What is the normalized wave function for the ground state and what are the allowed energies for this system?

(b) An electric field E parallel to the one-dimensional motion is suddenly applied. What is the new Hamiltonian, the new ground state wave function and the new allowed energies?

(c) Assume that the system was initially in its ground state and that the electric field was turned on so suddenly that the system remains in that initial state. What is the probability that a measurement of the energy of the system after the electric field has been applied returns the new ground state value? (Hint: this can be determined by computing the inner product of the ground state wave functions for the initial and final Hamiltonian or by relating the lowering operators for the initial and final systems and exploiting the coherent-state construction developed in the previous problem.)