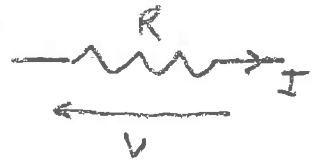
$$\text{Ohm} = \frac{\text{Volt}}{\text{Coulomb/sec}} = \frac{(1/300)\,\text{statvolt sec}}{3\times10^{9}\ \text{esu}}$$

$$= \frac{1}{3\times10^{11}}\ \frac{\text{sec}}{\text{cm}}$$

Now the esu unit of $\frac{\text{sec}}{\text{cm}}$ is impractically large, In general use



$V = IR$

2. Circuit theory - introduce discrete elements with two terminals,

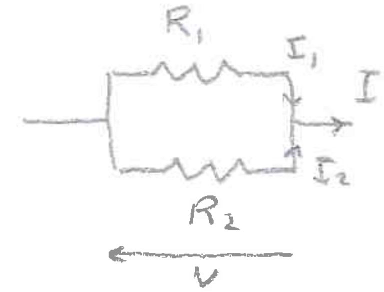
a) Connect with high conductivity wires So far we have

b) note

$V_1 = IR_1$, $V_2 = IR_2$

$$V = V_1 + V_2 = I\underbrace{(R_1+R_2)}_{R}$$

parallel



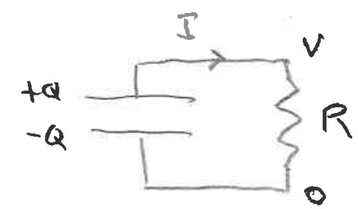
$\underline{I = I_1 + I_2}$

$$I = \frac{V}{R_1} + \frac{V}{R_2} = \left[\frac{1}{R_1} + \frac{1}{R_2}\right]V = \frac{V}{R}$$

$$\therefore \frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$$

01/27/22

c) Combine R & C



Assuming the conductivity of the wire is large the same potential difference is found between the terminals of C as betwen those of R

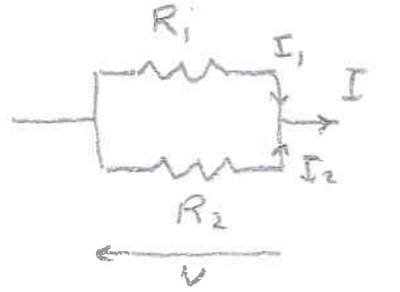
$$V = \frac{Q}{C}, \quad V = IR$$

Since the total charge in a closed system never changes

$$I = -\frac{dQ}{dt}$$

$$-\frac{dQ}{dt} = I = \frac{V}{R} = \frac{Q}{RC} \Rightarrow Q(t) = Q_0 e^{-\frac{t}{RC}}$$

parallel



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01/27/22

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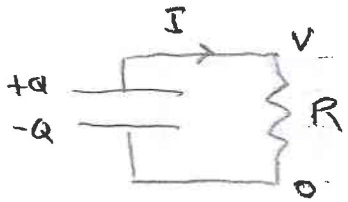

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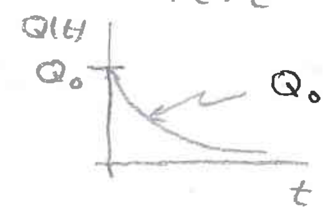
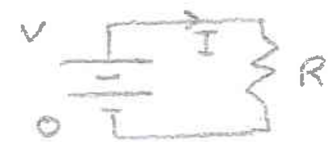
Where $Q(t=0) = Q_0$



$Q_0 e^{-t/RC}$ An "RC" circuit

Units $RC \sim \left(\frac{sec}{cm}\right) \cdot cm = sec$

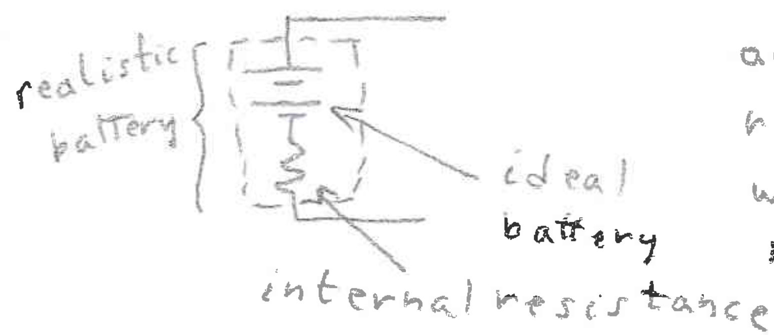
d) Add a battery



$I = V/R$

Flow of charge is powered by chemical reactions taking place in battery.

For an ideal battery V is fixed and does not change with I. A more realistic description would add an "internal" resistance in series with an ideal battery



Such a souce of energy powering a flow of current is said to provide an "electromotive force".

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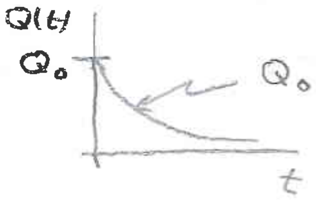

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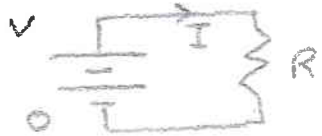

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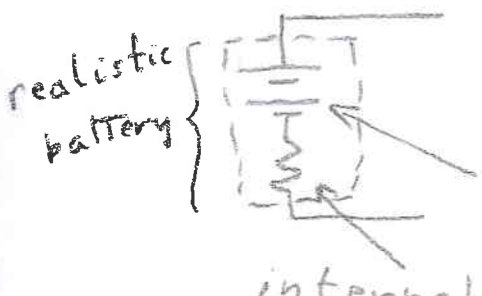

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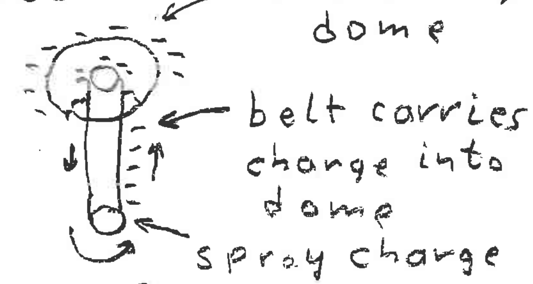
Recognize two types of EMF:

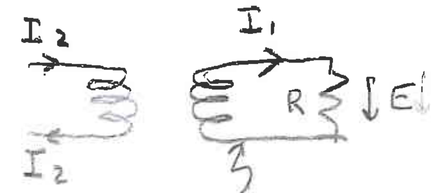
① $\oint \vec{E} \cdot d\vec{r} = 0$ and charge is forced to move against $\vec{E}$ field by quantum mechanics

Battery



Van de Graaf generator

② $\oint \vec{E} \cdot d\vec{r} = -\frac{1}{c}\frac{d\Phi_B}{dt}$



$$EMF = M_{12}\frac{dI_2}{dt} = I_1 R$$

Here $\vec{\nabla} \times \vec{E} \neq 0$ and force is not conservative!

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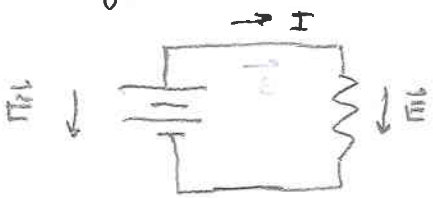

battery

conducting dome

belt carries charge into dome

spray charge onto moving belt

Van de Graaf generator

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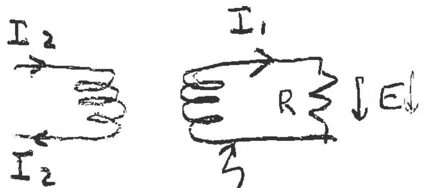


$EMF = M_{12}\frac{dI_2}{dt}$

$= I_1 R$

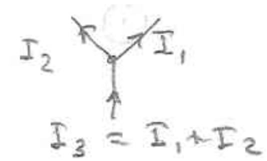
wire with large σ - no $\vec{E}$ in wire!

Here $\vec{\nabla}\times\vec{E} \neq 0$ and force is not conservative!

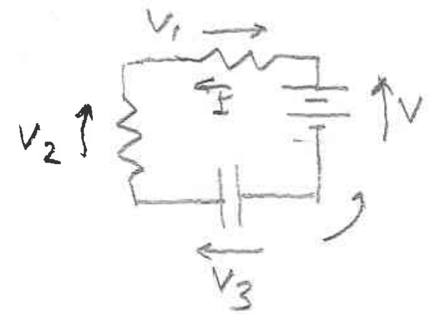
Circuit theory: a variation of electrostatics

- all circuit elements remain electrically neutral
- Charge is allowed to flow
- currents can change with time but not too fast!

Result is Kirchoff's laws



1. Sum of all currents entering a node is zero



2. Sum of voltage differences around every closed loop is zero.

$V - V_1 - V_2 - V_3 = 0$

Solve two problems:

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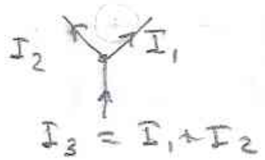

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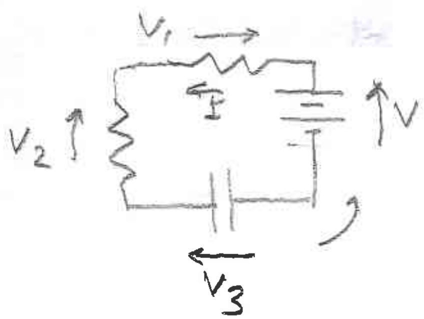

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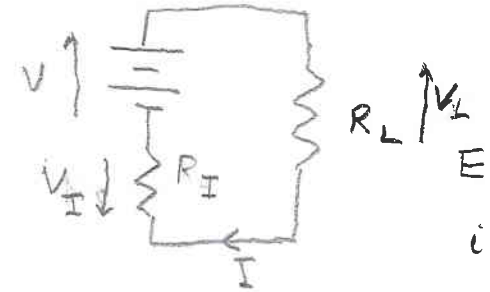
① Given a battery with internal resistances R_I what value of a load resistance R_L will result in the largest dissipation of energy in R_L

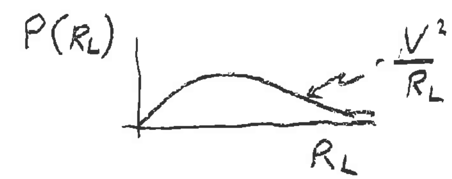


$$I(R_I + R_L) = V$$

Energy transferred to R_L in time Δt

$$\Delta U = \Delta Q\, V_L = (I \Delta t)\, I R_L$$

power lost $\quad P = \dfrac{\Delta U}{\Delta t} = I^2 R_L = \dfrac{V^2 R_L}{(R_I + R_L)^2}$



O

$$0 = \frac{dP}{dR_L} = V^2 \left\{ \frac{1}{(R_I + R_L)^2} - \frac{2R_L}{(R_I + R_L)^3} \right\}$$

$$= \frac{V^2}{(R_I + R_L)^3} (R_I - R_L)$$

$\Rightarrow R_L = R_I$ "impedance matching"

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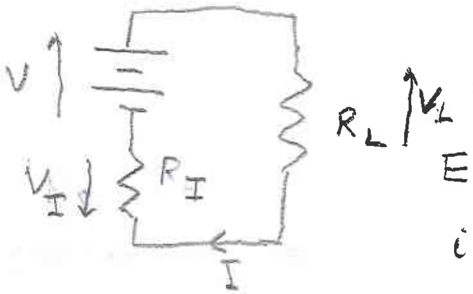

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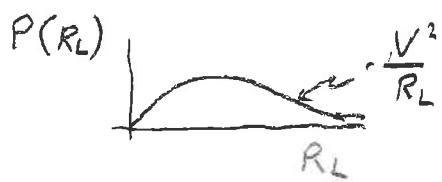

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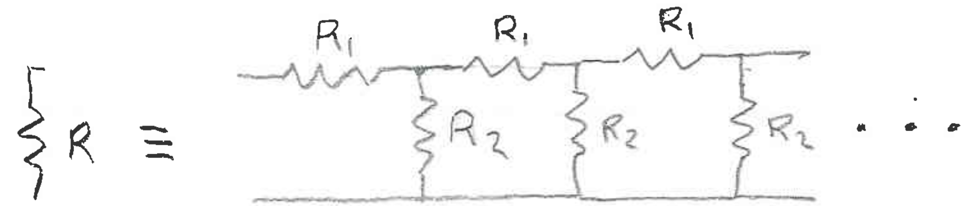
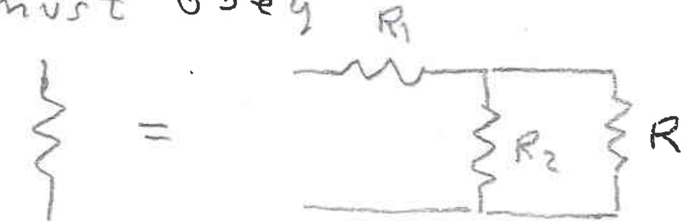
(2) Find the equivalent resistance of the infinite series of resistors



R must obey



Since adding an extra rung to an infinite ladder can have no effect

Apply V to both and equate current

$$R = R_1 + \frac{1}{\frac{1}{R} + \frac{1}{R_2}} = R_1 + \frac{R R_2}{R + R_2}$$

$$R(R + \cancel{R_2}) = R_1(R + R_2) + \cancel{R R_2}$$

$$R^2 - R R_1 - R_1 R_2 = 0$$

$$R = \frac{R_1 + \sqrt{R_1^2 + 4R_1R_2}}{2} = R_1\left\{\frac{1}{2} + \left[\frac{1}{4} + \frac{R_2}{R_1}\right]^{1/2}\right\}$$

$\to R_1$ as $R_2 \to 0$ ✓

$\to \infty$ as $R_2 \to \infty$ ✓

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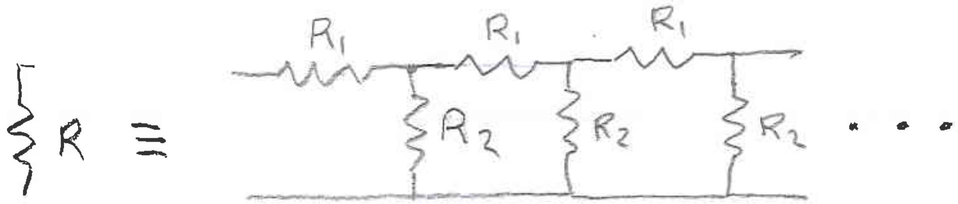

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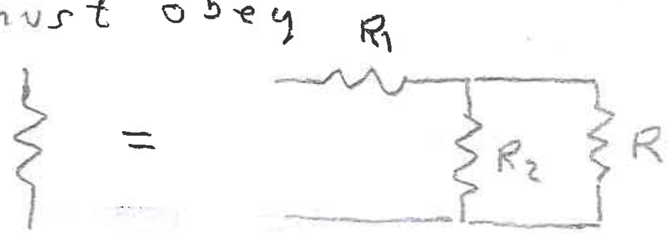

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V Electrodynamics

A "Motivation"

1. Overview

(1) $\vec{\nabla} \cdot \vec{E} = 4\pi\rho$ True in general!

$\vec{\nabla} \times \vec{E} = 0$ True only when charges move slowly

$\downarrow$

(2) $\vec{\nabla} \times \vec{E} + \frac{1}{c}\frac{\partial \vec{B}}{\partial t} = 0$ Faraday's law

$\frac{d\vec{p}}{dt} = q\vec{E}(\vec{r})$ Newton's law for slowly moving charge at $\vec{r}$.

$\downarrow$

Lorentz force law $\rightarrow$ $\frac{d\vec{p}}{dt} = q\vec{E}(\vec{r}) + q\frac{\vec{v}}{c} \times \vec{B}$

now $\vec{p} = m\gamma_v \vec{v}$

Need more equations to determine $\vec{B}$:

(3) $\vec{\nabla} \cdot \vec{B} = 0$

(4) $\vec{\nabla} \times \vec{B} - \underbrace{\frac{1}{c}\frac{\partial \vec{E}}{\partial t}}_{\text{Maxwell's displacement current}} = \underbrace{4\pi \vec{J}}_{\text{Ampere's law}}$

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Equations ①, ②, ③, ④ are Maxwell's equations – the complete classical theory (no quantum effects) of electricity and magnetism including light!

Discovered over ~100yrs from experiment and theoretical insight. A fascinating story which we will skip in order to emphasize the requirements of special relativity

2. Assume $\vec{\nabla}\cdot\vec{E}(\vec{r}) = 4\pi\rho(\vec{r})$ requires no modification. Is it consistent with relativity?

$$\int_{\partial V}\vec{E}(\vec{r},t)\cdot\hat{n}\,dS = 4\pi\int_V \rho(\vec{r},t)\,dV$$

appears to be inconsistent with causality! Charge $\rho(\vec{r},t)$ inside V

$\Rightarrow$ $\vec{E}(\vec{r}',t)$ changes instantaneous on the surface!?

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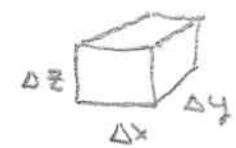
appears to be inconsistent with causality! Change $\rho(\vec{r},t)$ inside V $\Rightarrow$ $\vec{E}(\vec{r}',t)$ changes instantaneous on the surface!?

However, this inconsistency can be resolved if we require that electric charge is conserved. Q_{tot} can only change if current flows through the surface and $\vec{E}(\vec{r},t)\cdot\hat{n}$ can change there too no need for "action at a distance".

In equations:

$$\frac{d}{dt}\int_V \rho(\vec{r},t)\,dV = -\int_{\partial V} \vec{J}(\vec{r},t)\cdot\hat{n}\,dS$$

This can be written as a nice local equation if V is a small box



$\times\,\Delta x\,\Delta y\,\Delta z$

$$\frac{d}{dt}\,\rho(\vec{r},t)\,\Delta x\Delta y\Delta z = -\left(\frac{\partial J_x}{\partial x} - \frac{\partial J_y}{\partial y} - \frac{\partial J_z}{\partial z}\right)$$

or $\dfrac{\partial\rho}{\partial t} + \dfrac{\partial J_x}{\partial x} + \dfrac{\partial J_y}{\partial y} + \dfrac{\partial J_z}{\partial z} = 0$

the "continuity equation"

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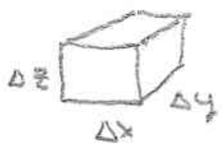

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the "continuity equation"

We can learn two important things from the continuity equation

1. $\vec\nabla\cdot\vec E = 4\pi\rho$ need not be inconsistent with causality
2. $(j_x, j_y, j_z, c\rho)$ transform under a charge of coordinate system the the components of a four vector.

Proof of 2

$$0 = \vec\nabla\cdot\vec j + \frac{\partial \rho}{\partial t} = \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z} - \left(-\frac{\partial}{\partial ct}\,c\rho\right)$$

must be true in all inertial systems and $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, -\frac{1}{c}\frac{\partial}{\partial t}\right)$ transforms as a four-vector. This implies $(\vec j, c\rho)$ must also be a four-vector

---

Why is $\left(\vec\nabla, -\frac{1}{c}\frac{\partial}{\partial t}\right)$ a four-vector?

familiar 3-vector

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familiar 3-vector

?

Minus sign comes from minus sign in construction of invariant length! Assume $f(x)$ is a scalar function. Then

$$f(x+\delta x) - f(x) =$$

$$\left[\frac{\partial}{\partial x_1}f(x)\right]\delta x_1 + \left[\frac{\partial}{\partial x_2}f(x)\right]\delta x_2 + \left[\frac{\partial}{\partial x_3}f(x)\right]\delta x_3 - \left[-\frac{\partial}{\partial ct}f(x)\right]\left[c\delta t\right]$$

must also be a scalar and therefore a dot product between two four-vectors. Since $(\delta\vec{x}, ct)$ is a four vector $\left(\vec{\nabla}, -\frac{\partial}{\partial ct}\right)$ must also transform as a four-vector. Useful language:

x_μ is a contravariant vector

$\frac{\partial}{\partial x_\mu}$ is a covariant vector

(with $x_4 = ct$)

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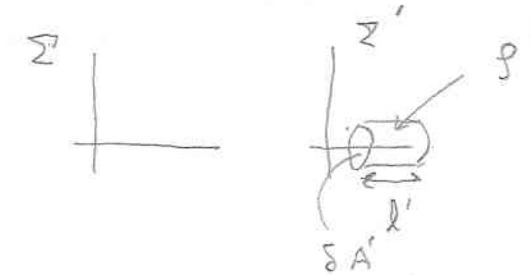
(with $x_4 = ct$)

Knowing how $(\vec{J}, c\rho)$ transform we can determine how charge transforms:



in Σ'

$Q' = \delta A' l' \rho'$

$\vec{J}' = 0$

Σ calculates the charge Q in the moving volume

$$Q = l\,\delta A\,\rho = \frac{l'}{\gamma}\,\delta A'\,\rho$$

where $c\rho = \gamma(c\rho' + \frac{v}{c}\underset{0}{\underset{\shortparallel}{J'}}) = c\gamma\rho'$

$$\therefore Q = \frac{l'}{\gamma}\delta A'\,\gamma\rho' = Q'$$

and charge is Lorentz invariant!

We began with the idea that charge must be conserved $\sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} J_\mu = 0$ and concluded that charge was a Lorentz scalar - this is too strong!

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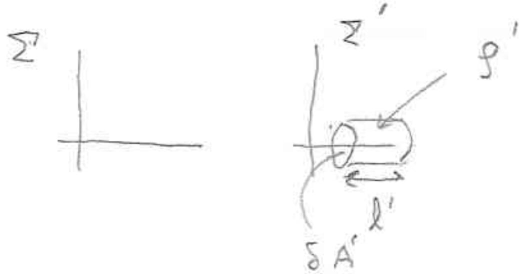

in Σ'

$Q' = \delta A' \ell' \rho'$

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$$\& \quad Q = \frac{\ell'}{\gamma} \delta A' \gamma \rho' = Q'$$

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$\sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} j_\mu = 0 \Rightarrow j_\mu$ is a 4-vector is the simpest possibility. There could be an extra index:

$$\sum_{\mu=1}^{4} \frac{\partial}{\partial x_\mu} j_{\mu\nu} = 0 \qquad \nu = 1, 2, 3, 4$$

extra index

This would require 4 types of "charges" not one - inconsist with Nature.

However, if we are not discussing charge but momentum and energy there __ARE__ four quantities and

$T_{\mu,4}(\vec{r})$ is the vector of currents ($\mu = 1,2,3$) and density of energy/c

an $T_{\mu,i}(\vec{r})$ is the vector of currents ($\mu = 1,2,3$) and density of momentum in the i^{th} direction

$T_{\mu\nu}$ is the energy-momentum tensor