F1

Final Exam

1. A π^- meson strikes a proton and creates a K^+ meson and a Σ^- baryon:

$$\pi^- + p \to K^+ + \Sigma^-$$

The masses of these particles are $m_\pi^- = 140$, $m_p = 938$, $m_{\Sigma^-} = 1116$ and $m_{K^+} = 494$ all in units of MeV/c^2. What minimum laboratory energy is required for the incoming π^- if the reaction is to take place? (The numerical values for the masses are provided for context. Only an algebraic solution is needed.) [20 points]

Use momentum four vectors for each particle.

- four vectors for initial particles easy in lab system

choose $\hat{x} \parallel \vec{p}_\pi$

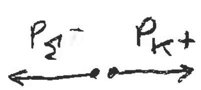
$$P_\pi = (m_\pi v \gamma_v, 0, 0, m_\pi c \gamma_v)$$

$$P_p = (0, 0, 0, m_p c)$$

- four vectors for final particle easy in cm system.

$$P_{\Sigma^-} = (0, 0, 0, m_\Sigma c) \qquad P_{K^+} = (0, 0, 0, m_K c)$$

(smallest total energy result when relative Σ^- & K^+ momentum in cm is zero



F2

#1 continued

Strategy since $P_\pi + P_p = P_\Sigma + P_K$

Their invariant lengths must be equal. and since they are frame-invariant they can be computed in different frames.

- $$(P_\pi + P_p)^2 = P_\pi^2 + P_p^2 + 2P_\pi \cdot P_p$$

$$= -m_\pi^2 c^2 - m_p^2 c^2 - 2\underbrace{m_\pi \gamma_v c}_{E_\pi/c} m_p c$$

- $$(P_\Sigma + P_K)^2 = -(m_\Sigma + m_K)^2 c^2$$

$$m_\pi^2 c^2 + m_p^2 c^2 + 2E_\pi m_p = (m_\Sigma + m_K)^2 c^2$$

$$E_\pi = \frac{(m_\Sigma + m_K)^2 c^2 - m_\pi^2 c^2 - m_p^2 c^2}{2m_p}$$

Intended to be an easy 20 points.

F2

#1 continued

Strategy since $P_\pi + P_p = P_\Sigma + P_K$

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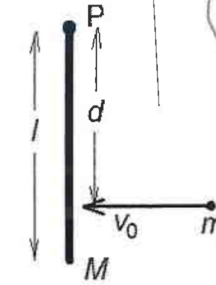
Intended to be an easy 20 points.

F3

2. A thin rod of length l and mass M swings without friction from a fixed pivot P. Initially the rod is vertical and at rest. A second small mass m moving horizontally to the left with velocity v_0 collides elastically with the rod at a distance d below P. After the collision m again moves horizontally but now to the right with velocity v'. The rod is initially at rest but immediately after the collision rotates around P with angular velocity Ω_0.



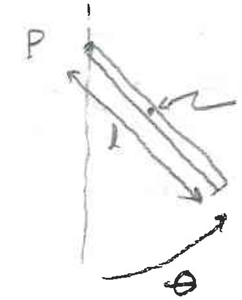
(a) The hanging rod can be described as a physical pendulum. Find the oscillation frequency of this pendulum in the small angle approximation. [5 points]

(b) Find Ω_0 and v'. [25 points]

(c) If θ is the angle between the rod and the vertical direction defined so that $\theta = 0$ before the collision, find the maximum value of θ during the subsequent motion. 5 points]

(d) How long after the collision does θ first reach this maximum value? [5 points]

a) Standard "physical" pendulum problem



$$I_p \ddot{\theta} = -Mg\frac{l}{2}\sin\theta \approx -Mg\frac{l}{2}\theta$$

$$\ddot{\theta} = -\underbrace{\frac{Mgl}{2I}}_{\omega^2}\theta$$

$$I_p = I_{\text{about cm}} + I_{\text{cm about } P}$$

$$= \frac{1}{12}Ml^2 + M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2$$

$$\omega = \sqrt{\frac{Mgl}{2} \cdot \frac{3}{Ml^2}} = \sqrt{\frac{3}{2}\frac{g}{l}}$$

F3

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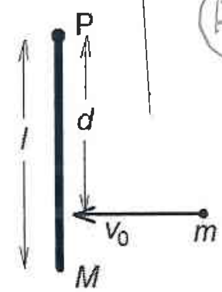

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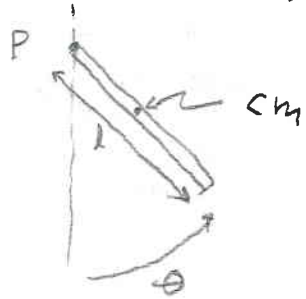

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F4

#2 continued

b) Conservation of angular momentum

$$mvd = mv'd + I\Omega_0 \quad \rightarrow \quad v - v' = \frac{I\Omega_0}{md} \quad (A)$$

Conservation of energy

$$\frac{1}{2}mv^2 = \frac{1}{2}mv'^2 + \frac{1}{2}I\Omega_0^2$$

Two equations and two unknowns, easiest to divide:

$$\frac{mv^2 - mv'^2}{mvd - mv'd} = \frac{I\Omega_0^2}{I\Omega} \rightarrow \frac{v+v'}{d} = \Omega_0 \quad (B)$$

add $\frac{A}{d} + B$: $\frac{2v}{d} = \left[\frac{I}{md^2} + 1\right]\Omega_0$

$$\Omega_0 = \frac{2v}{d}\frac{1}{1+\frac{1}{3}\frac{M}{m}\left(\frac{l}{d}\right)^2} \quad (C)$$

substitude C into B

$$v' = -v + \frac{2v}{1+\frac{1}{3}\frac{M}{m}\left(\frac{l}{d}\right)^2} = \frac{1-\frac{1}{3}\frac{M}{m}\left(\frac{l}{d}\right)^2}{1+\frac{1}{3}\frac{M}{m}\left(\frac{l}{d}\right)^2}v$$

c) $\theta(t) = A\sin\omega t \qquad \dot{\theta}(0) = \underbrace{\omega A\cos(0)}_{\Omega_0}$

$$\theta_{max} = A = \Omega_0/\omega$$

d) $\text{time} = \frac{\tau}{4} = \frac{1}{4}\frac{2\pi}{\omega} = \frac{\pi}{2}\sqrt{\frac{2l}{3g}}$

#2 continued (F4)

b) Consevation of angular momentum

$$mvd = mv'd + I\Omega_o \rightarrow v - v' = \frac{I\Omega_o}{md} \quad (A)$$

Conservation of energy

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Two equations and two unknowns, easiest to divide:

$$\frac{mv^2 - mv'^2 = I\Omega_o^2}{mvd - mv'd = I\Omega} \rightarrow \frac{v+v'}{d} = \Omega_o \quad (B)$$

add $\frac{A}{d} + B$: $\frac{2v}{d} = \left[\frac{I}{md^2} + 1\right]\Omega_o$

$$\Omega_o = \frac{2v}{d}\,\frac{1}{1 + \frac{1}{3}\frac{M}{m}\left(\frac{\ell}{d}\right)^2} \quad (C)$$

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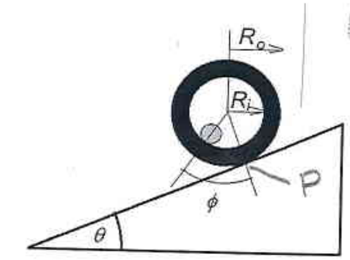
(F5)

3. A cylindrical shell of mass M, outer radius R_o and inner radius R_i rolls without slipping on an inclined plane making an angle θ with the horizontal. A smaller solid cylinder of radius r and mass m slides without friction on the inner surface of the cylindrical shell.

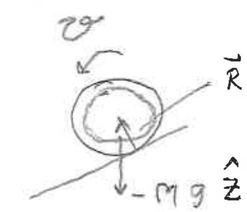


(a) If the inner cylinder is not present, find the acceleration of the center-of-mass of the cylindrical shell. [20 points]

(b) Next include the inner cylinder in the problem and assume that the angle ϕ identified in the figure is chosen so that ϕ does not change as the two cylinders move down the plane. Again find the acceleration of the center-of-mass of the cylindrical shell. [Hint: treat the pair of cylinders as a single object while explaining why such a treatment is justified.] [10 points]

(c) Determine the angle ϕ. [10 points]

a) Use point of contact P



$$I_P\ddot\vartheta = MgR_o\sin\theta$$

$$I_P = I_{cm} + MR^2$$

$$I_{cm} = \int_{R_i}^{R_o} \rho\, r^2\, 2\pi r\, dr$$

$$\rho = \frac{M}{\text{area}} = \frac{M}{\pi(R_o^2 - R_i^2)}$$

$$I_{cm} = \frac{2\pi}{4}(R_o^4 - R_i^4)\frac{M}{\pi(R_o^2 - R_i^2)} = \frac{1}{2}M(R_o^2 + R_i^2)$$

$I_{cm} \longrightarrow \frac{1}{2}MR_o^2$ for $R_i = 0$ ✓

MR_o^2 for $R_i = R_o$ ✓

F5

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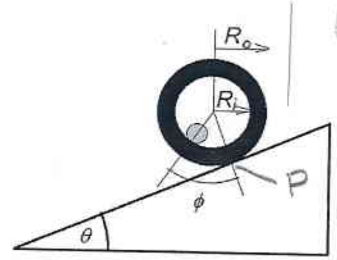

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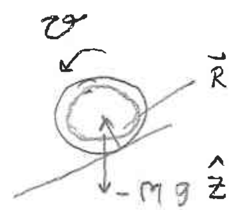

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$$MR_o^2 \quad \text{for } R_i = R_o \quad ✓$$

F6

3 a) continued

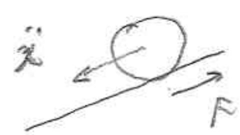
$$I_P = I_{cm} + MR_o^2 = \frac{1}{2}M(3R_o^2 + R_i^2)$$

$$\ddot{x}_{cm} = R\ddot{\vartheta} = \frac{R_o}{I_R} M g R_o \sin\theta$$

$$= \frac{2R_o}{M(3R_o^2 + R_i^2)} M g R_o \sin\theta$$

$$= \frac{2}{3 + (R_i/R_o)^2}\, g \sin\theta$$

b) Since inner cylinder slides w/o friction it exerts no torque about center of hollow cylinder. Don't use P but instead center of hollow cylinder



$$(M+m)\ddot{x} = Mg\sin\theta - F$$

$$I_{cm}\ddot{\vartheta} = R_o F$$

$$(M+m)\ddot{x} = (M+m)g\sin\theta - \frac{1}{2}M(R_o^2 + R_i^2)\frac{1}{R_o}\frac{\ddot{x}}{R_o}$$

$$\ddot{x} = \frac{(M+m)g\sin\theta}{\frac{3}{2}M + m + \frac{1}{2}M\left(\frac{R_i}{R_o}\right)^2}$$

3 a) continued

F6

$$I_P = I_{cm} + M R_0^2 = \frac{1}{2} M (3R_0^2 + R_i^2)$$

$$\ddot{x}_{cm} = R\ddot{\vartheta} = \frac{R_0}{I_R} M g R_0 \sin\theta$$

$$= \frac{2R_0}{M(3R_0^2 + R_i^2)} M g R_0 \sin\theta$$

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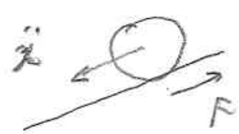
b) Since inner cylinder slides w/o friction it exerts no torque about center of hollow cylinder. Don't use P but instead center of hollow cylinder



$$(M+m)\ddot{x} = M g \sin\theta - F$$

$$I_{cm}\ddot{\vartheta} = R_0 F$$

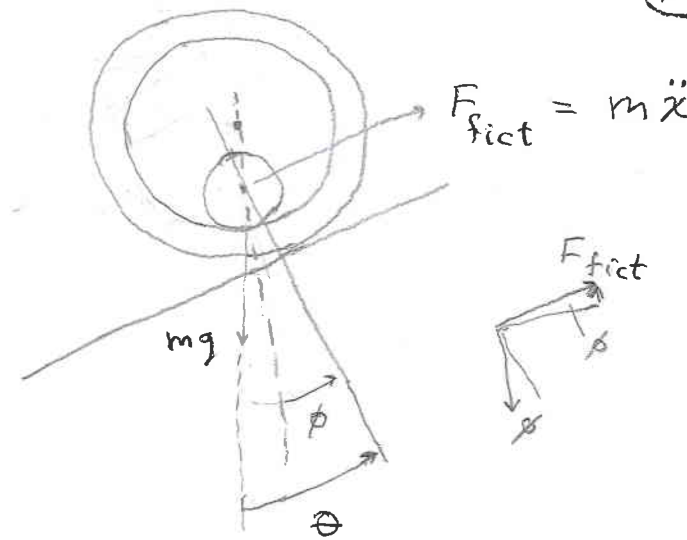
$$(M+m)\ddot{x} = (M+m) g \sin\theta - \frac{1}{2} M (R_0^2 + R_i^2) \frac{1}{R_0} \frac{\ddot{x}}{R_0}$$

$$\ddot{x} = \frac{(M+m) g \sin\theta}{\frac{3}{2} M + m + \frac{1}{2} M \left(\frac{R_i}{R_0}\right)^2}$$

3 continued

F7

c) Require components of $\vec{F}_{fict}$ and $-mg\hat{z}$ ⊥ to ϕ direction cancel



$$\cancel{m} g \sin(\theta - \phi) = \cancel{m} \ddot{x} \cos\phi$$

$$g(\sin\theta\cos\phi - \cos\theta\sin\phi) = \ddot{x}\cos\phi$$

$$\sin\theta - \frac{\ddot{x}}{g} = \cos\theta \tan\phi$$

$$\tan\phi = \left[1 - \frac{\ddot{x}/g}{\sin\theta}\right] \tan\theta$$

$$= \left[1 - \frac{M+m}{\frac{3}{2} M + m + \frac{1}{2} M \left(\frac{R_i}{R_0}\right)^2}\right] \tan\theta$$

$$= \tan\theta \; \frac{\frac{1}{2} M \left[1 + (R_i/R_0)^2\right]}{\frac{3}{2} M + m + \frac{1}{2} M \left(\frac{R_i}{R_0}\right)^2}$$