2. Simple theory of conduction:

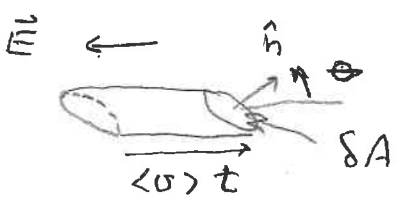
$$\vec{a}_e = -\frac{\vec{E} e}{m_e} \qquad \vec{v}_e(t) = -\frac{\vec{E} e}{m_e} t$$

However, $\vec{v}_e(t)$ does not increase forever - e^- collides with something & $\vec{v}_e(t)$ is randomized. If τ_{col} is the average time between collisions

$$\langle \vec{v}_e \rangle = \frac{\vec{E} e}{m_e} \frac{\tau_{col}}{2}$$

If the is a density n_e of such conduction electrons per unit volume, a current or flux of charge $\vec{J}(\vec{r}) = +e n_e \frac{\vec{E} e}{m_e} \frac{\tau_{col}}{2}$ results

$$= \underbrace{\frac{n_e e^2 \tau_{col}}{2 m_e}}_{\sigma \,=\, \text{conductivity}} \vec{E}(\vec{r})$$



charge passing through δA in time t

$$= -e n_e V = -e n_e \delta A \cancel{\cos\theta} \langle v \rangle t$$

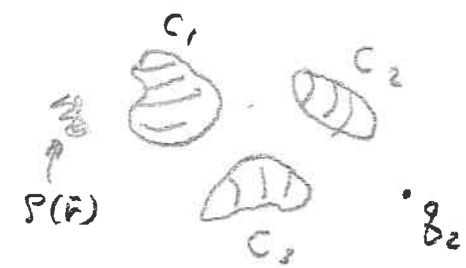
$$= +e n_e \frac{E e \tau_{col}}{2 m_e} \times \delta A \cancel{\cos\theta}\, t = \vec{J} \cdot \hat{n}\, \delta A\, t$$

$\vec{J}(\vec{r}) = \sigma \vec{E}(\vec{r})$ is the most basic form of Ohm's law

01/20/22

For electrostatics $\vec{J} = 0$ so Ohm's law requires $\vec{E}(\vec{r}) = \frac{1}{\sigma}\vec{J} = 0$ for $\vec{r}$ inside a conductor.

Now consider a new type of problem: Given the location of charges and conductors find $\vec{E}(\vec{r})$ everywhere.



This looks hard because we must determine how the charge is distributed on the conductors to make $\vec{E}(\vec{r}) = 0$ inside each and then compute resulting $\vec{E}(\vec{r})$ outside conductors!

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For problem to be well defined we must specify for example the value of $\Phi(\vec{r})$ on each conductor.

3. A good way to state such a problem with K conductors C_i $1 \le i \le K$, a given charge density $\rho(\vec{r})$ for $\vec{r}$ outside of the conductors and $\Phi(r) = \Phi_i$ on the surface of the conductor C_i:

"Solve Poisson's equation $\vec{\nabla}^2 \Phi = -4\pi\rho$ for $\vec{r}$ outside of the conductors at $\Phi(\vec{r}) = \Phi_i$ on C_i." [Also require $\Phi(\vec{r}) \to 0$ as $|\vec{r}| \to \infty$.]

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This looks obsure but is easy to solve numerically by relaxation and can often be guessed using a uniqueness theorem.:

There is only one function obeying there 3 conditions

① $\vec{\nabla}^2 \Phi(\vec{r}) = \rho(\vec{r}) 4\pi$ $\vec{r}$ outside all C_i

② $\Phi(\vec{r}) = \Phi_i$ for $\vec{r}$ on surface of C_i

③ $\lim_{|\vec{r}| \to \infty} \Phi(\vec{r}) = 0$

<u>Proof:</u> Imagine two functions $\Phi_1(\vec{r})$ and $\Phi_2(\vec{r})$ obey conditions ①, ② + ③ then $\delta\Phi(\vec{r}) = \Phi_1(\vec{r}) - \Phi_2(\vec{r})$ obeys

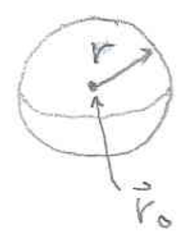
① $\vec{\nabla}^2 \delta\Phi = \vec{\nabla}^2\Phi_1 - \vec{\nabla}^2\Phi_2 = 4\pi(\rho(\vec{r}) - \rho(\vec{r})) = 0$

② $\delta\Phi(r) = 0$ for r on C_i

$\delta\Phi(\vec{r}) \to 0$ as $|\vec{r}| \to \infty$

prove $\delta\Phi(\vec{r}) = 0$ everywhere + $\Phi_1(\vec{r}) = \Phi_2(\vec{r})$

Based on an important property of Laplace equation solution



$$\frac{1}{4\pi r^2} \int_{\substack{\text{surface}\\ \text{of sphere}}} \Phi(\vec{r})\, dS = \Phi(\vec{r}_0)$$

↑ center of sphere

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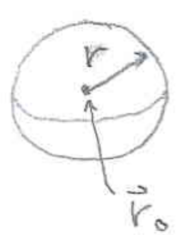

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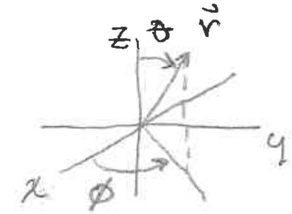
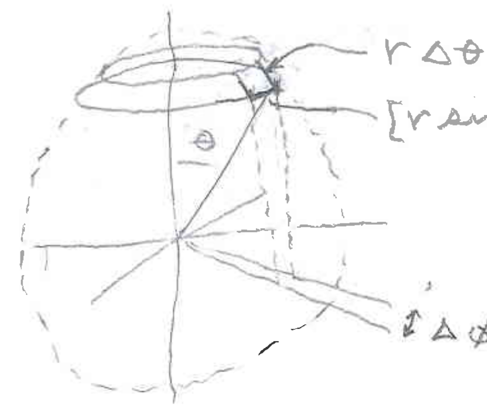
Define $A(r) = \frac{1}{4\pi r^2} \int_{\text{surface}} \Phi(\vec{r})\, dS$

Introduce polar coordinated to make the integral more concrete



replace $(x, y, z) \to (r, \theta, \phi)$

Express area in terms of $\Delta\theta$ + $\Delta\phi$



area $= r\Delta\theta \; r\sin\theta\Delta\phi$

$$A(r) = \frac{1}{4\pi r^2} \int \Phi(r, \theta, \phi)\, r\sin\theta\, d\phi\, r\, d\theta$$

$$= \frac{1}{4\pi} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\; \Phi(r, \theta, \phi) \sin\theta$$

$$\frac{d}{dr} A(\vec{r}) = \frac{1}{4\pi} \int_0^{\pi} d\theta \int_0^{2\pi} d\phi\; \frac{\partial \Phi}{\partial r}(r, \theta, \phi) \sin\theta$$

Define $A(r) = \frac{1}{4\pi r^2} \int_{\text{surface}} \varphi(\vec{r})\, ds$

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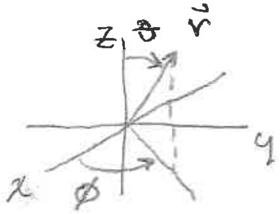

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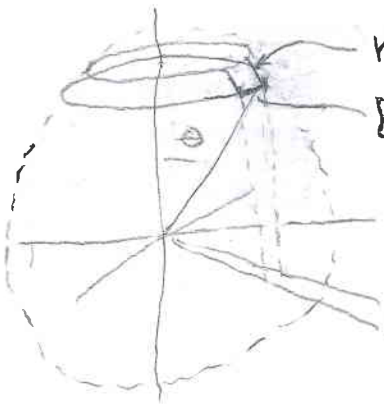
Expres area in terms of $\Delta\theta$ + $\Delta\phi$



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note $\frac{\partial \varphi}{\partial r} = \hat{r} \cdot \vec{\nabla} \varphi$

[recall $\Delta r \frac{\partial \varphi}{\partial r}(\vec{r}) = \varphi(\vec{r} + \hat{r}\Delta r) - \varphi(\vec{r}) = \Delta r\, \hat{r} \cdot \vec{\nabla}\varphi$]

Thus $\frac{dA}{dr}(r) = \frac{1}{4\pi r^2} \int_{\partial S} \vec{\nabla}\varphi \cdot \hat{n}\, dS$

$$= \frac{1}{4\pi r^2} \int_S \vec{\nabla}^2 \varphi\, dV = 0$$

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$$= \varphi(r_0) \frac{1}{4\pi} 2\pi \underbrace{\int_0^{\pi} d\theta \sin\theta}_{\int_{-1}^{1} d(\cos\theta) = 2}$$

$$= \varphi(r_0) \quad \text{Q.E.D.}$$

$\delta\varphi(\vec{r}) = \varphi_1(\vec{r}) - \varphi_2(\vec{r}) = 0$ every where is now easy:

$$\varphi(r_0) = \frac{1}{4\pi r^2} \int_{\partial S \text{ centered at } r_0} \varphi(r)\, ds$$

implied neither a local maximum or minimum of $\varphi(\vec{r})$ can appear at a point in the volume between the conductors.

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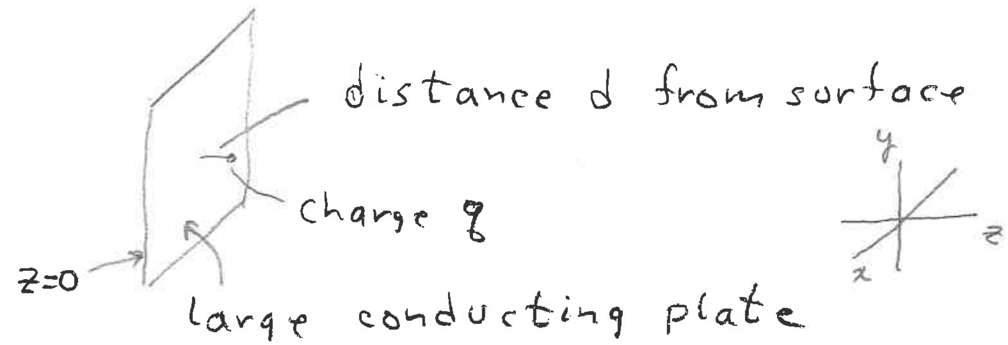
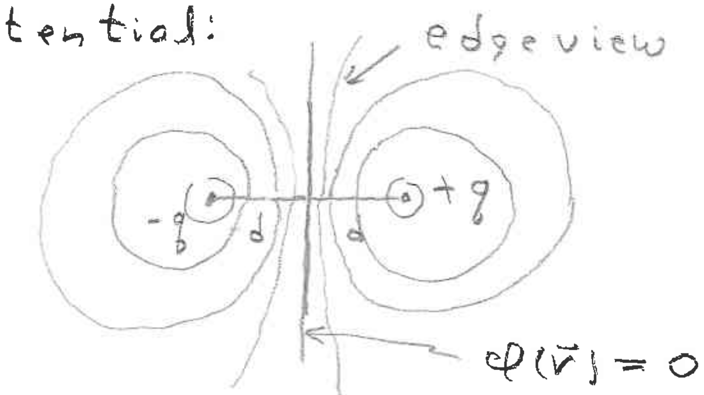
Therefore both the largest and smallest values of $\delta\varphi(\vec{r})$ must occur on the surface of a conductor or at ∞ where $\delta\varphi(r) = 0$!

4. Solve famous problem:



Find $\varphi(\vec{r})$ everywhere.

Easily solved by recognizing a second problem which matches that above for $z \geq 0$ and makes the $z=0$ plane an equipotential:



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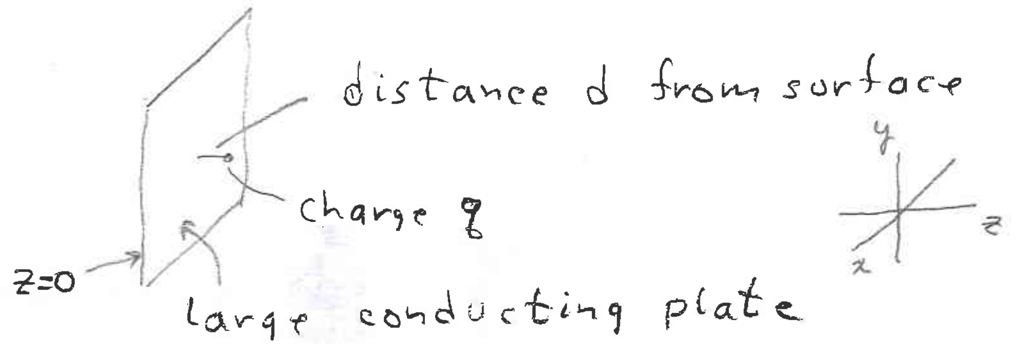

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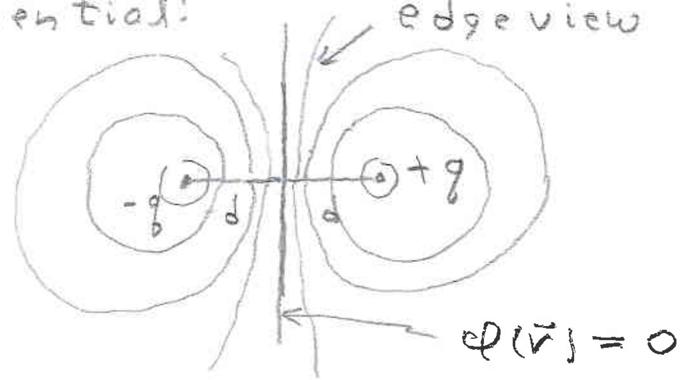
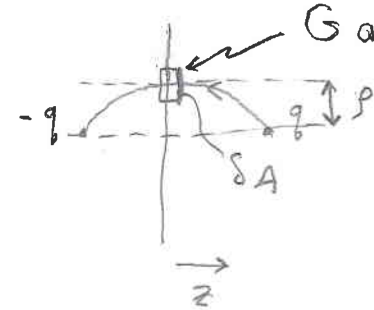


Solution
$$\varphi(\vec r) = -\frac{q}{|\vec r + d\hat z|} + \frac{q}{|\vec r - d\hat z|}$$
$$|\vec r + d\hat z| = \sqrt{x^2+y^2+(z+d)^2} \qquad |\vec r - d\hat z| = \sqrt{x^2+y^2+(z-d)^2}$$

These cancel and $\varphi(\vec r) = 0$ when $z = 0$

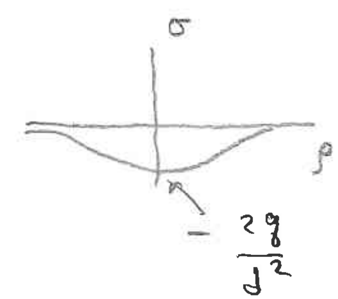
Find the surface charge density on the plate:



$$E_z = -\frac{\partial}{\partial z}\left[\frac{-q}{\sqrt{\rho^2+(z+d)^2}} + \frac{q}{\sqrt{\rho^2+(z-d)^2}}\right]$$
$$= -\left\{\frac{+q(z+d)}{[\rho^2+(z+d)^2]^{3/2}} - \frac{q(z-d)}{[\rho^2+(z-d)^2]^{3/2}}\right\}_{z=0}$$
$$= -\frac{2qd}{(\rho^2+d^2)^{3/2}}$$

$$\int_{\partial V} \vec E\cdot\hat n\, dS = 4\pi\int \rho(r)\, dV$$
$$-\frac{2qd}{\rho 3}\,\delta A = 4\pi\,\sigma\,\delta A$$

$$\sigma = -\frac{1}{2\pi}\,\frac{qd}{[\rho^2+d^2]^{3/2}}$$



Solution $\Phi(\vec{r}) = -\dfrac{q}{|\vec{r}+d\hat{z}|} + \dfrac{q}{|\vec{r}-d\hat{z}|}$

$|\vec{r}+d\hat{z}| = \sqrt{x^2+y^2+(z+d)^2}$, $\quad |\vec{r}-d\hat{z}| = \sqrt{x^2+y^2+(z-d)^2}$

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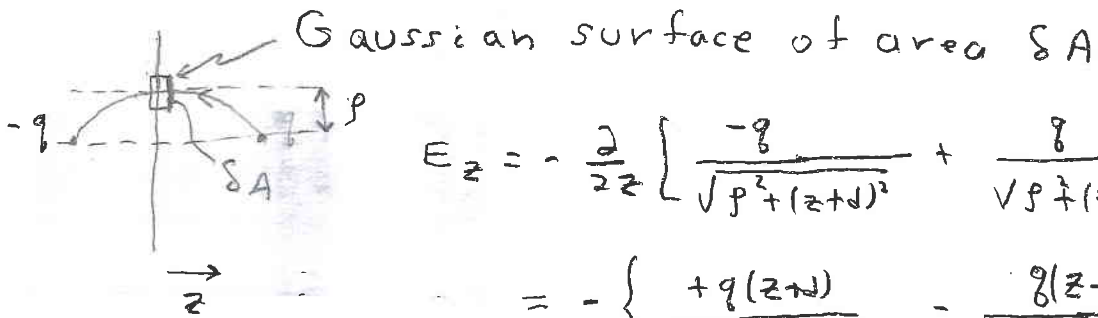

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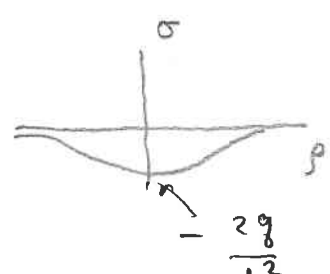
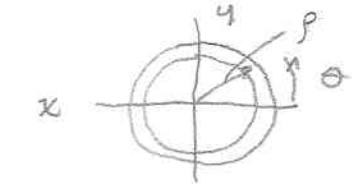


Finally calculate the total charge on the plate

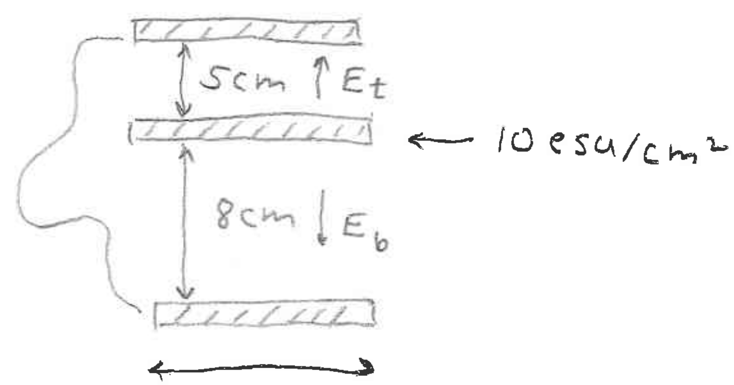


$$Q = \int \sigma(x,y)\, dx\, dy$$

$$= -\frac{qd}{2\pi}\int_0^\infty \frac{1}{(\underbrace{\rho^2+d^2}_{u})^{3/2}}(2\pi\rho)\, d\rho$$

$$= -qd\int_{d^2} \frac{1}{2}\frac{du}{u^{3/2}} = -qd\,\frac{1}{2}\cdot 2\frac{1}{(d^2)^{1/2}} = -q$$

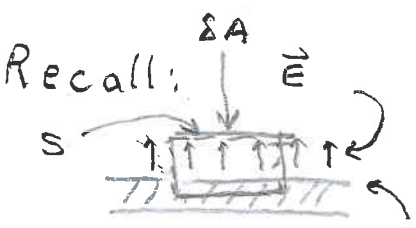
4. Solve an interesting problem from Purcell (# 3.8)



Top and bottom plate have total charge 0 but are joined by a wire

this dimension should be very large (not drawn to scale)

Recall:



$$\delta A\, E = 4\pi\sigma\,\delta A$$

$$E = 4\pi\sigma$$

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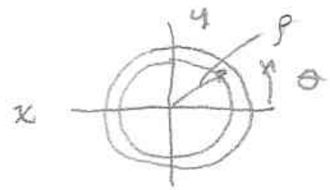

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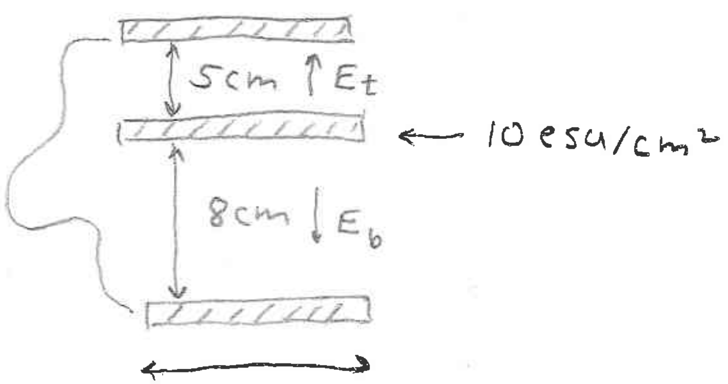

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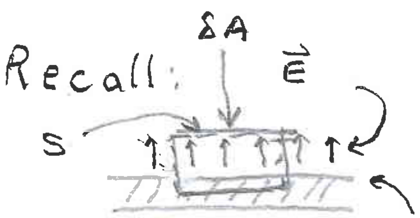

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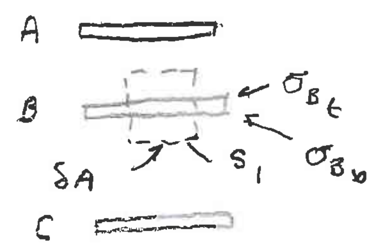
$$E = 4\pi\sigma$$

Use Gauss' law and properties of conductors



$$V_B - V_A = E_t\, 5\text{ cm}$$

$$V_B - V_C = E_b \cdot 8\text{ cm}$$

$$V_A = V_C$$

$$E_t \cdot 5 = E_b\, 8$$

Gauss' law for S_1 ⇒

$$4\pi\, \delta A\, \sigma_B = (E_t + E_b)\, \delta A$$

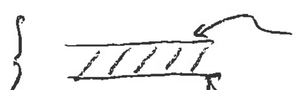
$$4\pi \sigma_{B_t} = E_t \qquad 4\pi \sigma_{B_b} = E_b \qquad \sigma_{B_t} + \sigma_{B_b} = \sigma_B = 10\text{ esu}$$

$$\sigma_t = \frac{8}{5}\sigma_B \qquad \sigma_{B_t} + \sigma_{B_b} = 10$$

$$\frac{8}{5}\sigma_{B_b} + \sigma_{B_b} = 10 \qquad \sigma_{B_b} = \frac{5}{13} \times 10$$

$$\sigma_{B_t} = \frac{8}{13} \times 10$$

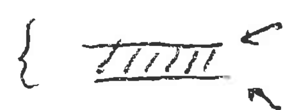
A



$$\sigma_{A_t} = \sigma_{B_t} + \Delta\sigma$$

$$\sigma_{A_B} = -\frac{1}{4\pi} E_t = -\sigma_{B_t}$$

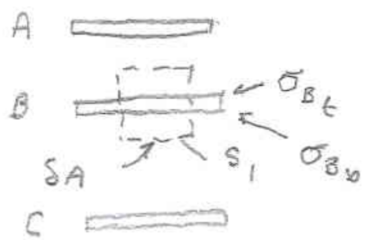
C



$$\sigma_{C_t} = -\frac{1}{4\pi} E_b = -\sigma_{B_b}$$

$$\sigma_{C_b} = +\sigma_{B_b} - \Delta\sigma$$

Use Gauss' law and properties of conductors



$V_B - V_A = E_t\ 5\,\text{cm}$

$V_B - V_C = E_b \cdot 8\,\text{cm}$

$V_A = V_C$

$E_t \cdot 5 = E_b\ 8$

Gauss' law for $S_1 \Rightarrow$

$$4\pi\,\delta A\,\sigma_B = (E_t + E_b)\,\delta A$$

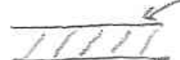
$4\pi\sigma_{B_t} = E_t \qquad 4\pi\sigma_{B_b} = E_b \qquad \sigma_{B_t} + \sigma_{B_b} = \sigma_B$

$= 10$ esu

$\sigma_t = \frac{8}{5}\sigma_B \qquad \sigma_{B_t} + \sigma_{B_b} = 10$

$\frac{8}{5}\sigma_{B_b} + \sigma_{B_b} = 10 \qquad \sigma_{B_b} = \frac{5}{13}\times 10$

$\sigma_{B_t} = \frac{8}{13}\times 10$

A $\Big\{$ 

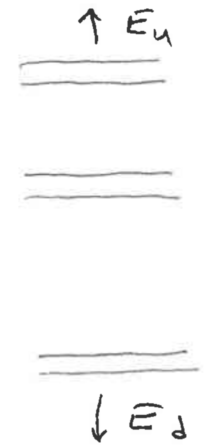
$\sigma_{A_t} = \sigma_{B_t} + \Delta\sigma$

$\sigma_{A_B} = -\frac{1}{4\pi}E_t = -\sigma_{B_t}$

C $\Big\{$ 

$\sigma_{C_t} = -\frac{1}{4\pi}E_b = -\sigma_{B_b}$

$\sigma_{C_b} = +\sigma_{B_b} - \Delta\sigma$

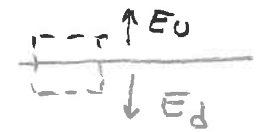


$E_U = (\sigma_{B_t} + \Delta\sigma)4\pi$

$E_d = (\sigma_{\sigma_{B_b}} - \Delta\sigma)4\pi$

How do we determine $\Delta\sigma$?

Expect at a distance system should look like



← $\sigma = 10$ esu

with $E_U = E_d$ & $E_U + E_d = 4\pi \cdot 10$ esu

$E_U = E_d = \frac{10\text{ esu}}{2\pi}$

$\therefore \quad \sigma_{B_t} + \Delta\sigma = 5$ esu

$\sigma_{B_b} - \Delta\sigma = 5$ esu

$\frac{5}{13}\times 10 + \Delta\sigma = 5\text{ esu} \Rightarrow \Delta\sigma = \overbrace{\left(\frac{1}{2} - \frac{5}{13}\right)}^{\frac{3}{26}}\times 10$

$\frac{8}{13}\times 10 - \Delta\sigma = 5\text{ esu} \Rightarrow \Delta\sigma = \left(\frac{8}{13} - \frac{1}{2}\right)\times 10$

If $E_u \neq E_d$ then $E_u = \frac{5\text{ esu}}{4\pi} + \Delta E$

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and we have applied extra uniform $\vec{E} = \Delta E\,\hat{z}$ not specified in problem.

$$E_u = (\sigma_{B_t} + \Delta\sigma)\, 4\pi$$

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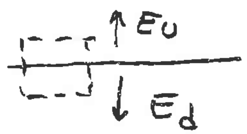

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E Capacitance

1. Given N conducts C_i $1 \leq i \leq N$ each carrying charge Q_i and at the potential Φ_i, how are the Q_i and Φ_i related?

(choose $\lim_{r \to \infty} \Phi(\vec{r}) = 0$)

Solve N different problems:

k^{th} problem has solution $\Phi^{(k)}(\vec{r})$

obeying: (1) $\vec{\nabla}^2 \Phi^{(k)}(r) = 0$ outside conductors

(2) on the i^{th} conductor $\Phi^{i}(r) = \delta_{ik} \Phi_k$

(3) $\lim_{|\vec{r}| \to \infty} \Phi^{(k)}(\vec{r}) = 0$

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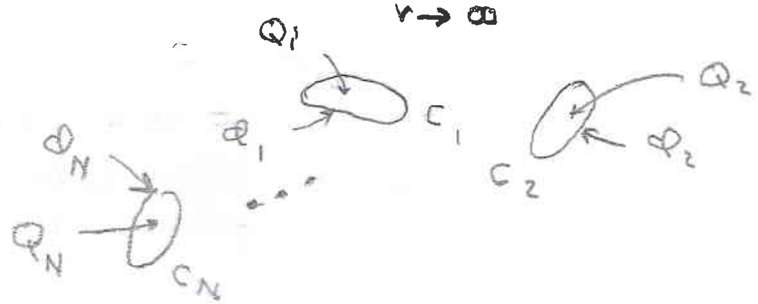

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③ $\lim_{|\vec{r}|\to\infty} \varphi^{(k)}(\vec{r}) = 0$

If $Q_i^{(k)}$ is the charge on the i^{th} conductor for the k^{th} problem,

$$Q_i^{(k)} = \frac{1}{4\pi}\int_{\partial C_i} \vec{E}^{(k)}(\vec{r})\cdot\hat{n}\, dS$$

$$= \frac{-\frac{1}{4\pi}\int_{\partial C_i} \vec{\nabla}\varphi^{(k)}(\vec{r})\cdot\hat{n}\, dS}{\varphi_k} \times \varphi_k$$

$$\underbrace{\qquad\qquad}_{\text{must not depend on } \varphi_k} \longrightarrow C_{ik}$$

Solve our general problem by superimposing these N problems

$$\varphi(\vec{r}) = \sum_{k=1}^{N} \varphi^{(k)}(\vec{r})$$

$$\varphi(r)\Big|_{\vec{r}\in C_i} = \sum_k \varphi^{(k)}(\vec{r})\Big|_{r\in C_i} = \varphi_i$$

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Thus if we can compute the N^2 coefficients of capacitances C_{ik} the charges Q_i are linearly related to the potentials φ_k

$$Q_i = \sum_{k=1}^{N} C_{ik}\varphi_k$$

2. Find the capacitance of a conducting sphere of radius R:

$$Q_1 = C_{11}\varphi_1$$

$$\varphi(\vec{r}) = \begin{cases} \frac{Q}{r} & r \geq R \\ \frac{Q}{R} & r \leq R \end{cases}$$

For a hollow sphere,

$$\varphi_1 = \frac{Q_1}{R} \quad \text{or} \quad C_{11} = R$$