Physics C2801 Professor Christ
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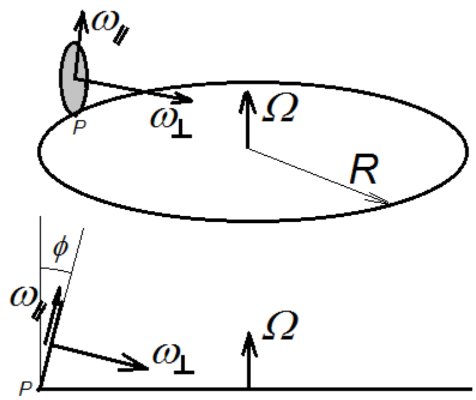
A solution to problem 7.6 without approximation

This diagram shows the coining rolling around a circle of radius R.



Since the motion of the coin has been specified, the problem can be solved in three steps:

1. Determine the angular velocity $\vec{\omega}(t)$ of the coin implied by the specified motion.

2. Use the angular velocity $\vec{\omega}(t)$ to determine the angular momentum $\vec{L}(t)$.

3. Require that $d\vec{L}/dt$ equal the torque $\vec{\tau}$ acting on the coin.

Here are these three steps:

1. The total angular velocity $\vec{\omega}$ can be viewed as a sum of the angular velocity about the axis perpendicular to the coin, $\vec{\omega}_\perp$, and the angular velocity $\vec{\omega}_\|$ along an axis parallel to the face of the coin passing through the point of contact P and the coin's center. These can be determined as follows:

   (a) View the coin as rotating with angular velocity $\vec{\omega}$ about the point of contact P. Define a vector $\vec{b}$ connecting the point of contact P to the center of mass of the coin.

   (b) Since $\vec{v} = \vec{\omega} \times \vec{b}$, the component of $\vec{\omega}$ perpendicular to $\vec{b}$ must have magnitude v/b. Thus, $|\vec{\omega}_\perp| = v/b$. (The parallel component, $\vec{\omega}_\|$ does not enter this cross product.)

   (c) This determines the angular velocity $\vec{\omega}$ except for the component $\vec{\omega}_\|$ that is parallel to $\vec{b}$ and cannot affect $\vec{\omega} \times \vec{b}$.

   (d) Since the vector $\vec{\omega}_\perp$ is fixed to the coin and thus must also rotate with $\vec{\omega}$:

   $$\frac{d\vec{\omega}_\perp}{dt} = \vec{\omega} \times \vec{\omega}_\perp = \vec{\omega}_\| \times \vec{\omega}_\perp. \tag{1}$$

   This equation is tricky to interprete because the vector $\vec{\omega}_\|$ itself also depends on time.

(e) However, $\vec{\omega}_\perp$ will perform a complete revolution about a vertical axis in the time $2\pi(R - b\sin(\phi))/v$. Thus, it precesses with a vertical angular velocity $\vec{\Omega}$:

$$\vec{\Omega} = \frac{v}{R - b\sin(\phi)}\hat{z}, \tag{2}$$

where $\hat{z}$ is an upward-pointing unit vector so

$$\frac{d\omega_\perp}{dt} = \vec{\Omega} \times \vec{\omega}_\perp. \tag{3}$$

Equating the right hand sides of Eqs. 1 and 3 we can determine $\omega_\parallel$:

$$\omega_\parallel \omega_\perp = \Omega \omega_\perp \cos(\phi). \tag{4}$$

or

$$\omega_\parallel = \Omega \cos(\phi) = \frac{v\cos(\phi)}{R - b\sin(\phi)}. \tag{5}$$

2. Since we now know the vector $\vec{\omega}$ we can compute $\vec{L}$.

(a) Recognizing that the moments of inertia about the perpendicular and parallel directions about the center of mass of the coin are given by $I_\perp = Mb^2/2$ and $I_\parallel = Mb^2/4$ we can determine $\vec{L}$ about the center of mass of the coin:

$$\vec{L}_{cm} = Mb^2\left(\frac{1}{2}\vec{\omega}_\perp + \frac{1}{4}\vec{\omega}_\parallel\right). \tag{6}$$

(b) If we plan to calculate the torque about the point of contact P, then we need the angular momontum about P which comes in the usual way from adding to $\vec{L}_{cm}$ the angular momemtum of the CM about P:

$$\vec{L}_P = Mb^2\left(\frac{3}{2}\vec{\omega}_\perp + \frac{1}{4}\vec{\omega}_\parallel\right). \tag{7}$$

3. Finally we equate the torque exerted by gravity about P and $d\vec{L}_P/dt$.

(a) As a vector equation:

$$\vec{b} \times (-Mg\hat{z}) = \vec{\Omega} \times \vec{L}_P. \tag{8}$$

(b) Since the cross product with $\vec{\Omega}$ picks out the horizontal components of $\vec{L}_\perp$ and $\vec{L}_\parallel$ we can easily equate the magnitudes of both sides of Eq. 8:

$$Mgb\sin(\phi) = \Omega Mb^2\left(\frac{3}{2}\omega_\perp \cos(\phi) + \frac{1}{4}\omega_\parallel \sin(\phi)\right) \tag{9}$$

(c) Substituting for Ω, $\omega_\perp$ and $\omega_\parallel$ and cancelling M gives the following equation for ϕ:

$$\sin(\phi) = \frac{v^2}{g(R - b\sin(\phi))}\left(\frac{3}{2}\cos(\phi) + \frac{1}{4}\frac{b\cos(\phi)\sin(\phi)}{R - b\sin(\phi)}\right). \tag{10}$$

Thus, the equation $\tan(\phi) = 3v^2/2gR$ holds when $R \gg b$.