Muddled Information*

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Abstract

We study a model of signaling in which agents are heterogeneous on two dimensions. An agent’s natural action is the action taken in the absence of signaling concerns. Her gaming ability parameterizes the cost of increasing actions. Equilibrium behavior muddles information about both dimensions. As incentives to take higher actions increase—due to higher stakes or more easily manipulated signaling technology—more information is revealed about gaming ability, and less about natural actions. A new externality emerges: showing agents’ actions to additional observers can worsen information for existing observers. We discuss applications to credit scoring, school testing, and web search.

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1. Introduction

In many signaling environments, there is a concern that agents’ gaming can lead to “muddled” information. Educators worry that rich students have better access to SAT tutoring and test preparation than do poor students, and so the test may be a flawed measure of underlying student quality. Indeed, in March 2014, the College Board announced plans to redesign the SAT test, in part to “rein in the intense coaching and tutoring on how to take the test that often gave affluent students an advantage.” (New York Times, 2014) Google tries to prevent undesirable search engine optimization from contaminating the relevance of its organic search results. The Fair Isaac Corporation keeps its precise credit scoring formula secret to make it more difficult for consumers to game the algorithm.

In canonical signaling models (e.g. Spence, 1973), the only welfare cost from gaming, if any, is through an increase in costly effort.\(^1\) Even though gaming may induce an inefficient rat race, it does not ultimately lead to a reduction in the information available to the market. In the absence of restrictions that may be viewed as artificial, there are typically equilibria that fully reveal agents’ private information under standard conditions, viz., when the Spence-Mirrlees single-crossing assumption holds.

Motivated by the aforementioned applications, this paper studies how gaming can worsen market information. We develop a model of signaling in which agents have two-dimensional types. Both dimensions affect an agent’s cost of sending a one-dimensional signal. The first dimension is an agent’s natural action, which is the action taken in the absence of signaling concerns.\(^2\) The second dimension is an agent’s gaming ability, which parameterizes the costs of increasing actions beyond the natural level. In the credit scoring application, the signal is an agent’s credit score; the natural action is the score the agent would obtain if this score would not be disseminated; and gaming ability determines how costly it is for an agent to increase her score. In the testing application, the natural action is the test score a student would receive without studying, and gaming ability captures how easily the student can increase her score by studying.

We assume that agents care about influencing a market’s belief about their quality on one of the two dimensions, which we refer to as the dimension of interest.\(^3\) Situations abound in

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\(^1\) By “gaming” we simply mean strategic behavior, which in a signaling context is an agent’s choice of what signal to send. More broadly, concerns about gaming have been studied in other contexts; for example, Ederer et al. (2014) recently address it in a multi-tasking moral-hazard problem.

\(^2\) Throughout the paper, we use the terms “signal” and “action” synonymously.

\(^3\) More generally, some combination of beliefs on both dimensions could be relevant. We illustrate in
which the dimension of interest is the natural action. For example, people with higher natural credit scores default less often on loans; the credit market does not care about gaming ability because this trait merely reflects one’s knowledge about how to manipulate credit scores. Similarly for search engine optimization, where higher natural actions correspond to more relevant web pages. Yet, there are contexts in which the dimension of interest is the gaming ability. In the testing environment, gaming ability would not be of interest to the market if it solely represents “studying to the test”, but colleges or employers might value gaming ability if it correlates with the ability to study more broadly. Or, in a job-market signaling model, gaming ability may be correlated with intelligence and work ethic, while the natural action—the amount of education that would be acquired if it were irrelevant to job search—may capture a dimension of preferences for schooling that is unrelated to job performance.\footnote{We explore how the combination of heterogenous gaming ability and natural actions interact in determining the market’s information. In our formulation, detailed in Section 2, each dimension of an agent’s type—natural action or gaming ability—satisfies a single-crossing property. Thus, the effects of heterogeneity on any one dimension alone are familiar. Indeed, if we were to assume homogeneity of natural actions and the dimension of interest to be gaming ability, then our model would be similar to a canonical signaling environment such as Spence (1973). If instead gaming ability were homogeneous and the dimension of interest were the natural action, then our model would share similarities with, for example, Kartik et al. (2007). In both cases, full separation would be possible.}

With two dimensions of heterogeneity, the market is typically faced with muddled information. Even though the market would like to evaluate an agent on his natural action (or gaming ability), the information revealed about this dimension of an agent’s type is muddled with irrelevant information about his gaming ability (or natural action). While agents who take higher actions will tend to have both higher natural actions and higher gaming ability, any observed action will generally not reveal either dimension. Intermediate actions might come from an agent with a high natural action and a low gaming ability, an agent with a low natural action and a high gaming ability, or an agent who is in-between on both.

\footnote{An agent’s gaming ability is sometimes also isomorphic to her private benefits from influencing the market’s beliefs, as elaborated subsequently. Heterogeneous private benefits may represent productivity differences, which in many contexts the market would obviously care about (e.g. Esteban and Ray, 2006).}

\footnote{Other signaling models with heterogeneity in natural actions include Bernheim (1994) and Bernheim and Severinov (2003). There is also a parallel in the literature on earnings management, wherein a market is assumed to observe a firm’s reported earnings but not its “natural earnings” (e.g. Stein, 1989).}

Section 5 how our main themes can be extended to the more general case.
One contribution of this paper is to identify a key relationship among cross types—pairs in which one has higher natural action but lower gaming ability than the other. For any such pair, we posit that the ordering by marginal cost of signaling turns on the level of signaling. Our central assumption, formalized in Assumption 1 (part 4), is that at low levels of signaling, differences in marginal cost are driven by differences in natural actions. At higher levels of signaling, they depend more and more on differences in gaming ability. In other words, as more gaming occurs, gaming ability becomes relatively more important in determining signaling costs.

The core of our analysis concerns comparative statics on equilibrium information on the dimension of interest. We establish that when agents’ incentives to take high actions increase—because the costs of signaling go down, for instance, or because the stakes in signaling go up—the muddled information reveals more about an agent’s gaming ability and less about her natural action. Hence, as a search engine like Google becomes more popular and the stakes for web sites to game its algorithm grow, Google searches can become less informative in equilibrium even after Google adjusts its algorithm to account for this extra gaming. We believe the mechanism captured by our model is compelling and robust. Moreover, our analysis clarifies that higher stakes do not always lead to less information. Rather, they simultaneously generate less information on one dimension and more information on the other. We uncover the general properties of signaling costs required for these comparative statics.

Section 3 illustrates the basic mechanics of the model, while Section 4 establishes the comparative statics precisely in a canonical “two-by-two” setting and provides general results for small and large signaling stakes. Section 5 develops a linear-quadratic-elliptical specification: signaling benefits are linear in the market belief; costs are quadratic; and the types are jointly elliptically distributed. This specification affords a sharp equilibrium characterization and additional comparative-statics results.

In Section 6 we consider the value of giving agents more information about how to manipulate signals, for example by making the inner working of the signaling technology more transparent. A more transparent algorithm will lower the costs of signaling for all agents, increasing the incentives to take higher actions. Therefore, when the dimension of interest is the natural action, the market becomes less informed as the algorithm is made more transparent. This analysis explains why evaluators often try to obscure the details of their evaluation metrics, such as the College Board keeping past SAT questions secret for many
years: it improves the informativeness of its test. (If the dimension of interest were gaming ability, our result would reverse, and one would want to make algorithms more transparent.)

It bears emphasis that it is not gaming per se that reduces information about natural actions; for example, if web sites were all equally prone to engage in search engine optimization, then their efforts could wash out and leave observers well informed. Rather, muddled information is driven by the fact that there is unobservable heterogeneity across agents in how prone they are to gaming. This provides an explanation for why, in addition to announcing changes to the SAT itself in March 2014, the College Board also announced provision of free online test preparation to “level the playing field”. Such a policy disproportionately helps those with low intrinsic gaming ability (i.e., poor families). By reducing heterogeneity on gaming ability, it should improve market information about natural actions.

In Section 7 we explore a novel tradeoff in making a signal available to new observers. The marginal new observers get information they didn’t previously have. However, with more observers tracking her actions, an agent’s stakes in signaling have grown. At higher stakes, the signal becomes less informative about the natural action. So there is a negative informational externality on those inframarginal observers who already had access to the signal. In the context of credit scoring, allowing employers and insurance companies to use credit reports will improve information in those markets, but at a cost of reducing the information available in the loan market. Among other results, we show that the social value of information across markets can decline after the signal is made available to new markets.

Muddled information—information loss on the dimension of interest owing to presence of other dimensions of private information—is not a new phenomenon in signaling environments. See, among others, Austen-Smith and Fryer (2005), Bénabou and Tirole (2006), Esteban and Ray (2006) and Bagwell (2007) in the economics literature, and Dye and Sridhar (2008) and Beyer et al. (2014) in the earnings-management accounting literature. As already mentioned, our main contribution is in developing the comparative statics of market

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6 Indeed, in discussing an update to Google’s ranking algorithm to combat undesirable search engine optimization, Google engineer Matt Cutts (2012) has blogged that “a few sites use techniques that don’t benefit users ... [Google] can’t divulge specific signals because we don’t want to give people a way to game our search results and worsen the experience for users.” (emphasis added)

7 Early work on signaling with multidimensional types (Quinzii and Rochet, 1985; Engers, 1987) establishes the existence of fully separating equilibria under suitable “global ordering” or single-crossing assumptions. As already noted, our model satisfies single-crossing within each dimension but not globally. See Araujo et al. (2007) for a multidimensional-type model in which it is effectively as though single-crossing fails even within a single dimension, which leads to “counter-signaling” equilibria. Feltovich et al. (2002), Chung and Esö (2013), and Sadowski (2013) make related points in models with a single-dimensional type.
information when it is muddled.

The closest antecedent in this respect is the innovative work of Fischer and Verrecchia (2000). They study a linear-quadratic-normal model (also seen in Bénabou and Tirole (2006)) that is related to our linear-quadratic-elliptical specification in Section 5. Their motivation is a manager’s report of firm earnings given private information on both true earnings (analogous to our natural action) and her own objectives (analogous to our gaming ability). Among other things, Fischer and Verrecchia show how “price efficiency”—the information on true earnings contained in reported earnings—changes with parameters. For reasons elaborated in Section 5, we cannot use normal distributions. Inspired by Gesche’s (2015) work in progress, we use elliptical distributions with bounded support; we study equilibria when agents’ payoffs depend on market beliefs not only about their natural action but also about their gaming ability. The linear-quadratic-normal and -elliptical specifications are appealing in their tractability. In particular, they yield a scalar measure of information which one can combine with explicit equilibrium calculations to deduce comparative statics. However, owing to the limitations of functional form assumptions and algebraic calculations, we believe that a proper understanding of the underlying forces requires a more general analysis based on more fundamental assumptions. We aspire for Sections 2 through 4 of the current paper to elucidate these more general forces.

We should note that there are arguments for incomplete revelation of information even when agents have one-dimensional types satisfying single crossing. Separation may be precluded if there are bounds on the signal space, in which case there can be bunching at the edges of the type space (Cho and Sobel, 1990). However, this does not seem relevant for applications such as school testing or credit scores where few people have perfect scores. Indeed, if bunching at the edges is ever a problem, it may be possible to simply expand the signal space: a test can be made more difficult. On the other hand, there are critiques of the focus on separating equilibria even when these exist (Mailath et al., 1993); recently, Daley and Green (2013) note that separating equilibria need not be strategically stable when the market exogenously receives sufficiently precise information about the agents. Another rea-

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8 See Section 5 for details on the relationship between the linear-quadratic-normal and -elliptical specifications of these papers. Among the other papers, only Fischer and Verrecchia (2000) looks at informativeness of equilibria. That paper only considers information about the natural action (we also study information about the gaming ability), and focuses on types being uncorrelated across dimensions (we allow for correlation).

9 In 2014, less than 0.1% of students taking the SAT got a perfect 2400; the 99th percentile score was 2250 (College Board, 2014b). Also in 2014, only about 1% of the U.S. population had a perfect FICO credit score of 850; less than 20% of people had a score between 800 and 850 (Wall Street Journal, 2015).
son why the market may not be able to perfectly infer the agent’s type is that the signaling technology may be inherently noisy (Matthews and Mirman, 1983), although this can again be a choice object (Rick, 2013).

2. The Model

Our model is a reduced-form signaling game. One or more agents independently take observable actions; for convenience, we sometimes refer to “the” agent. The agent has two-dimensional private information—her type—that determines her cost of taking a single-dimensional action. The agent then receives a benefit that depends on the public belief about her type that is formed after her chosen action is observed.

2.1. Types and signaling costs

The agent takes an action, \( a \in A \equiv \mathbb{R} \). The agent’s type, her private information, is \( \theta = (\eta, \gamma) \), drawn from a cumulative distribution \( F \) with compact support \( \Theta \subset \mathbb{R} \times \mathbb{R}^{++} \). We write \( \Theta_\eta \) and \( \Theta_\gamma \) for the projections of \( \Theta \) onto dimension \( \eta \) and \( \gamma \) respectively. The first dimension of the agent’s type, \( \eta \), which we call her natural action, represents the agent’s intrinsic ideal point, or the highest action that she can take at minimum cost.\(^{10}\) The second dimension, \( \gamma \), which we call gaming ability, parameterizes the agent’s cost of increasing her action above the natural level: a higher \( \gamma \) will represent lower cost. (It will be helpful to remember the mnemonics \( \theta \) for type, \( \eta \) for natural, and \( \gamma \) for gaming.) The cost for an agent of type \( \theta = (\eta, \gamma) \) of taking action \( a \) is given by \( C(a, \eta, \gamma) \), also written as \( C(a, \theta) \). Using subscripts on functions to denote partial derivatives as usual, we make the following assumption on signaling costs:

**Assumption 1.** The cost function \( C : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{++} \rightarrow \mathbb{R} \) is differentiable, twice-differentiable except possibly when \( a = \eta \), and satisfies:

1. For all \( \gamma \) and \( a \leq \eta \), \( C(a, \eta, \gamma) = 0 \).
2. For all \( \gamma \) and \( a > \eta \), \( C_{aa}(a, \eta, \gamma) > 0 \).
3. For all \( \gamma \) and \( a > \eta \), \( C_{a\eta}(a, \eta, \gamma) < 0 \) and \( C_{a\gamma}(a, \eta, \gamma) < 0 \).
4. For any \( \bar{\eta} < \eta \) and \( \underline{\gamma} < \gamma \), \( C_a(\cdot, \bar{\eta}, \underline{\gamma})/C_a(\cdot, \eta, \gamma) \) is strictly increasing on \( [\bar{\eta}, \infty) \) and there exists \( a^{oc} > \bar{\eta} \) such that \( C_a(a^{oc}, \bar{\eta}, \underline{\gamma}) = C_a(a^{oc}, \bar{\eta}, \gamma) \).

\(^{10}\)We will abuse notation by using the same symbols to denote both dimensions and realizations.
Together, Parts 1 and 2 of Assumption 1 say that (i) the natural action $a = \eta$ is an agent’s highest cost-minimizing action, with cost normalized to zero; (ii) the agent can costlessly take actions below her natural action (“free downward deviations”); (iii) the marginal cost of increasing her action is zero at her natural action; and (iv) the agent must pay an increasing and convex cost to take actions above this level. Part 3 of the assumption stipulates that the marginal cost of increasing one’s action is lower for agents with either higher natural actions or higher gaming ability. Consequently, $C(\cdot)$ satisfies decreasing differences (and hence a single-crossing property): if $a < \bar{a}$ and $\theta < \bar{\theta}$ in the component-wise order, then

$$C(\bar{a}, \bar{\theta}) - C(a, \theta) \geq C(\bar{a}, \theta) - C(a, \bar{\theta}),$$

where the inequality is strict if $a$ is no smaller than $\theta$’s natural action.

The key feature of our model is that owing to the multidimensionality of $\Theta$, the single-crossing property of $C(\cdot)$ does not, in general, hold globally on $A \times \Theta$. The fourth part of Assumption 1 places structure on how $C(\cdot)$ behaves for pairs of cross types, where one type, $(\eta, \pi)$, has a strictly higher natural action but a strictly lower gaming ability than the other, $(\bar{\eta}, \bar{\pi})$. We write $(\eta, \pi) \prec (\bar{\eta}, \bar{\pi})$ as shorthand for $\eta < \bar{\eta}$ and $\pi > \bar{\pi}$. At low actions, the type with the lower $\gamma$ (and higher $\eta$) has a lower marginal cost of increasing its action. But this type’s marginal cost grows faster than the other type’s, and there is some cutoff action, $a^{or}$, at which the marginal-cost ordering of the two types reverses: at higher actions the type with the higher $\gamma$ (and lower $\eta$) now has a lower marginal cost of increasing its action. We refer to the action $a^{or}$ as the order-reversing action.

Assumption 1 implies the existence of another cutoff action, one at which the cross types share an equal signaling cost. We denote this action by $a^{ce}$ and refer to it as the cost-equalizing action. For any action below $a^{ce}$, the type with lower $\gamma$ (but higher $\eta$) bears a lower cost, whereas the relationship is reversed for higher actions.

**Lemma 1.** For any $\eta < \bar{\eta}$ and $\gamma < \bar{\gamma}$, there exists $a^{ce} > a^{or}$ such that $C(a^{ce}, \eta, \gamma) = C(a^{ce}, \bar{\eta}, \gamma)$. Furthermore, for any $a > \bar{\eta}$, $\text{sign}[C(a, \eta, \gamma) - C(a, \bar{\eta}, \gamma)] = \text{sign}[a^{ce} - a]$.

(The proof of this and all other formal results can be found in the Appendices.)

Figure 1 summarizes the above discussion by illustrating the implications of Assumption 1 when $\Theta$ consists of four types: a low type, $(\eta, \gamma)$; two intermediate cross types, $(\bar{\eta}, \pi)$ and $(\bar{\eta}, \bar{\pi})$; and a high type $(\bar{\eta}, \bar{\pi})$. We discuss the economics captured by Assumption 1 more in Subsection 2.3.
Figure 1 – Cost curves for $\Theta = \{\eta, \bar{\eta}\} \times \{\gamma, \bar{\gamma}\}$ with $\eta < \bar{\eta}$ and $\gamma < \bar{\gamma}$. The solid red curve is $C(\cdot, \eta, \gamma)$ and the solid blue curve is $C(\cdot, \eta, \bar{\gamma})$; the dashed red curve is $C(\cdot, \bar{\eta}, \gamma)$ and the dashed blue curve is $C(\cdot, \bar{\eta}, \bar{\gamma})$.

Example 1. The canonical functional form to keep in mind is $C(a, \eta, \gamma) = \frac{c(a, \eta)}{\gamma}$. In this case the first three parts of Assumption 1 reduce to only requiring the analogous properties on $c(a, \eta)$ (with the second requirement of part 3 automatically ensured). Since $\frac{C_u(a, \eta, \gamma)}{C_u(a, \eta, \bar{\gamma})} = \frac{c_u(a, \eta, \gamma)}{c_u(a, \eta, \bar{\gamma})}$, a sufficient condition for the fourth part is that for any $\eta < \bar{\eta}$, $\frac{c_u(a, \eta)}{c_u(a, \eta, \bar{\gamma})}$ is strictly increasing on the relevant domain with $\lim_{a \to \infty} \frac{c_u(a, \eta)}{c_u(a, \eta, \bar{\gamma})} = 1$. In particular, given any exponent $r > 1$, the cost function $C(a, \eta, \gamma) = \frac{(a-\eta)^r}{\gamma}$ for $a > \eta$ (and 0 for $a \leq \eta$) satisfies Assumption 1. This family will be our leading example.$^{11}$

2.2. Beliefs, payoffs, and equilibrium

We suppose that there is one dimension of interest about the agent’s type, $\tau \in \{\eta, \gamma\}$. After observing the agent’s action, an observer or “market” forms a posterior belief $\beta_\tau \in \Delta(\Theta_\tau)$ over the dimension of interest, where $\Delta(X)$ is the set of probability distributions on a set $X$. We assume the market evaluates the agent by the expected value of her type on

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$^{11}$In this family, for any pair of cross types $(\eta, \gamma) < (\bar{\eta}, \bar{\gamma})$, $a^{or}$ and $a^{ce}$ can be computed as

$$a^{or} = \frac{\eta \gamma^k - \eta^k}{\gamma^k - \gamma^k} \quad \text{and} \quad a^{ce} = \frac{\eta \gamma^l - \eta^l}{\gamma^l - \gamma^l},$$

where $k = \frac{1}{r-1}$ and $l = \frac{1}{r}$. 

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dimension τ, which we denote ˆτ: ˆτ ⇐≡ Eβ(τ). We refer to ˆτ as the market belief about the agent. Gross of costs, the value or benefit from signaling for an agent who induces belief ˆτ is denoted V(ˆτ; s), where s ∈ R++ parameterizes the signaling stakes. This benefit is independent of an agent’s type. We maintain the following assumption about the benefit function.

**Assumption 2.** The benefit function, V(ˆτ; s), is continuous and satisfies:

1. For any s, V(·; s) is strictly increasing.

2. V(·) has strictly increasing differences, i.e., for any ˆτ′ > ˆτ, V(ˆτ′; ·) − V(ˆτ; ·) is strictly increasing.

3. For any ˆτ′ > ˆτ, V(ˆτ′; s) − V(ˆτ; s) → ∞ as s → ∞ and V(ˆτ′; s) − V(ˆτ; s) → 0 as s → 0.

In other words, the agent prefers higher market beliefs; higher beliefs are more valuable when the stakes are higher; as stakes grow unboundedly, the benefit of inducing any higher belief grows unboundedly; and analogously as stakes vanish. A simple example that we will sometimes refer to is V(ˆτ; s) = sv(ˆτ) for some strictly increasing function v(·). We highlight that a higher value of stakes does not represent greater direct benefits from taking higher actions; rather, it implies greater rewards to inducing higher market beliefs.

Combining the benefits and costs of signaling, an agent of type θ = (η, γ) who plays action a yielding beliefs ˆτ on dimension τ has net (von-Neumann Morgenstern) payoff V(ˆτ; s) − C(a, θ). This payoff function together with the prior distribution of types, F, induces a signaling game in the obvious way. We focus on Bayesian Nash equilibria—simply equilibria, hereafter—of this signaling game. Since the market does not take any actions, Bayesian Nash equilibria coincide with Perfect Bayesian equilibria in our setting. Given that the agent cares about the market belief on only one dimension of his type, equilibria cannot generally fully reveal both dimensions (cf. Stamland, 1999). We say that an equilibrium is separating if it fully reveals the agent’s private information on the dimension of interest; an equilibrium is pooling if it reveals no information on the dimension of interest; and an equilibrium is partially-pooling if it is neither separating nor pooling. We say that two equilibria are equivalent if they share the same mapping from types to (distributions over) the posterior belief, βτ, and the same mapping from types to (distributions over) signaling costs.

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12 One may be interested in functions V(η, ˆτ; s) that depend on beliefs about both dimensions of an agent’s type; we study one such specification in Subsection 5.3.
The assumption of free downward deviations implies that equilibrium beliefs must be monotone over on-path actions. More precisely, following the convention that $\sup\emptyset = -\infty$:

**Lemma 2.** In any equilibrium, if $a' < a''$ are both on-path actions, then $\hat{\tau}(a') \leq \hat{\tau}(a'')$. Moreover, for any equilibrium, there is an equivalent equilibrium in which (i) if $a' < a''$ are both on-path actions, then $\hat{\tau}(a') < \hat{\tau}(a'')$, and (ii) if $a$ is an off-path action, 

$$\hat{\tau}(a) = \max\{\min\Theta, \sup\{\hat{\tau}(a') : a' \text{ is on path and } a' < a\}\}.$$ 

The first statement of the lemma is straightforward. Property (i) of the second statement follows from the observation that if there are two on-path actions $a' < a''$ with $\hat{\tau}(a') = \hat{\tau}(a'')$, then one can shift any type’s use of $a''$ to $a'$ without altering either the market belief at $a'$ or any incentives. We will refer to this property as belief monotonicity, and without loss of generality, restrict attention to equilibria that satisfy it. Part (ii) of the second statement of Lemma 2 assures that there would be no loss in also requiring weak monotonicity of beliefs off the equilibrium path. However, we will not impose this property because it is simpler to instead assume that off-path actions are assigned “worst beliefs” (i.e. a belief corresponding to the lowest type on the dimension of interest).

**Remark 1.** By free downward deviations, there is always a pooling equilibrium in which all types play $a = \min\Theta$.

**Remark 2.** If either the agent’s natural action, $\eta$, or her gaming ability, $\gamma$, is known to the market—i.e., there is private information on only one dimension—then there is a separating equilibrium (supposing the dimension of interest is the dimension on which there is private information). More generally, if there are no cross types in $\Theta$ then there is a separating equilibrium due to the single-crossing property.

**Remark 3.** Consider our leading family of cost functions: for some $r > 1$, $C(a, \eta, \gamma) = \frac{(a-\eta)^r}{\gamma}$ for $a > \eta$ (and 0 for $a \leq \eta$). With the change of variables $e \equiv \frac{a-\eta}{\gamma^{1/r}}$, the agent’s net payoff function can be rewritten as $V(\hat{\tau}; s) - e^r \cdot 1_{\{e > 0\}}$. The problem is thus isomorphic to one where the agent chooses “effort” $e$ at a type-independent cost $e^r$ (if $e > 0$, and 0 otherwise) and the market observes “output” $a = \eta + \gamma^{1/r} e$ before forming its expectation of $\tau \in \{\eta, \gamma\}$. Here, $\eta$ can be interpreted as the agent’s baseline talent while $\gamma$ parameterizes her marginal product of effort.
2.3. Discussion of the model

Assumptions. Two of our assumptions warrant additional discussion. The first is that of free downward deviations (part 1 of Assumption 1): an agent can costlessly take any action below her natural action. As noted above, free downward deviations ensures that equilibrium beliefs are monotonic in actions and that a pooling equilibrium always exists. These two properties are common features of signaling games. The fact that free downward deviations guarantees the properties without further parametric assumptions or equilibrium refinements will simplify the analysis. In making the assumption, though, we are primarily motivated by applications. It is much easier to make web pages appear to be of lower than higher quality; it is obvious how to wreck one’s credit score but not how to raise it; and it is virtually costless to get questions wrong on a test, whereas getting more questions right is difficult.

Notwithstanding, in some settings there may be a direct cost of deviating downwards. An accountant manipulating financial reports, as in Fischer and Verrecchia (2000), cannot easily make them look worse than they truly are; lowering one’s credit score by failing to pay a bill on time may incur monetary costs; or, as in Bénabou and Tirole (2006), agents may dislike contributing less to charity. The mechanism explored in our paper does not turn on the assumption of free downward deviations. Indeed, take any equilibrium strategy profile in our game in which each type’s action is weakly above her natural action. This strategy profile would remain an equilibrium even with costly downward deviations—unplayed lower actions would now be even less attractive. Section 5 studies a specification of the model and a class of equilibria in which actions are in fact never below natural actions. Hence, the comparative statics there would be unchanged if downward deviations were made costly.

Our key assumption is part 4 of Assumption 1: for any pair of cross types, the ratio of marginal costs of the high natural action type to the high gaming ability type is increasing in the action. The interpretation is that as more gaming that occurs—that is, agents choose higher actions—cost differences become less driven by variation in natural actions, and more by variation in gaming ability. In the credit scoring example, suppose Anne has a natural credit score of 675 and low gaming ability, while Bob has a lower natural score, 600, but a higher gaming ability. If both agents aim for a credit score around 700, Anne’s marginal cost of score improvement is likely to be lower than Bob’s: Anne can address the most obvious flaws on her credit history while Bob has already made a lot of changes from his natural behavior. At higher scores around 800, though, Bob’s marginal cost of improvement is lower.
than Anne’s: both Anne and Bob must engage in a lot of gaming to reach this level, and Bob is the one who knows more about how to game or is better at it.

In many settings, over the relevant range of actions, we think our key assumption is likely to be valid. On the other hand, one can conceive of violations. For instance, no matter the amount of studying (gaming), only those with sufficiently high natural ability (natural actions) might be able to attain the highest scores on an IQ test. It turns out, however, that the assumption is crucial for the comparative statics on how market information responds to changes in stakes. Indeed, we view one of our main contributions as identifying the role and importance of such an assumption. In the conclusion of the paper we discuss how our analysis also helps shed light on situations where the assumption may not be satisfied.

Applications. The main applications we focus on are school tests, web search, and credit scores. Heterogeneity in natural actions is uncontroversial in these applications, as it simply reflects that agents would have different actions absent signaling concerns. Students get different test scores prior to studying; web sites are more or less relevant for a given query; and even without a formal credit score, people differ in their propensity to pay bills on time. In these applications, it is usually this natural action—the preexisting variation in the absence of gaming—which is the dimension of interest.

There are a number of (non-exclusive) sources for heterogeneity in gaming ability in these and other applications. An obvious one is underlying skills: some students may simply be more facile at studying, web designers may vary in their skill with respect to search engine optimization, or agents may incur different lying costs in strategic information transmission (cf. fn. 33). But there are other sources.

Differences in information. Agents are likely to vary in their understanding of how to game a signal. This may be due to experience or other factors.\textsuperscript{13} In his book “The Big Short”, Michael Lewis (2011, p. 100) describes how Wall Street firms raced to find ways to manipulate credit rating agencies’ algorithms during the subprime mortgage crisis (emphasis added): “The models... were riddled with... [gaming] opportunities. The trick was finding them before others did”—finding, for example, that both Moody’s and S&P favored floating-

\textsuperscript{13}Sometimes, experience with a signaling technology is endogenous and could be considered as part of gaming; e.g. a student must decide how many practice tests to take. But in other cases experience may be thought of as more akin to a fixed type: web designers have different levels of experience prior to their role at their current job; a consumer who has applied for a loan before may have a better understanding of the credit scoring market.
rate mortgages with low teaser rates over fixed-rate ones.” Alternatively, many “white hat” or benign search engine optimization methods are considered best practices for web design—using appropriate keywords, linking to relevant sites, coding web sites in ways that are easily parsed by search engines—but professional web designers would likely be more attuned to these techniques than amateurs.\(^{14}\)

Information differences may be especially relevant for consumers trying to improve their credit scores, but who may not know what strategies to use. One should obviously make on-time payments. But, according to the Fair Isaac Corporation, payment history counts for only 35% of one’s FICO score. The credit score also puts a 30% weight on credit utilization (having a higher credit limit and a lower balance on one’s card is helpful), 15% on length of credit history, 10% on recent searches for credit, and 10% on types of credit used. The formula is presumably derived from correlations with loan defaults, but not in ways that are transparent to consumers. A large genre of books (e.g. Weston, 2011) and other products is devoted to explicating strategies to improve one’s credit score.

\textit{Differences in preferences for gaming.} Students may vary in how much they enjoy or dislike studying. Those who enjoy it more will face a lower cost of increasing their test scores above their natural levels. When gaming involves monetary costs, we can also interpret those with access to more money as having a lower disutility of spending money relative to the signaling benefits. The College Board worries that richer students can better afford private tutoring and test prep courses for its SAT test.\(^{15}\) More profitable web sites have a lower relative cost of spending money on search engine optimization to improve their ranking.

Moreover, when there are somewhat unethical approaches to gaming signals, differences in “integrity” could affect agents’ preferences towards gaming. Students can not just study for exams, but also find ways to cheat. Some consumers pay “credit doctors” to raise credit scores in ambiguously legal ways, such as reporting fake accounts with great repayment records (Businessweek, 2008). Similarly, colleges can and sometimes do engage in dubious

\(^{14}\)Many of these white-hat techniques are encouraged by search engines as a way of improving the quality of web sites to users. We would model that by adding a direct payoff to agents from taking higher actions, analogously to making education intrinsically productive in the job market signaling game of Spence (1973). This direct benefit of actions on payoffs is outside of our model because it would effectively create a cost of deviating below one’s natural action. However, as discussed at the outset of Subsection 2.3, adding this effect would not fundamentally alter our results.

\(^{15}\)Our analysis will imply that this concern is only warranted when colleges or other interested parties are not able to directly observe all relevant aspects of students’ economic backgrounds, at least not perfectly. Indeed, to the extent that observers do obtain some exogenous information about his dimension, they would wish to treat students with identical scores differently—an issue that is much debated in college admissions.

Differences in signaling benefits. One can also interpret an agent’s gaming ability as parameterizing her private benefits from signaling. Specifically, if \( C(a, \eta, \gamma) = \frac{c(a, \eta)}{\gamma} \), then the net payoff function \( V(\hat{\tau}; s) - \frac{c(a, \eta)}{\gamma} \) is indistinguishable from the payoff function \( \gamma V(\hat{\tau}; s) - c(a, \eta) \). Intuitively, it is identical to state that rich students have a lower cost of paying for coaching relative to their benefit from higher scores, or that these students have a higher benefit (in dollar terms) relative to the monetary cost of such coaching. In addition to such differences driven by wealth or amounts of money at stake, web site owners might simply differ on their interest in attracting more hits, and colleges may differ in the benefits they obtain from boosting their US News rankings. Relatedly, managers may value the market’s evaluation of their firm’s earnings differently (Fischer and Verrecchia, 2000), and individuals may care differently about their social image (Bénabou and Tirole, 2006).

Gaming ability as the dimension of interest. There are contexts in which what we refer to as the gaming ability would in fact be the dimension of interest. Indeed, Spence’s (1973) framework of job-market signaling is precisely one in which the market values “gaming” ability, because the ability to “game” by completing undesirable schooling at lower cost is positively correlated with productivity. While the simplest version of Spence (1973) has a homogenous natural action of acquiring no education, there may in fact be underlying preference variation over education that is irrelevant to employers. Similarly, pre-existing variation in SAT scores might arise from differences in socioeconomic status or high school quality while “gaming” ability may correlate with a broader ability to study and learn new skills. (More generally, colleges or employers could value a mix of both dimensions; we consider such an extension in Subsection 5.3.\textsuperscript{16})

Still further applications for gaming ability as the dimension of interest emerge when gaming ability is isomorphic to private benefits. For instance, better students choose to attend better colleges; a supermodular match function would imply that better colleges—those that benefit more from attracting better students—also benefit more from raising students’ perceptions of their quality. See Esteban and Ray (2006) for a related point in the context of signaling quality for license procurement.

\textsuperscript{16}Note that even in this case we do not consider the market directly valuing actions, by contrast with signaling through productive education (cf. fn. 14).
2.4. Measuring information and welfare

The natural measure for agent welfare is the expected payoff across types, $E[V(\hat{\tau}; s) - C(a, \theta)]$. We say that allocative efficiency is the expected benefit from signaling gross of signaling costs: $E[V(\hat{\tau}; s)]$. Besides these standard quantities, our focus in this paper will be on the amount of information revealed about the dimension of interest of the agent’s type, $\tau$.

Recall that $\beta_\tau \in \Delta(\Theta_\tau)$ is the market posterior (the marginal distribution) over the dimension of interest, $\tau$. From the ex-ante point of view, any equilibrium induces a probability distribution over $\beta_\tau$, which is an element of $\Delta(\Delta(\Theta_\tau))$. In any equilibrium, the expectation over $\beta_\tau$ must be the prior distribution over $\tau$. Equilibria may differ, however, in the distribution they induce over $\beta_\tau$. A separating equilibrium is fully informative about $\tau$: after any on-path action, $\beta_\tau$ will be degenerate. A pooling equilibrium is uninformative about $\tau$: after any on-path action, $\beta_\tau$ is simply the prior over $\tau$. To compare informativeness of equilibria in between these two extremes, we will use the canonical partial ordering of Blackwell (1951, 1953). We say that a distribution of beliefs or posteriors is more informative than another if the former is a mean-preserving spread of the latter. An equilibrium $e'$ is more informative about $\tau$ than an equilibrium $e''$ if the distribution of $\beta_\tau$ under $e'$ is more informative than that under $e''$.

As the agent’s signaling benefit depends only on the market’s posterior mean on the dimension of interest, $\hat{\tau}$, we will also be interested in information specifically about $\hat{\tau}$ rather than about the entire distribution $\beta_\tau$. An equilibrium $e'$ is more informative about $\hat{\tau}$ than $e''$ if the distribution of $\hat{\tau}$ under $e'$ is a mean-preserving spread of the distribution under $e''$. An equilibrium is uninformative about $\hat{\tau}$ if the distribution it induces over $\hat{\tau}$ is a point mass at the prior mean of $\tau$; it is fully informative about $\hat{\tau}$ if every on-path action reveals the agent’s true mean on the dimension of interest, which can only happen if $\tau$ is revealed. Note, however, that an equilibrium can be uninformative about $\hat{\tau}$ even if the equilibrium is informative about $\tau$. In general, the partial order on equilibria generated by information about $\hat{\tau}$ is finer than that generated by information about $\tau$: more informative about $\tau$ implies more informative about $\hat{\tau}$, but more informative about $\hat{\tau}$ does not necessarily imply more informative about $\tau$.

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17 Throughout this paper, we use the terminological convention that binary comparisons are always in the weak sense (e.g., “more informative” means “at least as informative as”) unless explicitly indicated otherwise.

18 If $\Theta_\tau$ is binary, then the posterior mean $\hat{\tau}$ is a sufficient statistic for the posterior distribution $\beta_\tau$. In this case, more informative about $\hat{\tau}$ does imply more informative about $\tau$, and uninformative about $\hat{\tau}$ implies uninformative about $\tau$. 

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Comparing equilibria according to their informativeness is appealing—when possible—because of the fundamental connection between this statistical notion and economic quantities of interest. In particular, allocative efficiency $E[V(\hat{\tau}; s)]$ is tied to information. If the benefit function $V(\cdot; s)$ is convex, then for fixed stakes there is higher allocative efficiency when the beliefs about $\hat{\tau}$ are more informative (and hence also when the beliefs about $\tau$ are more informative). If $V(\cdot; s)$ is concave, the opposite holds: allocative efficiency is maximized by pooling all types and leaving the market belief at the prior. For a linear $V(\cdot; s)$, allocative efficiency is independent of the information about $\hat{\tau}$.

We are primarily motivated by situations where information has an allocative benefit, corresponding to a weakly convex benefit function. Consider, for instance, a market in which consumers (agents) bring differing service costs to a firm that provides them a product. Revealing information about consumer costs means that higher cost consumers will be offered higher prices. This information transfers surplus from high cost to low cost consumers but also improves the efficiency of the allocation (strictly, so long as demand is not perfectly inelastic). Appendix A provides an explicit example relating the demand curve for a product to the shape of a convex benefit function.

3. The Logic with Two Types

To illustrate the core mechanism and implications of muddled information, we begin with a simple two type example in which $\Theta = \{\theta_1, \theta_2\}$. If the two types are not cross types, then a separating equilibrium exists. Throughout this section, then, we consider the case of cross types: $\theta_1 = (\eta, \gamma) <> (\bar{\eta}, \gamma) = \theta_2$. For ease of exposition, we refer to $\theta_1$ as “the gamer” type (i.e., the one with the higher gaming ability), and $\theta_2$ as “the natural.”

3.1. Dimension of interest is the natural action

Let the dimension of interest be the natural action: $\tau = \eta$. When can agents’ private information be fully revealed? In a separating equilibrium, the gamer type $\theta_1$ plays some action $a_1$ while the natural type $\theta_2$ plays $a_2 > a_1$.\footnote{Due to free downward deviations, each type can be mixing over multiple actions in a separating equilibrium; however, such an equilibrium is equivalent to one in which the agent uses a pure strategy. While we set aside such details here to ease exposition, the full range of possibilities is covered by the formal results in Subsection 4.1 for a setting that subsumes the current one.} The incentive constraints that each type is willing to play its own action over the other’s are $C(a_2, \theta_1) - C(a_1, \theta_1) \geq V(\bar{\eta}; s) - V(\eta; s) \geq C(a_2, \theta_2) - C(a_1, \theta_2)$. The first inequality says that the gamer type’s incremental cost of
playing $a_2$ rather than $a_1$ is higher than the incremental benefit, while the second inequality says that the incremental benefit is greater than the incremental cost for the natural type. Without loss, we can take $a_1 = \eta$, as the gamer would never pay a positive cost to be revealed as the type with the lower natural action. Substituting $a_1 = \eta$ and rewriting the incentive constraints yields

$$C(a_2, \theta_1) \geq V(\eta; s) - V(\eta; s) \geq C(a_2, \theta_2). \quad (1)$$

We find that a separating equilibrium, or equivalently an action $a_2$ satisfying (1), exists if and only stakes are sufficiently low.\(^{20}\) The logic behind this result can be seen by considering the two cross types’ costs, as illustrated in Figure 1. For small $s$, separation can be obtained through the natural actions. For moderate values of $s$, the standard signaling mechanism operates: the natural or “high” type distorts its action upwards to a point that is too costly for the gamer or “low” type to mimic. But as stakes increase the natural type cannot continue to separate by taking ever higher actions, because the single-crossing order reverses and eventually the gamer has lower costs. Formally, we can keep raising stakes without violating the inequalities in (1) until the first inequality binds, and so we choose action $a_2$ to make $C(a_2, \theta_1)$ as large as possible; the constraint is that we must choose $a_2$ small enough such that $C(a_2, \theta_1) \geq C(a_2, \theta_2)$. As $C(a_2, \theta_1) \geq C(a_2, \theta_2)$ if and only if $a_2$ is below the cost-equalizing action $a^{ce}$ (Lemma 1), a separating equilibrium exists for stakes up to a cutoff $s^*_\eta$ that solves

$$V(\eta; s^*_\eta) - V(\eta; s^*_\eta) = C(a^{ce}, \theta_1).$$

Action $a^{ce}$ acts like an endogenous upper bound on actions insofar as separating the two types is concerned. At any action $a > a^{ce}$ the signaling cost is lower for the gamer type, and hence $a$ cannot be used as a credible signal of a high natural action by the natural type.\(^{21}\)

While separation is impossible at stakes $s > s^*_\eta$, there are in fact informative equilibria even at arbitrarily high stakes. Recall that both types have the same incremental cost of taking action $a_2 = a^{ce}$ rather than $a_1 = \eta$. (The subsequent logic can be applied to any pair of actions $a_1 < a_2$ for which $C(a_2, \theta) - C(a_1, \theta)$ is the same for both types $\theta$; for any $a_1$ in $[\eta, a^{or}]$, there is a corresponding $a_2$ in $(a^{or}, a^{ce}]$ at which this holds.) At high stakes,

\(^{20}\)We need not consider incentive constraints on taking actions other than $a_1$ or $a_2$, as we can set beliefs to be $\tilde{\eta} = \eta$ at all off-path actions.

\(^{21}\)At sufficiently high stakes, there will exist pooling equilibria in which both types always play $a > a^{ce}$. However, it can be shown that no action above $a^{ce}$ can be used in an informative equilibrium.
both types can mix over this pair of actions: there is an equilibrium where the natural type plays $a_2$ with a suitably higher probability than the gamer type so that the incremental benefit $V(\hat{\eta}(a_2); s) - V(\hat{\eta}(a_1); s)$ is equal to the incremental cost $C(a_2, \theta) - C(a_1, \theta)$. This partially-pooling equilibrium is informative, as $\hat{\eta}(a_2) > \hat{\eta}(a_1)$. Thus, even though $a^{ce}$ acts like an upper bound on separation, the current setting is very different from a canonical (one-dimensional) signaling model with a bounded signal space where at high stakes there would only be pooling equilibria.\footnote{Some readers may note that the partially-informative equilibrium constructed here violates strategic-stability-based refinements. For example, the D1 criterion (Cho and Kreps, 1987) would imply that no actions above $a^{ce}$ can be taken in equilibrium, since at actions $a > a^{ce}$ it would be the natural type rather than the gamer type who would have a greater cost savings from deviating slightly downwards. Hence, under D1, at high enough stakes both types would pool at $a^{ce}$—just as in a canonical model one-dimensional signaling model with maximum action $a^{ce}$. Our approach in this paper is to set aside questions of plausible out-of-equilibrium beliefs and to instead focus on gaining what we view as more robust insights by considering local or global comparative statics given an arbitrary equilibrium. In any case, richer type spaces—including the $2 \times 2$ type space considered in Subsection 4.1—will have informative equilibria that satisfy the D1 criterion for all stakes.}

**Observation 1.** In the two cross types setting, if the dimension of interest is the natural action, then there is a threshold $s^*_\eta > 0$ such that a separating equilibrium exists if and only if $s \leq s^*_\eta$. At higher stakes, a partially-pooling equilibrium exists.

Nevertheless, as $s \to \infty$, the informativeness of any equilibrium vanishes. In the equilibrium described above, for instance, it must hold that $\hat{\eta}(a_2) \to \hat{\eta}(a_1)$ as $s \to \infty$ in order to maintain $V(\hat{\eta}(a_2); s) - V(\hat{\eta}(a_1); s) = C(a_2, \theta) - C(a_1, \theta)$. Otherwise, by Part 3 of Assumption 2, the left-hand side goes off to infinity while the right-hand side stays fixed.

Is it globally the case that higher stakes result in less equilibrium information about natural actions? We show formally in Subsection 4.1 that this is indeed true in a setting that embeds the current two cross types. It is instructive, however, to explain the basic logic in this simpler setting. Take some equilibrium at a given stakes, and consider a marginal reduction in stakes. We perturb the old equilibrium to construct a more informative equilibrium at the lower stakes. In particular, we alter mixing probabilities to shift gamer types left, from high actions to low ones, and shift natural types right, from low to high. As the beliefs on the dimension of interest are increasing in actions, this perturbation moves high natural action types from lower to higher beliefs, and moves low natural action types from higher to lower beliefs. This generates a mean-preserving spread of the distribution of posteriors, i.e., a more informative equilibrium.
To illustrate the approach, consider a partial-pooling equilibrium of the form depicted in Figure 2: both types (denoted in the figure by $\theta_1 = \eta \gamma$ and $\theta_2 = \eta \gamma$) put positive probability on the two on-path actions $a_1$ and $a_2$, with $a_1 < a_2$ and $\hat{\eta}(a_1) < \hat{\eta}(a_2)$.\footnote{As depicted in the figure, it must hold that $\eta \leq a_1 < a^{or} < a_2 \leq a^{ce}$ in order for both types $\theta$ to be indifferent because they must have the same cost difference $C(a_2, \theta) - C(a_1, \theta)$.} As stakes decrease, we aim to perturb the equilibrium in a way that only moves gamer types left and natural types right.

![Figure 2 - Increasing informativeness as $s$ decreases, when $\tau = \eta$.](image)

There are two relevant cases to consider. The first case is that both types are indifferent over actions $a_1$ and $a_2$, but no other incentive constraints bind. As stakes decrease, $a_2$ would become less attractive relative to $a_1$ (by Part 2 of Assumption 2) if we held strategies fixed. We can recover the indifference over the two actions, and thus the equilibrium, if we make $a_2$ relatively more attractive by decreasing beliefs at $a_1$ and increasing beliefs at $a_2$. Figure 2 shows exactly how to do this: move natural types right from $a_1$ to $a_2$, and/or move gamer types left from $a_2$ to $a_1$.

The second case is when there is another binding constraint: suppose $a_1 > \eta$ and that the gamer type is originally indifferent not just over $a_1$ and $a_2$ at beliefs $\hat{\eta}(a_1) < \hat{\eta}(a_2)$, but also over its ideal point $a = \eta$ at belief $\hat{\eta} = \eta$.\footnote{Since the marginal cost of signaling is higher for the gamer on the domain $(\eta, a^{or})$, and because $a_1 < a^{or}$, the gamer type prefers to deviate from $a_1$ to a lower action before the natural type does.} A perturbation of the previous form would no longer yield an equilibrium. Lowering the stakes makes $a = \eta$ more appealing than $a_1$, and then shifting types to reduce beliefs at $a_1$ would make $\eta$ even more appealing; gamers would no longer be willing to play their equilibrium action $a_1$. Instead, we use a different perturbation as depicted in Figure 3. First shift some of the gamer types left from action $a_1$ to $\eta$ (see the horizontal arrow). This shift holds beliefs at $\eta$ at $\hat{\eta} = \eta$, but increases the belief at $a_1$. After a small reduction in $s$, we can increase beliefs at $a_1$ in this manner until the gamer is again indifferent between playing $\eta$ and $a_1$. But both types now prefer $a_1$ to $a_2$, since stakes have gone down while beliefs at $a_1$ have gone up. The next step is to increase beliefs at $a_2$ by moving gamer types left to action $\eta$ (see the curved arrow of Figure 3).
Beliefs at $\eta$ are unchanged at $\hat{\eta} = \eta$, and so this second shift can recover the indifference of both types between $a_1$ and $a_2$ without affecting the indifference of gamer types between $a_1$ and $\eta$. These two steps of the perturbation only ever move gamer types left, from high-belief actions to low-belief actions, so the perturbation again improves informativeness.

![Figure 3](image_url) – Increasing informativeness with two steps as $s$ decreases, when $\tau = \eta$.

Similar techniques (see the proof of Lemma 3) show that, starting from any equilibrium, it is always possible to weakly improve informativeness as stakes decrease.

### 3.2. Dimension of interest is the gaming ability

Now suppose the dimension of interest is the gaming ability: $\tau = \gamma$. A separating equilibrium will exist if and only if the stakes are sufficiently large, by contrast to the previous subsection. In a separating equilibrium, without loss the natural type $\theta_2 = (\eta, \gamma)$ plays $a_2 = \eta$ and the gamer type $\theta_1 = (\eta, \gamma)$ plays $a_1 > \eta$. (If $a_1 \leq \eta$, free downward deviations implies that the natural would simply mimic the gamer.) So a separating equilibrium exists if and only if there is an $a_1 > \eta$ such that

$$C(a_1, \theta_2) \geq V(\gamma; s) - V(\gamma; s) \geq C(a_1, \theta_1).$$

The first inequality is the incentive constraint for the natural not to mimic the gamer, while the second inequality is the gamer’s incentive constraint not to deviate to its natural action. Combining the implication of (2) that $C(a_1, \theta_2) \geq C(a_1, \theta_1)$ with $a_1 > \eta$ shows that the gamer’s action $a_1$ must in fact be above the cost-equalizing action $a^{ce}$. Furthermore, the value of separation must be high enough that the gamer is willing to bear the cost of taking an action sufficiently above $a^{ce}$ to deter the natural from mimicking it. Formally, a separating equilibrium exists if and only if $s \geq s^*_\gamma$, where $s^*_\gamma$ solves

$$V(\gamma; s^*_\gamma) - V(\gamma; s^*_\gamma) = C(a^{ce}, \theta_1).$$

Next, we consider whether an informative equilibrium exists at low stakes. At $s$ small
enough such that $V(\gamma; s) - V(\gamma; s) \leq C(\eta, \theta_1)$, it is clear that the gamer type would never take any action above $\eta$ in equilibrium, and so the natural type could costlessly mimic any action taken by the gamer; hence, informative equilibria cannot exist. Indeed, under any informative equilibrium the gamer type must actually play some action above $a^{or}$, the order-reversing action. Otherwise, say that the highest action she plays is $a_1 \leq a^{or}$; it follows that this action has the highest on-path market belief $\hat{\gamma}$. But because the natural type has uniformly lower marginal costs at actions below $a^{or}$, the natural type must uniquely play $a_1$, so $a_1$ has the lowest equilibrium belief as well; hence, the equilibrium is uninformative. No informative equilibrium exists at stakes below $s^{**}_\gamma$, where $s^{**}_\gamma$ satisfies $V(\gamma; s^{**}_\gamma) - V(\gamma; s^{**}_\gamma) = C(a^{or}, \theta_1)$.

For moderate stakes between $s^{**}_\gamma$ and $s^{*}_\gamma$, partially-pooling equilibria do exist. For instance, the two types can both mix over a pair of actions $a_1$ and $a_2$, with $a_1 < a^{or} < a_2$ and $C(a_2, \theta) - C(a_1, \theta)$ equal across the two types. At stakes just above $s^{*}_\gamma$, we would take $a_1$ and $a_2$ close to each other and make the belief difference $\hat{\gamma}(a_2) - \hat{\gamma}(a_1)$ close to zero; at stakes just below $s^{*}_\gamma$, we would take $a_1$ close to $\eta$, $a_2$ close to $a^{ce}$ and make the belief difference $\hat{\gamma}(a_2) - \hat{\gamma}(a_1)$ close to $\eta - \gamma$.

**Observation 2.** In the two cross types setting, if the dimension of interest is gaming ability, then there are thresholds $0 < s^{**}_\gamma < s^{*}_\gamma$ such that a separating equilibrium exists if and only if $s \geq s^{*}_\gamma$, and all equilibria are pooling when $s \leq s^{**}_\gamma$. At stakes $s \in (s^{**}_\gamma, s^{*}_\gamma)$ there exist partially-pooling equilibria.

We establish exact comparative statics that for $\tau = \gamma$, information increases in stakes (rather than decreases as when $\tau = \eta$) in Subsection 4.1, which embeds the current cross-types setting. To see the intuition, suppose that at some stakes $s$, there is a partially-pooling equilibrium with both types mixing over two on-path actions, $a_1 < a_2$. Consider a marginal increase in stakes to $s' > s$. Here we want to perturb the equilibrium to be more informative by moving gamer types right, from low beliefs on gaming ability to high beliefs, and/or moving natural types left.

There are two important differences with the logic given in the previous subsection, where stakes decreased (recall Figure 2 and Figure 3). As stakes have increased, both types find their natural actions less attractive relative to actions $a_1$ and $a_2$. This difference simplifies matters, as we will be able to ignore any incentives to deviate down to one’s natural action. However, because action $a_2$ has also gotten more attractive relative to $a_1$, maintaining indifference by altering the behavior only across the two actions would require reducing $\hat{\gamma}(a_2)$ and increasing $\hat{\gamma}(a_1)$. That shift would involve moving gamers left and naturals right, which
would induce a mean-preserving contraction in the distribution of market beliefs—a less informative equilibrium. Instead, as depicted by the arrow in Figure 4, we move gamers right from $a_2$ to some new higher action $a_3$. By moving gamers right, from a low belief $\hat{\gamma}(a_2)$ to a higher belief $\bar{\gamma}$, we produce a more informative equilibrium.

Specifically, to get the new strategies to be an equilibrium, we shift gamer types right to reduce beliefs at $a_2$ until we recover the indifference of both types between $a_1$ and $a_2$. The action $a_3$ is then set to make gamers indifferent between $a_2$ at the new beliefs, and $a_3$ at beliefs $\hat{\gamma} = \bar{\gamma}$. Because $a_2 > a^\alpha$ and because gamers have lower marginal costs above $a^\alpha$, if the gamers are indifferent between $a_2$ and $a_3$, the naturals strictly prefer to stick with $a_2$.

![Figure 4](image)

**Figure 4** – Increasing informativeness as $s$ increases, when $\tau = \gamma$.

### 4. The Effect of Stakes on Muddled Information

The previous section provided intuition for why higher stakes tend to decrease equilibrium information when the dimension of interest is the natural action, and the reverse when the market is interested in gaming ability. It is, however, artificial to only consider two types that are cross types. With this in mind, we turn to our main results.

#### 4.1. A $2 \times 2$ type space

The minimal non-trivial setting where $\Theta$ is a product set is what we refer to as the $2 \times 2$ setting: $\Theta = \{\eta, \bar{\eta}\} \times \{\gamma, \bar{\gamma}\}$ with $\eta < \bar{\eta}$ and $\gamma < \bar{\gamma}$. Here, we are able to establish global comparative statics on the informativeness of equilibria with respect to the stakes.

For any given stakes, there are typically multiple equilibria; moreover, these equilibria need not be ranked according to (Blackwell) informativeness. We seek to compare the informativeness of the equilibrium sets at different stakes. We do so using the weak set order: equilibrium set $Q$ is *more informative* than equilibrium set $Q'$, and $Q'$ is *less informative* than $Q$, if (i) for any equilibrium $e \in Q$ there exists $e' \in Q'$ with $e$ more informative than $e'$, and (ii) for any $e' \in Q'$ there exists $e \in Q$ with $e$ more informative than $e'$. In our
setting, condition (i) is satisfied for any pair of equilibrium sets by taking $e' \in Q'$ to be a pooling equilibrium. So it is only condition (ii) that in fact has bite: for any equilibrium in the less informative set, there is an equilibrium in the more informative set that is more informative. Note that there may exist no “most informative” equilibrium in either set because the Blackwell (1951) informativeness criterion is a partial order.\footnote{If we were to extend the Blackwell partial ordering to a complete ordering on informativeness of equilibria—for example, by taking any convex function $U(\beta_\tau)$ and comparing equilibria by $E[U(\cdot)]$—then our notion of equilibrium set $Q$ being more informative than equilibrium set $Q'$ would correspond to the most informative element of $Q$ being more informative than that of $Q'$.}

We now formalize the sense in which information decreases in the stakes when the dimension of interest is the natural action, and increases in the stakes when the dimension of interests is gaming ability.

**Proposition 1.** In the $2 \times 2$ setting, consider stakes $s < \bar{s}$:

1. If $\tau = \eta$, the set of equilibria under $s$ is more informative about $\eta$ than the set of equilibria under $\bar{s}$.

2. If $\tau = \gamma$, the set of equilibria under $s$ is less informative about $\gamma$ than the set of equilibria under $\bar{s}$.

The first part of Proposition 1 is a consequence of the following lemma.

**Lemma 3.** Consider the $2 \times 2$ setting with $\tau = \eta$. Let $Q(s)$ be the set of equilibria at stakes $s$, and fix some equilibrium $e_0$ at stakes $s_0 > 0$. There exists a function $e(s)$ from stakes $s > 0$ to equilibria in $Q(s)$ that satisfies (i) $e(s_0) = e_0$; (ii) the distribution of $\beta_\eta$ under $e(s)$ is continuous in $s$;\footnote{That is, for a sequence $s \to s^*$, the corresponding distributions under $s$ converge weakly to that under $s^*$.} and (iii) $e(s'')$ is less informative than $e(s')$ for any $s'' > s'$.

In words, Lemma 3 says that starting from any equilibrium at some stakes, say $s_0$, we can find a path of equilibria that is continuous in information—i.e. in the distribution of the market posterior—and decreases information as stakes increase from $s_0$, and increases information as stakes decrease from $s_0$. This result is substantially stronger than the first part of Proposition 1 because of the continuity in information that is assured from the arbitrary starting point.

Similarly, the second part of Proposition 1 is a consequence of the following lemma.
Lemma 4. Consider the $2 \times 2$ setting with $\tau = \gamma$. Let $Q(s)$ be the set of equilibria at stakes $s$, and fix some equilibrium $e_0$ at stakes $s_0 > 0$. There exists a function $e(s)$ from stakes $s \in [s_0, \infty)$ to equilibria in $Q(s)$ that satisfies (i) $e(s_0) = e_0$; (ii) the distribution of $\beta_\gamma$ under $e(s)$ is continuous in $s$; and (iii) $e(s'')$ is more informative than $e(s')$ for any $s'' > s'$.

This lemma is a slightly weaker counterpart of Lemma 3: starting from any equilibrium at some stakes, it only assures a continuous increase in information when stakes increase, but not necessarily a continuous decrease in information when stakes decrease.

The proofs of Lemma 3 and Lemma 4 are constructive, building on and generalizing the arguments illustrated in Section 3. We note that we would conjecture the conclusion of Proposition 1 to also hold when attention is restricted to equilibria satisfying stability-based refinements such as D1 or divinity (Cho and Kreps, 1987; Banks and Sobel, 1987).27

4.2. General type spaces

For more general type spaces, $\Theta \subset \mathbb{R} \times \mathbb{R}^+$, we are unable to get exact comparative statics on information as stakes vary. Instead, to extend the theme that observers tend to be more informed about the natural action at low stakes and more informed about gaming ability at high stakes, we focus on the limiting cases when stakes get arbitrarily small or large. For these limits, we provide results on either the existence of separating equilibria, or on conditions guaranteeing that equilibria become approximately uninformative about $\hat{\tau}$. Formally, we say that equilibria are approximately uninformative about $\hat{\tau}$ at high (low) stakes if for any sequence of equilibria $e_s$ at stakes $s$, it holds that as $s \to \infty$ ($s \to 0$), the distribution of $\hat{\tau}$ under $e_s$ approaches the uninformative distribution, a point mass at $\mathbb{E}[\tau]$.

4.2.1. Dimension of interest is the natural action

Let the dimension of interest be the natural action: $\tau = \eta$. We first generalize the idea of Observation 1 by establishing conditions under which full information can only be revealed at low stakes.

Proposition 2. Assume $\tau = \eta$. If $|\Theta_\eta| < \infty$, then for sufficiently small stakes, an equilibrium exists that is fully informative about $\eta$. If $\Theta$ has any cross types, then a fully-informative equilibrium does not exist for sufficiently large stakes.

27 See Bagwell (2007) for a thorough analysis of equilibria satisfying the intuitive criterion in a model with $2 \times 2$ types; he does not analyze comparative statics of informativeness.
If there were no cross types, then standard arguments based on the single-crossing property imply that a fully-informative equilibrium would exist for any stakes.

We next seek to formalize the idea that for larger $s$, the information on $\eta$ becomes limited. The first step is to take an arbitrary pair of cross types and show that the equilibrium belief difference between the high-$\eta$ type and the low-$\eta$ type is bounded above by a function that decreases in $s$.

**Lemma 5.** Assume $\tau = \eta$ and fix any two cross types, $\theta_1 = (\eta_1, \gamma) < > \theta_2 = (\eta, \gamma)$, with the corresponding cost-equalizing action, $a^{ce}$. Across all type spaces $\Theta$ containing $\{\theta_1, \theta_2\}$ and across all equilibria, it holds that if $\hat{\eta}_i$ is some belief that $\theta_i$ induces in equilibrium ($i = 1, 2$), then

$$V(\hat{\eta}_2; s) - V(\hat{\eta}_1; s) \leq C(a^{ce}, \theta_1) = C(a^{ce}, \theta_2).$$

The important consequence of Lemma 5 (and Part 3 of Assumption 2) is that for any pair of cross types, $\theta_1 < > \theta_2$, as stakes get arbitrarily large, the type with lower natural action and higher gaming ability ($\theta_1$) cannot induce a belief much lower than any belief induced by the type with higher natural action and lower gaming ability ($\theta_2$). Depending on the type space and the parameters, there may be equilibria in which $\theta_1$ induces a strictly higher belief than $\theta_2$, because $\theta_1$ may (partially-)pool with types with even higher natural actions than $\theta_2$; this possibility does not arise in a $2 \times 2$ setting. In the limit of $s \to \infty$, any type with strictly higher gaming ability than another type must induce a weakly higher belief about its natural action.

This monotonicity of beliefs in the limit provides an upper bound on how informative an equilibrium can be about natural actions at very large stakes: any limiting distribution of beliefs on $\eta$ must be “ironed” so that the set of $\gamma$ types consistent with a belief $\hat{\eta}$ is weakly increasing in $\hat{\eta}$ (in the sense of the strong set order). In certain salient cases any limiting distribution is necessarily uninformative about the posterior mean $\hat{\eta}$; in these cases any equilibrium is approximately uninformative about $\hat{\eta}$ at large stakes.

**Proposition 3.** Assume $\tau = \eta$. If the marginal distribution of $\gamma$ is continuous and if $E[\eta | \gamma]$ is non-increasing in $\gamma$ over the support of $\Theta$, then equilibria are approximately uninformative about $\hat{\eta}$ for large stakes.

A simple condition which implies that $E[\eta | \gamma]$ is non-increasing in $\gamma$ is that the natural action and gaming ability are independent. Note that equilibria becoming uninformative
about $\hat{\eta}$ does not imply their becoming uninformative about $\eta$. In the limit, observers learn nothing about the mean of the agent’s natural action—as this moment is what the agent cares about—but may still learn something about other belief moments, such as the variance.

Two observations help explain why some conditions are needed in Proposition 3. First, if the distribution of $\gamma$ were not continuous, then, for example, a mass of types with the lowest $\gamma$ and a low $\eta$ (or the highest $\gamma$ and a high $\eta$) may separate from other types even in the limit of $s \to \infty$, revealing information about their $\eta$. Second, even with a continuous distribution of $\gamma$, if the expectation $\mathbb{E}[\eta|\gamma]$ were increasing—e.g. $\eta$ and $\gamma$ are positively correlated—then types with higher $\gamma$ may be able to signal their higher average $\eta$ by taking higher actions.

4.2.2. Dimension of interest is the gaming ability

Let the dimension of interest be the gaming ability: $\tau = \gamma$. The mirror image of Proposition 2 now holds:

**Proposition 4.** Assume $\tau = \gamma$. If $|\Theta_\gamma| < \infty$, then for sufficiently large stakes, an equilibrium exists that is fully informative about $\gamma$. If $\Theta$ has any cross types, then a fully-informative equilibrium does not exist for sufficiently small stakes.

We can also find bounds on belief differences between a pair of cross types $\theta_1 = (\eta, \gamma) <> \theta_2 = (\bar{\eta}, \bar{\gamma})$, similarly to Subsection 4.2.1. In particular, we will show that at sufficiently low $s$, the type $\theta_2$ with lower $\gamma$ but higher $\eta$ can at worst replicate the beliefs of $\theta_1$, and potentially can do even better.

**Lemma 6.** Assume $\tau = \gamma$. Fix any pair of cross types in $\Theta$, $\theta_1 <> \theta_2$, with order-reversing action $a^{or}$. If $V(\max \Theta_\gamma; s) - V(\min \Theta_\gamma; s) \leq C(a^{or}, \theta_1)$, then in any equilibrium it holds that $\hat{\gamma}_1 \leq \hat{\gamma}_2$ for any beliefs $\hat{\gamma}_1$ and $\hat{\gamma}_2$ that $\theta_1$ and $\theta_2$ respectively induce.

By compactness of $\Theta_\gamma$ and Assumption 2, $V(\max \Theta_\gamma; s) - V(\min \Theta_\gamma; s) \to 0$ as $s \to 0$. Since $a^{or} > \bar{\eta} > \bar{\gamma}$ for any pair of cross types, $(\eta, \gamma) <> (\bar{\eta}, \bar{\gamma})$, Lemma 6 implies that for small enough stakes, a low-$\gamma$ cross type must induce a weakly higher belief about its gaming ability than its high-$\gamma$ counterpart. In the limit as $s \to 0$, we see that the set of $\eta$ types compatible with some belief $\hat{\gamma}$ must be increasing in $\hat{\gamma}$; as before, only “ironed” limiting distributions are possible. When such ironing is only compatible with an equilibrium that is uninformative about $\hat{\gamma}$, it follows that the informativeness of equilibria about the mean of the gaming ability must vanish as stakes gets small. In particular:
Proposition 5. Assume $\tau = \gamma$. If the marginal distribution of $\eta$ is continuous and if $\mathbb{E}[\gamma|\eta]$ is non-increasing in $\eta$ over the support of $\Theta$, then equilibria are approximately uninformative about $\hat{\gamma}$ for small stakes.

5. A Linear-Quadratic-Elliptical Specification

We next turn to a specification of our general framework that permits explicit equilibrium characterization and additional comparative statics. In this section, we specialize to a linear signaling benefit $V(\hat{\tau}; s) = s\hat{\tau}$ and a quadratic signaling cost $C(a, \eta, \gamma) = (\max\{a - \eta, 0\})^2/2\gamma$. Given stakes $s > 0$, the agent’s payoff is then

$$s\hat{\tau} - \frac{(\max\{a - \eta, 0\})^2}{2\gamma}.$$ (3)

We take the agent’s type $\theta = (\eta, \gamma)$ to be drawn from an elliptical distribution, i.e. a distribution in which there is a constant probability density on each concentric ellipse about a mean. More formally, an (absolutely continuous) elliptical distribution $\mathcal{E}(\mu, \Sigma, g)$ over a two-dimensional realization $x = (x_1, x_2)$ is defined by $\mu = (\mu_1, \mu_2) \in \mathbb{R}^2$, $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$ a positive definite matrix, and $g(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ a measurable function called the density generator. The probability density of this distribution is

$$f(x) = k|\Sigma|^{-1/2}g((x - \mu)\Sigma^{-1}(x - \mu)^\prime)$$

with $k = 1/(\pi \int g(t)dt) \in \mathbb{R}_{++}$ a constant of integration. We take $\theta$ to be drawn from $\mathcal{E}(\mu_\theta, \Sigma_\theta, g_\theta)$, where $\mu_\theta = (\mu_\eta, \mu_\gamma)$ and $\Sigma_\theta = \begin{pmatrix} \sigma_\eta^2 & \rho\sigma_\eta\sigma_\gamma \\ \rho\sigma_\eta\sigma_\gamma & \sigma_\gamma^2 \end{pmatrix}$ with $\sigma_\eta > 0$, $\sigma_\gamma > 0$, and $\rho \in (-1, 1)$. Our maintained assumption of a compact support corresponds to a requirement that $g_\theta$ has compact support; without loss, we take the support of $g_\theta$ to be contained in $[0, 1]$. Finally, in order to guarantee our maintained assumption that $\gamma > 0$ for all types in the support, we assume that $\mu_\gamma > \sigma_\gamma$. We refer to this specification of preferences and type distribution as the linear-quadratic-elliptical, or LQE, specification.

In an elliptical distribution with a given density generator $g$, the marginal distribution of component $i = 1, 2$ depends only on $\mu_i$ and $\sigma_i^2$. (See Gómez et al. (2003) for an accessible

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28 Hence, we assume $\int g(t)dt \in (0, \infty)$. The notation $|\Sigma|$ refers to the determinant of $\Sigma$ and $(x - \mu)^\prime$ refers to the transpose of $(x - \mu)$. Vectors $x$ and $\mu$ are row vectors prior to transposition.
introduction to this and other properties of elliptical distributions.) The vector of means is \( \mu \). The covariance matrix is \( \alpha \Sigma \) for some constant \( \alpha > 0 \) which depends only on \( g \). The correlation coefficient between the two components is therefore \( \sigma_{12}/(\sigma_1\sigma_2) \). The coefficient of determination in a linear regression of one component on the other, what is commonly referred to as the \( R^2 \), is equal to the square of this correlation coefficient: \( R^2 \equiv \sigma_{12}^2/(\sigma_1^2\sigma_2^2) \). In our specification, the correlation coefficient between \( \eta \) and \( \gamma \) is given by the parameter \( \rho \) and the \( R^2 \) is \( \rho^2 \).

Elliptical distributions are a generalization of joint normal distributions. The normal family obtains when the density generator is \( g(t) = \exp\left\{(-1/2)t\right\} \). We cannot use normal distributions (and have ruled them out by requiring \( g_\theta \) to have compact support) because they would entail types with \( \gamma < 0 \); such types would reverse the sign of the cost component of (3), violating Assumption 1. That is, an agent with \( \gamma < 0 \) would obtain direct benefits rather than costs from taking higher actions. A simple example of an elliptical distribution with compact support is a uniform distribution over the interior of an ellipse, which corresponds to \( g(t) = 1_{\{t \leq 1\}} \). Elliptical distributions preserve many useful properties of joint normal distributions. Crucially for our purposes, when \( (\eta, \gamma) \) is elliptically distributed and the action \( a \) is an arbitrary linear function of \( \eta \) and \( \gamma \), it holds that \( E[\tau|a] \) is a linear function of \( a \). So a linear strategy in the agent’s type will imply a linear market belief. (To anticipate how equilibrium is closed, quadratic costs and a linear value function will make it optimal for the agent to use a linear strategy when the market belief is linear.)

Our analysis in this section is related to Fischer and Verrecchia (2000), Bénabou and Tirole (2006, Section II.B), and the work in progress of Gesche (2015). Fischer and Verrecchia (2000) and Bénabou and Tirole (2006) have previously studied related specifications to what we use; they take their type distribution to be bivariate independent normal and focus on the dimension of interest being (their analog of) \( \tau = \eta \).\(^{29}\) Bénabou and Tirole study equilibrium actions. We emphasize comparative statics of equilibrium informativeness, which does appear in Fisher and Verrechia’s Corollary 3.\(^{30}\) To these papers our analysis adds

\(^{29}\)Recall from Subsection 2.3 that \( \gamma \) can be reinterpreted as parameterizing private benefits rather than gaming ability. Specifically, when \( \gamma > 0 \), the objective (3) is equivalent to \( s\gamma \hat{\tau} - \left(\max\{a-\eta, 0\}\right)^2 \). This latter objective is still meaningful when some agents have \( \gamma < 0 \), under the interpretation that some agents prefer lower market beliefs to higher. If the cost function is then modified from \( \left(\max\{a-\eta, 0\}\right)^2 \) to \( (a-\eta)^2 \), i.e., to stipulate downward deviations from the natural action being symmetrically costly to upward deviations, one recovers the objective function analyzed by Fischer and Verrecchia (2000), Bénabou and Tirole (2006), and Gesche (2015). In that specification, agents with \( \gamma < 0 \) will take actions below their ideal point \( \eta \) at a positive cost in order to reduce market beliefs.

\(^{30}\)In ongoing work using a related specification, Ali and Bénabou (2014) also discuss equilibrium informa-
broader dimensions of interest (allowing for $\tau = \gamma$ and even a mixture) as well as correlation of types. Correlation is important for our applications. For example, types are correlated when students from a higher socioeconomic class can more easily pay for effective test preparation (higher $\gamma$) and also tend to be better prepared for college (higher $\eta$). We owe the idea of using elliptical distributions to Gesche (2015); his equilibrium characterization is related to ours for the case of $\tau = \eta$, but he makes somewhat different assumptions than we do and our comparative statics are different. In particular, he does not focus on market information.

Consistent with the common practice in models with normal distributions, we will focus on linear equilibria in our LQE specification: equilibria in which an agent of type $(\eta, \gamma)$ takes action $a = l_\eta \eta + l_\gamma \gamma + b$, for some constants $l_\eta$, $l_\gamma$, and $b$. In such an equilibrium, the market belief, $\hat{\tau}(a)$, is a linear function of the agent’s action.\textsuperscript{31} Moreover, (i) the vector $(\tau, a)$ is elliptically distributed with the same density generator $g_\theta$ as $(\eta, \gamma)$, and (ii) the ex-ante distribution of posteriors $\beta_\tau$ on $\tau$ given $a$ is determined entirely by $R^2_{\tau a}$, the $R^2$ between $\tau$ and $a$. Fixing $g_\theta$ and fixing the prior distribution of $\tau$, a higher $R^2_{\tau a}$ implies an equilibrium that is (Blackwell) more informative about the market belief $\hat{\tau}$. An $R^2_{\tau a}$ of 1 implies a fully informative equilibrium about $\hat{\tau}$ (and hence also about $\tau$). An $R^2_{\tau a}$ of 0 is uninformative about $\hat{\tau}$.\textsuperscript{32} See Lemma 7 and Lemma 8 in Appendix E for details.

We begin by characterizing linear equilibria. As in our general analysis earlier, free downward deviations ensures that any equilibrium must have a market posterior $\hat{\tau}(\cdot)$ that is non-decreasing in $a$ (at least on the equilibrium path) and that there is a pooling equilibrium in which $\hat{\tau}(\cdot)$ is a constant. An increasing linear equilibrium has

$$\hat{\tau}(a) = La + K$$

for some $L > 0$. (A linear equilibrium is informative about $\hat{\tau}$ only if it is increasing.) Given a market belief of the form in (4), the agent’s optimal action is unique:

$$a = \eta + sL\gamma.$$  

\textsuperscript{31} Strictly speaking, the market belief is only pinned down at on-path actions. It is without loss for our purposes to stipulate a globally linear market belief.

\textsuperscript{32} $R^2_{\tau a} = 0$ implies that $\tau$ and $a$ are uncorrelated: $\hat{\tau}$, the posterior mean conditional on $a$, is constant with respect to $a$. However, while the equilibrium is uninformative about $\hat{\tau}$, it is still informative about $\tau$. The support of $\tau$ depends on $a$, for instance. Indeed, when two random variables are jointly elliptically distributed, they can be independent only if the distribution is joint normal (Kelker, 1970).
When the agent plays a linear strategy, the market’s posterior beliefs will be elliptically distributed with mean linear in the agent’s action. Increasing equilibria are determined by solving for a fixed point: values of \( L > 0 \) and \( K \) under which the market’s induced beliefs have mean equal to that hypothesized. While we relegate the details to Lemma 9 in Appendix E, it is useful to note that an equilibrium value of \( L > 0 \) is determined as:

\[
L = \frac{\mathcal{L}(s, L, \tau)\sigma^2 + \mathcal{L}(s, L, -\tau)\rho\sigma\gamma}{\sigma^2 + s^2L^2\sigma^2 + 2sL\rho\gamma},
\]

where \(-\tau\) refers to the dimension other than the dimension of interest (e.g. \(-\tau = \gamma\) when \(\tau = \eta\)), and \(\mathcal{L}(s, \bar{L}, \eta) \equiv 1\) and \(\mathcal{L}(s, \bar{L}, \gamma) \equiv s\bar{L}\). By Equation 4, the equilibrium constant \( L \) measures the responsiveness of the market belief \( \hat{\tau} \) to the agent’s action.

**Remark 4.** It is clear from Equation 5 that in any increasing linear equilibrium the agent takes an action above his natural action. Consequently, increasing linear equilibria are unaffected by relaxing free downward deviations: making it costly for the agent to take actions \( a < \eta \) would only make some deviations even less attractive. Such specifications can be applied, for example, to strategic information transmission with heterogenous lying costs.\(^{33}\)

**Remark 5.** Fixing the joint distribution of \( \eta \) and \( \gamma \), an equilibrium will be more informative about \( \eta \), and less informative about \( \gamma \), when the coefficient \( sL \) in Equation 5 is larger. If we fix the marginal distribution of \( \tau \) and vary \( \rho \) or \( \sigma_{-\tau} \), though, \( sL \) is no longer a sufficient statistic for information. One will need to look at \( R^2_{\tau a} \), with explicit formulas given in Equations 12 and 13 of Appendix E.

### 5.1. Dimension of interest is the natural action

Assume \( \tau = \eta \). As described above, the informativeness of the equilibrium about \( \hat{\eta} \) is captured by the one-dimensional value \( R^2_{\eta a} \). To streamline the exposition, the following proposition restricts attention to non-negative correlation; we comment on negative correlation subsequently.\(^{34}\)

**Proposition 6.** In the LQE specification, assume \( \tau = \eta \) and \( \rho \geq 0 \).

\(^{33}\)In this sender-receiver application, \( \eta \) represents a payoff-relevant state for the receiver (who takes a decision \( d \) with payoff \(-(d - \eta)^2\), for instance), \( a \) is the sender’s message, and the sender’s lying cost is given by, say, \(-(a - \eta)^2/(2\gamma)\). The parameter \( \gamma \) captures how much the sender dislikes lying. Kartik (2009) studies a related model without heterogeneity on \( \gamma \).

\(^{34}\)For the case of \( \rho = 0 \), Fischer and Verrecchia (2000) have obtained similar conclusions to Proposition 6 using their specification.
1. There is a unique increasing linear equilibrium.

2. The following limits hold for that equilibrium: (a) As \( s \to 0 \), \( R_{\eta a}^2 \to 1 \); (b) As \( s \to \infty \), \( R_{\eta a}^2 \to \rho^2 \).

3. The following comparative statics hold for that equilibrium: (a) \( \frac{d}{ds} R_{\eta a}^2 < 0 \); (b) \( \frac{d}{d\mu_\gamma} R_{\eta a}^2 = 0 \); (c) \( \frac{d}{d\sigma_\gamma} R_{\eta a}^2 < 0 \); (d) \( \frac{d}{d\rho} R_{\eta a}^2 > 0 \).

Part 1 of Proposition 6 is self-explanatory. Part 2(a) says that as stakes vanish, the increasing linear equilibrium is approximately fully informative about \( \hat{\eta} \); this intuitive result is a counterpart to the first portion of Proposition 2. Part 2(b) implies that as stakes grow unboundedly, the equilibrium becomes uninformative about \( \hat{\eta} \) when \( \rho = 0 \), which is consistent with Proposition 3. Part 2(b) also implies that for any \( \rho > 0 \), there is some information revealed about \( \hat{\eta} \) even in the limit of unbounded stakes; as explained after Proposition 3, the intuition is that when \( \mathbb{E}[(\eta|\gamma)] \) is increasing in \( \gamma \), higher \( \gamma \) types can signal their higher average \( \eta \) by taking higher actions. As this intuition suggests, the expression in Part 2(b) verifies that the equilibrium becomes arbitrarily close to fully informative as \( \rho \to 1 \) and \( s \to \infty \).

Part 3 of Proposition 6 provides comparative statics for any value of stakes. Part 3(a) confirms our fundamental theme that higher stakes reduce information about natural actions. The other three parts address comparative statics that we have not touched on so far. Part 3(b) notes that the ex-ante mean of the gaming ability has no effect on equilibrium informativeness about \( \hat{\eta} \); rather, changes in \( \mu_\gamma \) only shift the agent’s action and the market belief function by a constant. This observation underscores that muddled information is not due to gaming \textit{per se}, but rather heterogeneity in gaming ability.

Part 3(c) of the proposition says that greater ex-ante uncertainty about \( \gamma \) reduces equilibrium information about \( \hat{\eta} \). To understand why, notice that by dividing the payoff expression (3) by \( s \), the parameter \( s \) can be viewed as simply scaling \( \gamma \); thus, an increase in \( s \) is isomorphic to increasing both the ex-ante mean \( \mu_\gamma \) and the ex-ante variance \( \sigma^2_\gamma \) (albeit differently). Since we have just seen that \( \mu_\gamma \) has no effect on equilibrium informativeness, the directional effect of increasing \( \sigma^2_\gamma \) must be same as that of increasing \( s \). Finally, Part 3(d) says that increasing an already non-negative correlation between \( \eta \) and \( \gamma \) leads to more equilibrium information about \( \hat{\eta} \). An intuition is that a greater non-negative correlation reduces the amount of heterogeneity in \( \gamma \) conditional on any \( \eta \); for elliptical distributions, \( \text{Var}(\gamma|\eta) = \alpha \sigma^2_\gamma (1 - \rho^2) \). At the limit when \( \rho = 1 \), the type space is effectively one-dimensional and the equilibrium
fully reveals all private information.

Although Proposition 6 is stated for \( \rho \geq 0 \), the key points also extend to \( \rho < 0 \). The complication is that when \( \rho < 0 \) and stakes are intermediate, there can be multiple (increasing linear) equilibria. Nevertheless, the comparative statics in \( s, \mu_\gamma, \) and \( \sigma_\eta \) all generalize subject to the caveat of focusing on the appropriate equilibria.\(^{35}\)

5.2. Dimension of interest is the gaming ability

Assume \( \tau = \gamma \). The informativeness of the equilibrium about \( \hat{\gamma} \) is now captured by \( R_{\gamma a}^2 \). As before, we focus on \( \rho \geq 0 \).

Proposition 7. In the LQE specification, assume \( \tau = \gamma \) and \( \rho \geq 0 \).

1. (a) When \( \rho = 0 \), there is an increasing linear equilibrium if and only if \( s > \sigma_\eta^2/\sigma_\gamma^2 \); the increasing linear equilibrium is unique when it exists.

(b) When \( \rho > 0 \), there is a unique increasing linear equilibrium.

2. The following limits hold for the increasing linear equilibrium: (a) When \( \rho > 0 \) and as \( s \to 0 \), \( R_{\gamma a}^2 \to \rho^2 \); (b) As \( s \to \infty \), \( R_{\gamma a}^2 \to 1 \).

3. The following comparative statics hold for the increasing linear equilibrium: (a) \( \frac{d}{ds} R_{\gamma a}^2 > 0 \); (b) \( \frac{d}{d\mu_\eta} R_{\gamma a}^2 = 0 \); (c) \( \frac{d}{d\sigma_\eta} R_{\gamma a}^2 < 0 \); (d) \( \frac{d}{d\rho} R_{\gamma a}^2 > 0 \).

Part 1 of Proposition 7 says that an increasing linear equilibrium exists only when either \( \rho = 0 \) and stakes are sufficiently large, or when \( \rho > 0 \). In fact, when \( \rho = 0 \) (but not when \( \rho > 0 \)), for any level of stakes there is an equilibrium in which the agent plays \( a = \eta \): the agent takes her natural action, at no cost, and the market learns nothing about \( \hat{\gamma} \) (while fully learning \( \eta \), which is immaterial to the agent).\(^{36}\)

\(^{35}\) Three points bear clarification. First, when there are multiple such equilibria, there are generically three of them. In these cases, the comparative statics hold for the equilibria with the highest and lowest solutions to Equation 6. Second, for \( \rho < 0 \), \( R_{\gamma a}^2 \to 0 \) as \( s \to \infty \) (for large enough \( s \), there is an increasing linear equilibrium), as is consistent with the “ironing” logic of Proposition 3. Third, comparative statics on \( \rho \) when \( \rho < 0 \) are not clear-cut.

\(^{36}\) As \( \rho \to 0_+ \), the unique increasing linear equilibrium converges to the increasing linear equilibrium of \( \rho = 0 \) if \( s > \sigma_\eta^2/\sigma_\gamma^2 \), while it otherwise converges to the equilibrium where the agent plays \( a = \eta \). The equilibrium when \( \rho = 0 \) where \( a = \eta \) is uninformative about \( \hat{\gamma} \), but is not uninformative about \( \gamma \); see fn. 32. Note that this equilibrium exists irrespective of free downward deviations. There also exists a fully uninformative equilibrium (just as when \( \tau = \eta \)) supported using free downward deviations, in which the agent pools on a sufficiently low action independent of type.
To interpret Part 2(a) of Proposition 7, observe that as $s \to 0$ the agent’s play in an increasing linear equilibrium must converge to $a = \eta$; at the limit, the market simply learns $\eta$, which implies $R^2_{\gamma a} = R^2_{\eta \gamma} = \rho^2$. Consequently, how much information is obtained about $\hat{\gamma}$ as stakes vanish turns on the degree of correlation between $\eta$ and $\gamma$. Part 2(b) says that as stakes grow unboundedly, the equilibrium becomes arbitrarily informative about $\hat{\gamma}$; this is a counterpart to the first portion of Proposition 4. Part 3(a) confirms our fundamental theme that, even away from limiting stakes, higher stakes increase information about gaming ability. The remaining comparative statics in part 3 are analogous to those discussed in the context of Proposition 6. Note, however, that a change in $\sigma^2_\eta$ cannot be viewed as equivalent to a change in $s$ (unlike a change in $\sigma^2_\gamma$). Instead, the intuition for part 3(c) is that greater ex-ante uncertainty about $\eta$ makes the market belief about $\hat{\gamma}$ less responsive to the agent’s action for any given agent’s strategy, which in turn causes the agent’s action to become less responsive to her gaming ability, thereby generating less information about $\hat{\gamma}$.

5.3. Mixed dimensions of interest

The tractability of the LQE specification makes it possible to study a number of additional questions within this setting. Here, we only take up one: what if the agent cares about the market’s belief about both $\eta$ and $\gamma$? To this end, generalize the agent’s signaling benefit in the LQE specification to $s\gamma[\kappa \hat{\gamma} + (1 - \kappa) \hat{\eta}]$, where $\kappa \in (0, 1)$ is a commonly known parameter; call this the LQE specification with mixed dimensions of interest.

When the agent uses a linear strategy, the market updates on each dimension as given in Lemma 9. Given any pair of linear market beliefs $\hat{\gamma}(a) = L_\gamma a + K_\gamma$ and $\hat{\eta}(a) = L_\eta a + K_\eta$ with $\kappa L_\gamma + (1 - \kappa) L_\eta > 0$, the agent’s unique best response is to play $a = \eta + s (\kappa L_\gamma + (1 - \kappa) L_\eta) \gamma$. Routine substitutions imply that an equilibrium is now characterized by a pair of constants, $(L_\eta, L_\gamma)$, that simultaneously solve

$$L_\eta = \frac{\sigma^2_\eta + s(\kappa L_\gamma + (1 - \kappa) L_\eta) \rho \sigma_\eta \sigma_\gamma}{\sigma^2_\eta + s^2(\kappa L_\gamma + (1 - \kappa) L_\eta)^2 \sigma^2_\gamma + 2s (\kappa L_\gamma + (1 - \kappa) L_\eta) \rho \sigma_\gamma \sigma_\eta},$$

$$L_\gamma = \frac{s(\kappa L_\gamma + (1 - \kappa) L_\eta) \sigma^2_\gamma + \rho \sigma_\eta \sigma_\gamma}{\sigma^2_\eta + s^2(\kappa L_\gamma + (1 - \kappa) L_\eta)^2 \sigma^2_\gamma + 2s (\kappa L_\gamma + (1 - \kappa) L_\eta) \rho \sigma_\gamma \sigma_\eta}.$$

Consider, for simplicity, $\rho = 0$. Proposition 11 in the Supplementary Appendix shows that for any $\kappa \in (0, 1)$, a linear equilibrium exists where both $L_\gamma > 0$ and $L_\eta > 0$, and that in such equilibria, (i) $R^2_{\gamma a}$ is decreasing in stakes and ranges from 1 (as $s \to 0$) to 0 (as $s \to \infty$), and (ii) $R^2_{\gamma a}$ is increasing in stakes and ranges from 0 (as $s \to 0$) to 1 (as $s \to \infty$). Thus,
our main themes about how changes in stakes affect the market’s information about both
the natural action and gaming ability extend to this setting where the agent cares about the
market belief on both dimensions.

6. Manipulability and Information Provision

While our main results are couched in terms of changes in the signaling stakes, which affect
the benefits of signaling, additional insights into applications are gained by interpreting the
comparative statics in terms of changes in the costs of signaling. Consider adding a parameter
to the model, the manipulability of the signal $M > 0$, so that an agent’s payoff function is
now given by

$$V(\hat{\tau}; s) - \frac{C(a, \theta)}{M}. \quad (7)$$

Manipulability parameterizes the ease with which agents can affect their signaling action:
higher manipulability reduces the signaling costs for all types of agents. As this payoff
function is isomorphic to $MV(\hat{\tau}; s) - C(a, \theta)$, it follows that increasing $M$ is, from an agent’s
perspective, identical to increasing the marginal benefit of inducing a higher market belief,
which is the same effect that an increase in stakes has. Thus, our previous analysis already
implies the relevant comparative statics on information due to changes in $M$. When the
dimension of interest is the natural action, increasing manipulability reduces information.
When the dimension of interest is the gaming ability, increasing manipulability increases
information.

In some cases manipulability might be a fundamental property of the signaling technology.
Google’s PageRank algorithm for web site ranking is considered harder to game than some
earlier search engines based primarily on keyword density. Different types of tests might
cover material that is easier or harder to study for; it is often thought that tests which
measure “aptitude” rather than “achievement” are particularly difficult to study for.\footnote{The
College Board’s “SAT” was originally an acronym for Scholastic Aptitude Test, but the name
was changed in 1994 to remove the connotation that it measured something innate and
unchangeable. The College Board now insists that “SAT” is no longer an acronym for anything.
Without endorsing the distinction between aptitude and achievement, Malcolm Gladwell (2001) writes in the New Yorker:

[After World War II, the SAT] was just beginning to go into widespread use. Unlike existing
academic exams, it was intended to measure innate ability—not what a student had learned
but what a student was capable of learning—and it stated clearly in the instructions that
cramming or last-minute reviewing was pointless.}

Manipulability may also depend on the agents’ collective knowledge about the signaling
technology. One aspect of this is familiarity—how much signaling experience the agents have had. A designer trying to increase market information about natural actions may benefit from changing the signaling technology (e.g., moving to a new test format, or switching out the credit scoring, credit rating, or search engine ranking algorithm) even if the new one is inherently no less manipulable than the old one. Simply by virtue of being new, agents may know less about how to game it, at least in the short run. This leads to a constant battle between designers and agents; it is reported that Google tweaks its search algorithm as often as 600 times a year partly to mitigate undesirable search engine optimization.

Designers could also make agents more or less knowledgeable by controlling how much information is given out about the workings of the signaling technology. Google and Fair Isaac keep their precise algorithms secret; past SAT questions were kept hidden until the 1980s; and US News sometimes changes its ranking algorithm with an explicit announcement that the algorithm will not be revealed until after the rankings are published (Morse, 2010). Our model gives a straightforward reason for such practices: a more opaque signaling technology may be less manipulable or more costly to game than a transparent one. When the dimension of interest is the natural action, less manipulable implies more informative.38

It is important, to recognize, however that revealing information about the workings of an algorithm might not increase gaming ability uniformly across agents. As discussed earlier, information differences may have been a source of the original heterogeneity on gaming ability. In that case, revealing information need not act like an increase in $M$ in expression (7); rather, it may serve to effectively increase gaming ability for all types, but moreso for the low-$\gamma$ types than the high-$\gamma$ ones. In general, this has an ambiguous effect on information. Consider expression (7) with the specification $C(a, \eta, \gamma) = c(a, \eta)/\gamma$. On one hand, increasing all gaming abilities by a constant factor is identical to increasing $M$, which would lead to a decrease in information when the dimension of interest is natural actions. On the other hand, raising $\gamma$ differentially to reduce its heterogeneity can increase information about natural actions. This is always true at the extreme when heterogeneity is eliminated altogether because that would allow for a separating equilibrium with full information. Away from the extreme, we found in the LQE specification that reducing the variance of $\gamma$ improves information about the natural action no matter how the mean of $\gamma$ is altered. So, in some

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38 Weston (2011, pp. 8–9) writes, prior to legislation that required the Fair Isaac Corporation to reveal some details about its FICO algorithm, “Fair Isaac . . . wanted to keep the information secret. The company said it worried that consumers wouldn’t understand the nuances of credit scoring, or they would try to ‘game the system’ if they knew more. Fair Isaac feared that its formulas would lose their predictive abilities if consumers started changing their behavior to boost their scores.”
cases, it might make sense to reveal past SAT questions to everyone if the College Board is concerned that some people already have access to them.

Similarly, subsidizing direct monetary costs of gaming would effectively increase the manipulability of the mechanism by raising the gaming ability of all agents. This would seem to be a bad policy for a designer interested in increasing information about agents’ natural actions. But if heterogeneity on gaming ability had been driven by wealth differences in the first place, then such a policy would increase ability more for the low-γ poor students than the high-γ rich ones. This is our interpretation of the College Board’s recent move to produce and publicize free test prep material. Rich students were already paying for this sort of material, and so the first order effect of the subsidy would be to level the playing field across agents and reduce heterogeneity. That is, this intervention does not just transfer surplus by advantaging poor students at the expense of rich ones: it improves the information content of the test results.

7. Informational Externalities Across Markets

In this section we consider a designer who can choose whether or not to reveal agents’ actions to observers in different segments of a market or in different markets. There is an obvious direct informational benefit of revealing actions to observers who would otherwise lack access to this information about agents’ types. However, the previous analysis shows there is also an indirect informational effect. As the signal is used in more markets, agents have stronger incentives to signal that they are of higher quality: signaling stakes increase. When the dimension of interest is the gaming ability, this extra incentive makes the signal more informative. But when the dimension of interest is the natural action, the information value of the signal is reduced as the stakes go up. In other words, preexisting observers are now less informed. In this latter case, a designer who is considering revealing the action to additional markets must trade off the benefits to the marginal observers with the negative informational externality imposed on the inframarginal observers.

39 The College Board is teaming up with an online education service, Khan Academy, to produce the free test prep. The following excerpt from the College Board’s press release (2014a) is telling:

“For too long, there’s been a well-known imbalance between students who could afford test-prep courses and those who couldn’t,” said Sal Khan, founder and executive director of Khan Academy. “We’re thrilled to collaborate closely with the College Board to level the playing field by making truly world-class test-prep materials freely available to all students.”

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To make the point concrete, consider markets in which credit reports may be used. Lenders use credit scores to determine how much credit to offer borrowers and at what terms. Landlords run credit checks to screen prospective tenants. A growing number of employers check credit reports during the hiring process to assess risks like employee theft or general trustworthiness; surveys suggest that only 19% of employers used credit checks in 1996 but 60% did in 2010. Automobile and homeowners insurance companies are now also commonly using information from credit reports to help determine insurance rates.

Some uses of credit reports, such as for targeted marketing, are federally prohibited. States may pass additional regulations. In recent years, a number of states have considered or passed legislation restricting the use of credit history in insurance underwriting. As of November 2013, 10 states, including California and Illinois, limit or ban employers from using credit history in hiring. Many of the arguments in favor of these laws are based on some notion of “fairness”—it is not fair to deny someone employment or to raise their insurance rates based on missed mortgage payments or high credit card debt.

We suggest a very different concern. Making credit information available to markets such as insurance and employment could alter consumers’ gaming behavior in a way that dilutes the information contained in credit scores, thereby harming the efficiency of the loan market.

7.1. Two markets

We first formalize this informational externality in a setting where there are two markets. Let the dimension of interest be $\eta$, let stakes $s \in \{1, 2\}$ be binary, and let the value function be given by

$$V(\hat{\eta}, s) = v_1(\hat{\eta}) + w_2(\mathbb{E}[\eta]) + (s - 1)(w_2(\hat{\eta}) - w_2(\mathbb{E}[\eta]))$$

$$= v_1(\hat{\eta}) + v_2(\hat{\eta}, s), \text{ for } v_2(\hat{\eta}, s) \equiv w_2(\mathbb{E}[\eta]) + (s - 1)(w_2(\hat{\eta}) - w_2(\mathbb{E}[\eta])) \quad (8)$$

for some strictly increasing functions $v_1(\cdot)$ and $w_2(\cdot)$.

The interpretation is that there is some benchmark first market that always observes the signal—e.g., the loan market in the credit scoring application. In this market the agent receives a gross payoff $v_1(\hat{\eta})$. Agents also participate in a second market in which actions may be hidden or shown—e.g., an insurance market in the credit scoring application. If the signal is hidden from this second market ($s = 1$), the agent gets a payoff $v_2(\hat{\eta}, 1) = w_2(\mathbb{E}[\eta])$; if it is shown ($s = 2$), the agent gets a payoff $v_2(\hat{\eta}, 2) = w_2(\hat{\eta})$ in that market. Allocative efficiency in market $i = 1, 2$ is the expectation of $v_i$, and total allocative efficiency is the sum

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of these two terms.

Showing the action to the second market will unambiguously increase information in that market. However, it will also raise the stakes for the agent. As we have argued throughout the paper in a number of ways, raising the stakes will tend to decrease the information in the signal, and therefore will reduce information in the first market. Hence, there is a negative informational externality. If both \( v_1 \) and \( v_2 \) are convex (where \( v_2 \) is convex if and only if \( w_2 \) is)—that is, information is valuable for efficiency in these two markets—then a designer who wants to maximize allocative efficiency faces a tradeoff. Revealing information will improve allocative efficiency in the second market, while harming it in the first market.

But suppose instead that \( v_1(\cdot) \) is strictly convex while \( v_2(\cdot, s) \) is linear. (Taking \( v_2 \) to have a small amount of convexity would give similar results; a concave \( v_2 \) would only strengthen the point.) In the first market more information implies higher allocative efficiency. For example, it is efficient to make more or larger loans to lower-financial-risk consumers. In the second market, though, any information just redistributes surplus across types without affecting allocations. In the market for auto or homeowners insurance, say, lower-financial-risk consumers receive lower rates because they tend to file fewer claims, but no (or few) consumers are on the margin of ownership based on their insurance rates. It follows that showing the signal to the second market would only impact overall allocative efficiency through its indirect negative effect on the first market. So to support the efficiency of the “loan” market where the information is socially valuable (strictly convex \( v_1 \)), the action should be hidden from the “insurance” market in which there is no social value (linear \( v_2 \)).

In the following proposition we formalize this discussion for the \( 2 \times 2 \) model, where results follow directly from Proposition 1.

**Proposition 8.** Consider the \( 2 \times 2 \) setting with dimension of interest the natural action, and value function given by \( V(\hat{\eta}, s) = v_1(\hat{\eta}) + v_2(\hat{\eta}, s) \) as in Equation 8. If \( v_1 \) and \( v_2 \) are convex in \( \hat{\eta} \), then for any equilibrium under \( s = 2 \), there exists an equilibrium under \( s = 1 \) with higher allocative efficiency \( \mathbb{E}[v_1] \) in the first market and lower allocative efficiency \( \mathbb{E}[v_2] \) in the second market. If \( v_1 \) is convex and \( v_2 \) is linear in \( \hat{\eta} \), then for any equilibrium under \( s = 2 \), there exists an equilibrium under \( s = 1 \) with higher total allocative efficiency \( \mathbb{E}[V] \).

There is, of course, a converse: if the dimension of interest were gaming ability rather than natural actions, then revealing one’s action to a second market would generate a positive rather than a negative informational externality to the first market. The designer might
want to show the information to the purely redistributive second market just to ramp up the stakes for the agents and thereby increase gaming.

### 7.2. Many homogeneous markets

We next consider a case in which there are many markets, and information is equally valuable across markets: all markets have the same convex value function. There will be a tradeoff between providing socially-valuable information to additional observers and imposing negative information externalities on existing observers. We investigate whether allocative efficiency is maximized by showing signals to as many observers as possible, or whether there is instead an interior optimum.

Throughout this section, let the dimension of interest be the natural action, $\eta$, and let

$$V(\hat{\eta}; s) = sv(\hat{\eta}) \text{ for } v(\hat{\eta}) = w(\hat{\eta}) - w(\mathbb{E}[\eta]),$$

where $w(\cdot)$, and therefore $v(\cdot)$, is strictly increasing, convex, and twice differentiable. The designer chooses stakes $s \geq 0$, and is interested in maximizing allocative efficiency $\mathbb{E}[V(\hat{\eta}; s)]$. For any given $s$, we focus on equilibria that maximize the allocative efficiency.

The interpretation of the value function $V$ and the stakes $s$ is that there is a large number of markets in which the agent participates. The agent’s value as a function of beliefs is $w(\cdot)$ in each market, scaled by the size of the market. In some of these markets, the agent’s action is observed, in which case the beliefs on the agent’s type are $\hat{\eta}$. In other markets, the agent’s action is hidden, and beliefs remain at the prior $\mathbb{E}[\eta]$. Showing the signal to a given market turns it from uninformed to informed, leading to a net agent benefit of $v(\hat{\eta}) = w(\hat{\eta}) - w(\mathbb{E}[\eta])$.

(We omit the constant payoff that the agent receives in all of the other uninformed markets.) The stakes $s$ then corresponds to the mass of markets in which the signal is observable. The total value of information aggregated across markets, $\mathbb{E}[V(\hat{\eta}; s)]$, is the value per observation of $\mathbb{E}[v]$ times the mass of observers $s$: $s\mathbb{E}[v]$.

The designer chooses the mass of markets $s$ in which the signal is observable. The key assumption here is that the function determining the value of information, $w(\cdot)$, is homogeneous across markets. In the consumer pricing example (Example 2 of Appendix A), the corresponding assumption would be that, up to linear scalings, the demand curve is identical across markets, and that beliefs on $\eta$ yield the same prices across markets.

It is clear that if the market doesn’t become uninformative in the limit as the stakes $s$ gets unboundedly large—if $\mathbb{E}[w(\hat{\eta})]$ stays bounded away from $w(\mathbb{E}[\eta])$, or equivalently $\mathbb{E}[v(\hat{\eta})]$ is
bounded away from 0—then the designer who only cares about maximizing the social value of information (the allocative efficiency) should increase \( s \) to infinity.\(^4\) The tradeoff between the marginal benefit and inframarginal cost of adding observers is interesting when the market becomes uninformed in the limit.

We analyze this question for two simple settings in which signals become uninformative in the limit. The first, as a proof of concept, shows that information is maximized at an interior solution to \( s \) when the type space is composed of a pair of cross types. The second shows that with a richer type space in which \( \eta \) and \( \gamma \) are independently distributed, we should in fact take \( s \) as large as possible: the direct effect of increasing the information for marginal observers dominates the indirect effect of worsening the information for inframarginal observers. We consider this latter result to be the takeaway message from this section.

It is important to emphasize that our focus below will be solely on the social value of information, i.e., the allocative efficiency. If the designer’s welfare criterion also includes signaling costs, then he may not want to take \( s \) to infinity even if that maximizes information. Rather, the negative informational externality due to information muddling would instead be one factor that could lead the designer to reveal the signal to fewer observers than he or she otherwise might.

**Two Cross Types.** When there are only two cross types, information is maximized at an interior value of the stakes \( s \):

**Proposition 9.** Assume \( \Theta = \{\theta_1, \theta_2\} \) with \( \theta_1 = (\eta, \gamma) \leftrightarrow (\overline{\eta}, \overline{\gamma}) = \theta_2 \) and \( V(\hat{\eta}; s) = sv(\hat{\eta}) \) with \( v(\hat{\eta}) = w(\hat{\eta}) - w(\mathbb{E}[\eta]) \) for some strictly convex \( w \). The allocative efficiency \( \mathbb{E}[V] \) is maximized over choice of \( s \) and choice of equilibrium at some finite \( s \in (0, \infty) \).

Recall from Subsection 3.1 that in a two-cross-types setting, there is an informative equilibrium for any \( s > 0 \) but the informativeness vanishes as \( s \to \infty \). In the current context, this implies that \( \mathbb{E}[v] > 0 \) for any \( s > 0 \) and \( \mathbb{E}[v] \to 0 \) as \( s \to \infty \). The question is what happens to \( \mathbb{E}[V] = s\mathbb{E}[v] \) as \( s \to \infty \). We establish in the proof of Proposition 9 that \( s\mathbb{E}[v] \) goes to 0 at a rate faster than \( \frac{1}{s} \) as \( s \to \infty \). Allocative efficiency is positive at any positive \( s \), and since it goes to 0 as \( s \to 0 \) or as \( s \to \infty \), it has an interior optimum.

---

\(^4\)This result applies, for example, when \( \Theta \) contains an extreme type—one below or above all other types in both natural action and gaming ability—that has positive probability. The extreme type(s) can separate from the other types even in the limit, and the positive mass hypothesis keeps \( v(\hat{\eta}) - v(\mathbb{E}[\eta]) \) bounded away from zero.
Independent $\eta$ and $\gamma$. Now consider the more natural case when $\eta$ and $\gamma$ are independently distributed. The simplest specification with independent types that gets an uninformative limiting equilibrium ($\mathbb{E}[v] \to 0$ as $s \to \infty$) is when $\eta$ has a binary distribution while $\gamma$ is continuously distributed; see Proposition 3. In this benchmark setting, we find that under general conditions the designer would increase $s$ without bound if he could, because even though allocative efficiency increases at a less than linear rate with $s$, it still grows without bound.

**Proposition 10.** Assume $\eta$ and $\gamma$ are independent, $\Theta_\eta = \{\eta, \bar{\eta}\}$, and $\gamma$ is continuously distributed. Let $V(\hat{\eta}; s) = sv(\hat{\eta})$ with $v(\hat{\eta}) = w(\hat{\eta}) - w(\mathbb{E}[\eta])$ for some strictly convex $w$, and let $C(a, \eta, \gamma) = \frac{(a-\eta)^r}{\gamma}$ on $a \geq \eta$ for some $r > 1$. As $s \to \infty$, there exists a sequence of equilibria with allocative efficiency $\mathbb{E}[V] \to \infty$.

We prove Proposition 10 by constructing a class of two-action equilibria—all types take actions at either $a = \eta$, or at some higher stake-dependent action—in which $\mathbb{E}[v]$ goes to 0 at a limiting rate of $s^{\frac{2}{r+1}}$ in $s$, and hence allocative efficiency $s\mathbb{E}[v]$ increases at a limiting rate of $s^{\frac{r}{r+1}}$.\textsuperscript{41} In fact, the proof is more general than independent types, and establishes some conditions under which allocative efficiency increases linearly in the stakes when the distribution of gaming abilities for high-$\eta$ agents is in some sense above that of low-$\eta$ agents.

The analysis of homogeneous markets suggests that while it is possible to construct examples in which the social value of information is maximized by obscuring information from some observers, in more plausible settings the social value of information is maximized by showing signals to as many observers as possible. In other words, even though signals do become less informative as the number of observers grows, the inframarginal cost is outweighed by the marginal benefit. We stress, however, that our discussion ignores signaling costs; a designer who takes those into account might prefer to hide information from some observers.

### 8. Conclusion

This paper has studied a model of signaling with two-dimensional types and one-dimensional actions. Equilibria typically confound the two dimensions of an agent’s type: there is muddled information. In a nutshell, we find that when stakes increase, there is less equilibrium information about the agent’s natural action and more information about her gaming ability. We have argued that this simple but robust finding provides insight into a number of

\textsuperscript{41} We do not suggest this rate is optimal; there are examples in which $\mathbb{E}[V]$ increases at a faster rate.
applications and yields a novel tradeoff regarding how information, e.g. credit scores, should be regulated across markets. We close by noting some additional issues.

Our comparative statics on information stem from costs of signaling being driven by the natural action at low actions and the gaming ability at high actions. Specifically, we assumed that for any given pair of cross types, the ratio of marginal costs for the type with higher natural action relative to the type with higher gaming ability is monotonically increasing in the action (Assumption 1 part 4). As discussed in Subsection 2.3, we believe this assumption is a good approximation for many applications, including those we have highlighted.

However, our framework also provides insight into settings where this marginal cost ratio is not everywhere increasing, or where signaling costs depend on more than two dimensions of a type. While obtaining global comparative statics is generally infeasible, it is possible to derive limiting results on information as stakes get large or small. Essentially, at low (resp., high) stakes the market learns about the dimension that determines marginal costs at low (resp., high) actions. Suppose, for instance, that musicians differ on natural talent and on quality of coaching. For untrained musicians, performance quality—or the cost of performance improvement—is determined by natural talent. As musicians begin to train and reach higher levels of performance, the quality of coaching becomes important. Finally, those at the highest levels have absorbed all coaching lessons, and performance improvements are again driven by differences in talent. We would then expect a non-monotonic relationship between stakes and information about natural talent: very low and very high stakes generate precise information about natural talent.

A simplification of our model is that an agent’s action is directly observed by the market. In practice, markets sometimes only observe a noisy signal of an agent’s action. For example, in test taking, agents choose effort which stochastically translates into the test score. Such noisy signaling would complicate the analysis, but, we believe, not fundamentally change our main points. That said, it would be interesting to explore whether greater noise can sometimes to lead to more equilibrium information.

Finally, our analysis of information yields direct implications for the allocative efficiency of equilibria. Other quantities may also be of interest. In ongoing work, we are exploring some of the broader implications of multidimensional heterogeneity and muddled information on the equilibria of signaling games.
Bibliography


Appendices

A. Relating the Signaling Benefit to Consumer Demand

Here we illustrate how the benefit function \( V(\cdot; s) \) can be derived from the demand curve for a product by consumers with heterogeneous costs. In the example below, allocative efficiency corresponds to consumer surplus. The allocative benefit of information on consumer costs corresponds to the convexity of the induced benefit function.

Example 2. A consumer chooses an action \( a \) that is observed by a competitive market of firms before they offer the consumer a price \( p \). The consumer then (mechanically) purchases a quantity given by a demand curve \( D(p, s) \), where \( s \) captures the importance of this market to the consumer. The firms’ expected cost of transacting with or serving the consumer depends on the consumer’s type on the dimension of interest, \( \tau \), with higher types having lower cost. Specifically, expected service cost is \( k - \alpha \tau \), with the most costly consumer having \( \tau = \tau_\text{max} \). As the market is competitive, the price offered to a consumer will equal the expected cost: \( p = k - \alpha \hat{\tau} \).

A consumer who is thought to be type \( \hat{\tau} \) receives gross consumer surplus

\[
V(\hat{\tau}; s) = \int_{k - \alpha \hat{\tau}}^{k - \alpha \tau} D(k - \alpha t, s) dt.
\]

Allocative efficiency—the expected consumer surplus—measures the full social value of information in this example, because firms receive no surplus. Agent (consumer) welfare, which is allocative efficiency minus expected total costs of signaling, is also total welfare here.

If the consumer’s demand curve were completely inelastic—\( D(p, s) \) independent of \( p \)—then \( V(\cdot, s) \) would be linear. Additional information in the market would just transfer surplus from low to high types: a consumer of higher type would see lower prices and a consumer of lower type would see higher prices, but purchases would be unaffected. With a downward sloping demand curve, however, information becomes socially valuable. A consumer with higher cost of service purchases less, while a consumer with lower cost of service purchases more, leading to more efficient allocations. This induces a convex value function \( V(\cdot; s) \).

B. Proofs for Section 2

Proof of Lemma 1. For any \( a > \eta \),

\[
C(a, \eta, \gamma) - C(a, \eta, \gamma) = \int_{\eta}^{a} C_a(\hat{a}, \eta, \gamma) \left[ 1 - \frac{C_a(\hat{a}, \eta, \gamma)}{C_a(\hat{a}, \eta, \gamma)} \right] d\hat{a}.
\]

Since \( C_a(\cdot) > 0 \) on the relevant region, Part 4 of Assumption 1 implies that the sign of the intergrand above is the same as that of \( a^{o_r} - \hat{a} \), where \( a^{o_r} \) is the order-reversing action for the given cross types. This implies that on the domain \( a > \eta \), the integral is strictly quasi-concave with a maximum at \( a = a^{o_r} \). Furthermore, on the domain \( a > \eta \), the integral is strictly positive for \( a \leq a^{o_r} \) and can only change sign at most once. To see that it must change sign at some point, observe that Assumption 1 also implies that there is some \( \varepsilon > 0 \) such that the integrand is strictly less than \(-\varepsilon \) for all \( a > a^{o_r} + \varepsilon \).

\[Q.E.D.\]

Thus, one can think of a market of consumers each having identical continuous demand curves, where the cost of service is per-unit transaction; or, the demand curve may represent the stochastic demand for a single unit of the good, where this demand is realized after the consumer’s action is chosen.
Proof of Lemma 2. Fix an equilibrium in which \( a' < a'' \) are two on-path actions with \( \hat{\tau}(a') \geq \hat{\tau}(a'') \). Part 1 of Assumption 2 implies that
\[
\hat{\tau}(a') = \hat{\tau}(a''),
\]
for otherwise, any type would strictly prefer \( a'' \) to \( a' \). Furthermore, any type \( \theta = (\eta, \gamma) \) that uses \( a'' \) must have \( \eta \geq a'' \), and hence
\[
C(a', \theta) = C(a'', \theta) = 0,
\]
for otherwise, Assumption 1 implies \( \theta \) would strictly prefer \( a' \) to \( a'' \). Now consider a new strategy in which the agent behaves identically to the given equilibrium except for playing \( a' \) whenever she was to play \( a'' \). By Equation 9, the induced belief at \( a' \) does not change; hence, by Equation 10, this new strategy also constitutes an equilibrium in which \( a'' \) is off path.

Finally, note that in any equilibrium, assigning any off-path action \( \tilde{a} \) the belief specified in item (ii) of the lemma preserves the property that no type has a strict incentive to use \( \tilde{a} \).

Q.E.D.

C. Proofs for Section 3

Proof of Observation 1. From (1), there is a separating equilibrium with \( a_2 = \overline{\eta} \) (and \( a_1 = \eta \)) if \( s < s^*_{\eta} \), where \( V(\overline{\eta}; s^*_{\eta}) - V(\eta; s^*_{\eta}) = C(a_2, \theta_1) \). So restrict attention to \( s \geq s^*_{\eta} \). For the existence of a separating equilibrium, there is no loss in assuming the second inequality in (1) holds with equality. This implicitly defines a strictly increasing and continuous function, \( a_2(s) \), whose range is \([\overline{\eta}, \infty)\) for \( s \in [s^*_{\eta}, \infty) \). A separating equilibrium exists at stakes \( s \) if and only if \( H(a_2(s)) \geq 0 \), where
\[
H(a) := C(a, \theta_1) - C(a, \theta_2)
\]
is continuous. Since \( \text{sign}[H(a)] = \text{sign}[a^{ce} - a] \) by Lemma 1, it follows from the definition of \( s^*_{\eta} \) that a separating equilibrium exists if and only if \( s \leq s^*_{\eta} \).

Now suppose \( s > s^*_{\eta} \) and consider a strategy where each type mixes (possibly degenerately) between actions \( \eta \) and \( a^{ce} \). By choosing the mixing probabilities suitably, we can induce via Bayes rule any \( \hat{\eta}(a^{ce}) \) and \( \hat{\eta}(\eta) \) such that \( \overline{\eta} \geq \hat{\eta}(a_2) > \mathbb{E}[\eta] > \hat{\eta}(a_1) \geq \eta \). By the definition of \( s^*_{\eta} \), there is a pair \( \hat{\eta}(a^{ce}) \) and \( \hat{\eta}(\eta) \) satisfying these inequalities such that \( V(\hat{\eta}(a^{ce}); s) - V(\hat{\eta}(\eta); s) = C(a^{ce}, \theta_1) = C(a^{ce}, \theta_2) \); the corresponding mixing probabilities define an equilibrium strategy when belief \( \eta \) is assigned to any off-path actions. Q.E.D.

Proof of Observation 2. Define \( a_1(s) \) by setting the second inequality of (2) to hold with equality; \( a_1(s) \) is strictly increasing and unbounded above. It is straightforward that a separating equilibrium exists at stakes \( s \) if and only \( a_1(s) > \overline{\eta} \) and \( H(a_1(s)) \leq 0 \), where \( H(\cdot) \) was defined in (11). By continuity of \( H(\cdot) \) and Lemma 1, there is separating equilibrium if and only if \( s \geq s^*_{\eta} \), where \( V(\overline{\eta}; s^*_{\eta}) - V(\gamma; s^*_{\eta}) = C(a^{ce}, \theta_1) \).

Next, suppose \( s \leq s^*_{\eta} \) where \( s^*_{\eta} \) solves \( V(\overline{\eta}; s^*_{\eta}) - V(\gamma; s^*_{\eta}) = C(a^{or}, \theta_1) \). To contradiction, consider any informative equilibrium. Let \( a' \) and \( a'' \) be two on path actions such that \( \hat{\gamma}(a') < \hat{\gamma}(a'') \). It follows that \( a' < a'' \leq a^{or} \), by belief monotonicity and that \( \theta_1 \) will not play any \( a > a^{or} \). Since \( C(a, \theta_2) < C(a, \theta_1) \) for \( a \in (\eta, a^{or}) \), it further follows that type \( \theta_2 \) will not play \( a' \). But this implies \( \hat{\gamma}(a') = \overline{\eta} \), contradicting \( \hat{\gamma}(a') < \hat{\gamma}(a'') \).

Finally, consider \( s \in (s^*_{\eta}, s^*_{\gamma}) \). Let the two types mix over two actions, \( a_2 > a_1 \). By choosing the mixing probabilities suitably, we can induce via Bayes rule any beliefs \( \hat{\gamma}(a_2) \) and \( \hat{\gamma}(a_1) \) such that \( \hat{\gamma}(a_2) - \hat{\gamma}(a_1) \in [0, \overline{\gamma} - \gamma] \). By Lemma 1, for any \( x \in (0, C(a^{or}, \theta_1)] \), we can find \( a_1 \) and \( a_2 \) such that \( \overline{\eta} \leq a_1 < a^{or} < a_2 \leq a^{ce} \).
and \( C(a_2, \theta_1) - C(a_1, \theta_1) = C(a_2, \theta_2) - C(a_1, \theta_2) = x \). It follows that by assigning off-path actions the belief \( \gamma \), there is an informative equilibrium with two on-path actions so long as \( V(\gamma; s) - V(\gamma; \tilde{s}) \leq C(a^{ce}, \theta_1) \) and \( \tilde{V}(\gamma; s) - V(\gamma; \tilde{s}) > C(a^{ar}, \theta_1) \). These two inequalities are assured by the hypothesis that \( s \in (s^{\gamma}_a, s^{\gamma}_s) \). Q.E.D.

D. Proofs for Section 4

D.1. Proofs for Subsection 4.1

We first establish some straightforward preliminaries for the analysis of equilibria of the \( 2 \times 2 \) model. Throughout this section, in addition to referring to \((\eta, \gamma)\) as the gamer type and \((\eta, \gamma)\) as the natural type (with \( a^{ar} \) and \( a^{ce} \) being the order-reversing and cost-equalizing actions with respect to these cross types), we also refer to \((\tilde{\eta}, \tau)\) as the low type and \((\eta, \tilde{\tau})\) as the high type.

**Claim 1.** For any finite type space \( \Theta \), up to equivalence, there is a finite upper bound on the number of actions used in equilibrium over all \( s \) and all equilibria.

**Proof.** First, for any type \( \theta \), there can only be a single action (up to equivalence) which is played by \( \theta \) alone in a given equilibrium; otherwise that type would be playing two actions with the same beliefs but different costs. Second, for any pair of types, there can be at most two distinct actions that both types are both willing to play. Consequently, an upper bound on the number of equilibrium actions is \(|\Theta| + \frac{1}{2} |\Theta| \). Q.E.D.

**Claim 2.** Fix any finite type space \( \Theta \) and let \( s_n \rightarrow s^* < 0 \) be a sequence of stakes. If \( e_n \rightarrow e^* \), where each \( e_n \) is an equilibrium (strategy profile) at stakes \( s_n \) with corresponding distribution of market beliefs \( \delta_n \in \Delta(\min \Theta, \max \Theta) \), then (i) there exists \( \delta^* \) such that \( \delta_n \rightarrow \delta^* \), and (ii) \( e^* \) is an equilibrium at \( s^* \).

**Proof.** The claim follows from standard upper-hemicontinuity arguments, with one caveat. We need to show that if as \( s_n \rightarrow s^* \) there are two sequences \( a_n \rightarrow a^* \) and \( a'_n \rightarrow a^* \), where \( a_n \) and \( a'_n \) are each on-path actions in \( e_n \), then the respective equilibrium beliefs \( \hat{\tau}(a_n) \) and \( \hat{\tau}(a'_n) \) converge to the same limit. This ensures that the belief at \( a^* \) under \( e^* \) is equal to the limiting belief along both \( a_n \) and \( a'_n \), and therefore that the payoff of \( a^* \) under \( e^* \) is equal to the limit of the payoffs along any sequence of actions approaching \( a^* \), whereverafter routine arguments apply.

Suppose, to contradiction, that \( \hat{\tau}(a_n) \rightarrow h \) and \( \hat{\tau}(a'_n) \rightarrow l \) with \( h > l \). For any \( \theta \in \Theta \), \( C(a_n, \theta) \rightarrow C(a^*, \theta) \) and \( C(a'_n, \theta) \rightarrow C(a^*, \theta) \); hence, for any \( \theta \) and sufficiently large \( n \), \( V(\hat{\tau}(a_n); s_n) - C(a_n, \theta) > V(\hat{\tau}(a'_n); s_n) - C(a'_n, \theta) \), which contradicts \( a'_n \) being on path. Q.E.D.

Now, for the \( 2 \times 2 \) model specifically, we formally establish that moving high-\( \tau \) types from actions with low beliefs to ones with high beliefs, or moving low-\( \tau \) types from high to low beliefs, increases information in the sense of Blackwell. By “moving” a type \( \theta \) from action \( a \) to \( a' \) we mean marginally altering the mixed strategy to slightly reduce the probability that \( \theta \) takes \( a \), and to correspondingly increase the probability that it takes \( a' \). (As established in Claim 1, there are finitely many actions in the support and each has strictly positive

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43 Given any market belief function, \( \hat{\tau}(a) \), type \( \theta \) is said to be willing to play action \( a' \) if \( a' \) is optimal for \( \theta \).

44 Convergence of probability distributions is in the sense of weak convergence. A sequence of equilibria converges if the corresponding mixed strategies of each type converge. Note that for any sequence of equilibria \( e_n \), there is an equivalent subsequence that converges. This is because as \( s_n \rightarrow s^* \), up to equivalence, equilibrium actions are contained in compact set that is bounded below by \( \min \Theta_n > -\infty \) and above by \( \tilde{a} < \infty \) satisfying \( V(\max \Theta; s^*) = C(\tilde{a}, \max \Theta_n) \).
probability.) We informally applied this result in the analysis of cross types in Section 3 (see Figure 2 through Figure 4), and we will again apply it in the $2 \times 2$ model which embeds the cross types.

**Claim 3.** In the $2 \times 2$ model, consider information on dimension $\tau$, where $\Theta_{\tau} = \{\tau, \overline{\tau}\}$. Take some distribution of types over actions, with two actions $a_l$ and $a_h$ in the support inducing respective beliefs $\hat{\tau}_l < \hat{\tau}_h$. If we move either a type with $\tau = \overline{\tau}$ from $a_l$ to $a_h$, or move a type with $\tau = \tau$ from $a_h$ to $a_l$, then the posterior beliefs become more informative about the dimension of interest.

Moving types across actions in the reverse way would lead to less informative rather than more informative beliefs.

**Proof of Claim 3.** One posterior distribution of beliefs is Blackwell more informative than another if and only if, for any continuous and convex function over beliefs, $U$, it holds that $\mathbb{E}[U(\beta_{\tau})]$ is higher under the first distribution than the second. Moreover, in the $2 \times 2$ setting, beliefs $\beta_{\tau}$ about the dimension of interest are fully captured by the expectation $\hat{\tau}$. So moving types increases Blackwell information if and only if for any continuous and convex function $U : [\underline{\tau}, \overline{\tau}] \to \mathbb{R}$, the move yields an increase in $\mathbb{E}[U(\hat{\tau})]$.

To calculate $\mathbb{E}[U(\hat{\tau})]$, we first establish some notation. Let $f(\theta)$ be the probability of type $\theta$ under the prior distribution $F$, and let $p_\theta(a)$ indicate the probability that an agent of type $\theta$ chooses action $a$ under a specified strategy, and let $\tau(\theta)$ indicate the component of $\theta$ on the dimension of interest. Then the belief $\hat{\tau}(a)$ at action $a$ is

$$\hat{\tau}(a) = \frac{\sum_\theta \tau(\theta)f(\theta)p_\theta(a)}{\sum_\theta f(\theta)p_\theta(a)},$$

and $\mathbb{E}[U(\hat{\tau})]$ is given by $\sum_a U(\hat{\tau}(a)) \sum_\theta f(\theta)p_\theta(a)$ where we sum over all actions $a$ in the support.

The effect on $\mathbb{E}[U(\hat{\tau})]$ of a marginal move from $a_l$ to $a_h$ of a type $\theta'$ is given by $\frac{d}{dp_{\theta'}(a_l)}\mathbb{E}[U(\hat{\tau})] - \frac{d}{dp_{\theta'}(a_h)}\mathbb{E}[U(\hat{\tau})]$. When $\tau(\theta') = \overline{\tau}$, we can evaluate these derivatives and simplify to get

$$\frac{d}{dp_{\theta'}(a_h)}\mathbb{E}[U(\hat{\tau})] - \frac{d}{dp_{\theta'}(a_l)}\mathbb{E}[U(\hat{\tau})] = f(\theta') \cdot [U(\tau_h) + (\overline{\tau} - \tau_h)U'(\tau_h)] - [(U(\tau_l) + (\overline{\tau} - \tau_l)U'(\tau_l))].$$

Convexity of $U$ combined with $\tau_l < \tau_h \leq \overline{\tau}$ guarantees that the bracketed term is nonnegative (positive under strict convexity), so as required the move increases $\mathbb{E}[U(\hat{\tau})]$.

Likewise, the effect on $\mathbb{E}[U(\hat{\tau})]$ of a marginal move from $a_h$ to $a_l$ of a type $\theta'$ is given by $\frac{d}{dp_{\theta'}(a_h)}\mathbb{E}[U(\hat{\tau})] - \frac{d}{dp_{\theta'}(a_l)}\mathbb{E}[U(\hat{\tau})]$. With $\tau(\theta') = \tau$ the expression evaluates to

$$f(\theta') \cdot [(U(\tau_l) - (\tau_l - \underline{\tau})U'(\tau_l)) - (U(\tau_h) - (\tau_h - \underline{\tau})U'(\tau_h))],$$

which is nonnegative (positive) under $\tau \leq \tau_l < \tau_h$ and (strict) convexity of $U$. \(Q.E.D.\)

To clarify terminology, say that a type plays an action if its strategy assigns positive probability to this action. An action is an equilibrium action if some type of positive measure plays this action. (Recall Claim 1.)

**Claim 4.** In the $2 \times 2$ model, all equilibria are equivalent to one in which either (1) for any pair of actions weakly above $\eta$, at most a single type is willing to play both of these; or (2) there exist actions $a_1$ and $a_2$ with $\eta \leq a_1 < a_2$ such that at least two types are willing to play both $a_1$ and $a_2$. Under these cases:
Case (1). Take any three actions $a_1 < a_2 < a_3$, with $a_1 \geq \eta$. If types $\theta_1$ and $\theta_2$ are both willing to play $a_3$, if $\theta_1$ is willing to play $a_1$, and if $\theta_2$ is willing to play $a_2$, then it cannot be the case that any type plays $a_2$.

Case (2). It holds that $\eta \leq a_1 < a^{or} < a_2 \leq \eta$. The gamer type $(\eta, \gamma)$ and the natural type $(\eta, \gamma)$ are both willing to play $a_1$ and $a_2$. No other type is willing to play both of these actions, and for any other pair of actions weakly above $\eta$ at most a single type is willing to play both.

Moreover, no action in $(a_1, a_2)$ is played in equilibrium. The natural type $(\eta, \gamma)$ only plays actions $a_1$ and/or $a_2$ in equilibrium, and is not willing to play any action $a$ with $a < a_1$ or $a > a_2$. The low type $(\eta, \gamma)$ is not willing to play any actions $a$ with $a > a_1$, and the high type $(\eta, \gamma)$ is not willing to play any action $a$ with $a < a_2$.

Proof of Claim 4. Up to equivalence, we can take all equilibrium actions to be weakly above $\eta$.

Case (1). Take three actions $a_1 < a_2 < a_3$, with types $\theta_1$ and $\theta_2$ both willing to play $a_3$, with $\theta_1$ willing to play $a_1$, and $\theta_2$ willing to play $a_2$. By assumption of Case (1), $\theta_2$ is not willing to play $a_1$ and $\theta_1$ is not willing to play $a_2$. So it must be that the types are not single-crossing ordered over the range of $[a_1, a_3]$; that is, the two types $\theta_1$ and $\theta_2$ must be the gamer $(\eta, \gamma)$ and the natural $(\eta, \gamma)$—not necessarily in that order—and it must be that $a_1 < a^{or} < a_3$. The low type $(\eta, \gamma)$ can only take actions up to $a_1$, and the high type $(\eta, \gamma)$ can only take actions down to $a_3$. So we see that only type $\theta_2$ can be willing to play $a_2$.

Now suppose that type $\theta_2$, which is the only one willing to play $a_2$, does play $a_2$ with positive probability. If it did, then the beliefs at $a_2$ would reveal the type of $\theta_2$. But if $\theta_2$ has a high type on the dimension of interest, $\tau = \tau$, then $a_2$ would be at least as appealing to $\theta_1$ as $a_3$, so $\theta_1$ would be attracted to $a_2$.

On the other hand, if $\theta_2$ has a low type on the dimension of interest $(\tau = \tau)$, then $a_1$ would be at least as appealing to $\theta_2$ as $a_2$, so $\theta_2$ would be attracted to $a_1$. Either case yields a contradiction, since $\theta_1$ is not willing to play $a_2$ and $\theta_2$ is not willing to play to $a_1$.

Case (2). Take some pair of actions $a_1$ and $a_2$ that two types are both willing to play. This cannot hold for any pair of actions that are single-crossing ordered, and so it must be that the two types are $(\eta, \gamma)$ and $(\eta, \gamma)$.

Both players are willing to play both actions means that $C(a_2, \eta, \gamma) - C(a_1, \eta, \gamma) = C(a_2, \eta, \gamma) - C(a_1, \eta, \gamma)$. Since these cross types are single-crossing-ordered on the range of $a \geq a^{or}$ (where $(\eta, \gamma)$ has lower marginal costs) and on the range of $a \leq a^{or}$ (where $(\eta, \gamma)$ has lower marginal costs), it must be that one of these actions, $a_1$ must be strictly below $a^{or}$ and the other, $a_2$, must be strictly above. There can only be a single pair of such actions; otherwise the two types would both be willing to play two distinct actions above $a^{or}$ or below it, which we established cannot happen.

By assumption, we take the lower action $a_1$ to be weakly greater than $\eta$. And the higher action $a_2$ must be weakly below $a^{or}$ because for any $\tilde{a} > a^{or}$ and $a_1 < \tilde{a}$, it holds that $C(\tilde{a}, \eta, \gamma) - C(a_1, \eta, \gamma) < C(\tilde{a}, \eta, \gamma) - C(a_1, \eta, \gamma)$.

45 We can rule out that $(\eta, \gamma)$ and $(\eta, \gamma)$ are both indifferent over a pair of actions above $\eta$ but below $\eta$ because those actions would have the same costs and the same beliefs on the dimension of interest. So, up to equivalence, the two actions could be rolled into a single action. (We take our notion of equivalence to say that if two actions have the same costs for all types playing them and the same beliefs on the dimension of interest, but different beliefs on the opposite dimension, then it is “equivalent” to put them together into a single action.)
\((\eta, \gamma)\) cannot be willing to take any actions below \(a_1\) or else \((\eta, \gamma)\) would strictly prefer that action to \(a_1\). Additionally, the high type \((\bar{\eta}, \bar{\gamma})\) is unwilling to play any action below \(a_2\), and the low type is unwilling to play any action above \(a_1\); if one of these types were willing to play such an action, then another type currently playing \(a_1\) or \(a_2\) would strictly prefer to deviate to that action.

Finally, \((\bar{\eta}, \bar{\gamma})\) is unwilling to take any action in \((a_1, a_2)\) because if this type were willing to take such an action, then \((\bar{\eta}, \bar{\gamma})\) would strictly prefer it to \(a_1\) and \(a_2\). So only \((\eta, \gamma)\) can possibly be willing to take an action in \((a_1, a_2)\), but in equilibrium this type does not play any such actions because doing so would break the equilibrium. In particular, taking such an action in equilibrium would reveal her type. Under \(\tau = \eta\) this would mean she strictly preferred the intermediate action with \(\hat{\eta} = \eta\) to \(a_2\); and under \(\tau = \gamma\) she would strictly prefer \(a_1\) under beliefs \(\hat{\gamma} = \gamma\). \(Q.E.D.\)

We now proceed to prove the main lemmas. Throughout, we maintain the assumption that there is a positive measure of both natural and gamer types; otherwise, the type space is fully ordered and we can straightforwardly maintain the equilibrium information level of any equilibrium \(e_0\) at stakes \(s_0\) as stakes vary by sliding actions up and down. We allow for there to be a zero measure of high or low types, so that we subsume the Section 3 case of only two cross types.

**Proof of Lemma 3.** Starting from any given equilibrium at some stakes, we show that as stakes decrease the equilibrium can be continuously perturbed in a manner that increases information. We will give local arguments, which show the existence of a path of equilibria nearby the starting point. The upper-hemicontinuity of the equilibrium set (Claim 2) guarantees that this local construction around any given equilibrium \(e_0\) at any stakes \(s_0\) extends to a global path of equilibria on \(s \in (0, \infty)\).\(^{46}\)

There are two kinds of perturbations involved. One slides the location of actions up or down without changing the distribution of types across actions, which has no effect on information. The other follows steps similar to those discussed in Section 3. As stakes decrease, we move low types with \(\eta = \bar{\eta}\) down from actions with high beliefs to ones with low beliefs, and/or move high types with \(\eta = \bar{\eta}\) up from actions with low to high beliefs. (Recall that due to free downward deviations, higher equilibrium actions have strictly higher beliefs.) These moves spread beliefs out and, as formalized in Claim 3, increase information. As stakes increase, we can do the reverse moves to decrease information.

Using Claim 4, we can categorize all possible equilibria into a number of exhaustive cases (up to equivalence), and then address these cases separately.

**Case 1.** For any pair of distinct actions weakly above \(\bar{\eta}\), at most a single type is willing to play both actions.

**Case 2.** There exist actions \(a_1\) and \(a_2\) satisfying \(\bar{\eta} \leq a_1 < a_2 \leq \bar{\eta}\) such that the gamer type \((\eta, \gamma)\) and the natural type \((\bar{\eta}, \gamma)\) are both willing to play \(a_1\) and \(a_2\). No actions in \((a_1, a_2)\) are played in equilibrium. Natural types \((\bar{\eta}, \gamma)\) are not willing to play any action strictly below \(a_1\) or above \(a_2\), low types \((\eta, \gamma)\) are not willing to play any action above \(a_1\), and high types \((\bar{\eta}, \bar{\gamma})\) are not willing to play any action below \(a_2\). If there is an equilibrium action \(a_0\) strictly below \(a_1\), it can only be played by types with \(\eta = \eta\), and so would have the worst possible beliefs; hence it must be that \(a_0 = \bar{\eta}\).

We can divide this case into five subcases:

\(^{46}\)Our local arguments cover different cases separately, but extending to a global path may require patching together different cases as one leads in to another.
(a) The actions \( a_1 \) and \( a_2 \) are played in equilibrium. Either \( a_1 = \eta \); or, \( a_1 > \eta \), and no type playing \( a_1 \) is willing to play \( a = \eta \) or any equilibrium any action in \((\eta, a_1)\), and no type playing an equilibrium action below \( a_1 \) is willing to play \( a_1 \).

(b) Either \( a_1 = \eta \) or \( a_2 \) is not played in equilibrium. Up to equivalence, \( a_2 \) must be played in equilibrium; otherwise we could assign it low beliefs so that the natural and gamer type would strictly prefer \( a_1 \) to \( a_2 \). So it must be that \( a_2 \) is played, but \( a_1 \) is not played; moreover, up to equivalence, \( a_1 \) has the lowest possible beliefs \( \hat{\eta} = \eta \). For the gamer to be indifferent over \( a_1 \) and \( \eta \), then, it must be that \( a_1 = \eta \). Because low types play an action at least \( \eta \) and at most \( a_1 \), low types play \( a_1 \), and hence this case is only possible if the measure of low types is zero.

(c) It holds that \( a_1 > \eta \), and there is some type of positive measure that plays both actions \( a_0 = \eta \) and \( a_1 \). Such a type has \( \eta = \eta \), and can be a gamer or a low type. Beliefs \( \hat{\eta} \) at \( a_0 \) are at \( \eta \). Beliefs at \( a_1 \) are in \((\eta, \eta)\); the natural type plays \( a_1 \).

(d) It holds that \( a_1 > \eta \), and there is some type of positive measure that plays action \( a_0 = \eta \), and that does not play \( a_1 \) but is willing to play \( a_1 \). Such a type must be the low type—if it were the gamer, then only the natural type would play \( a_1 \), and the gamer would prefer \( a_1 \) over \( a_2 \). Beliefs at \( a_1 \) are in \((\eta, \eta)\), and so the natural and gamer types both play \( a_1 \).

(e) It holds that \( a_1 > \eta \), and there is some type of positive measure playing \( a_1 \) that is willing to play \( a_0 = \eta \) but does not play this action. Such a type has \( \eta = \eta \), and can be a gamer or a low type. Beliefs at \( a_1 \) are in \((\eta, \eta)\), and so the natural type must play \( a_1 \).

In all cases, we assume without loss that all equilibrium actions are weakly above \( \eta \).

**Case 1.** Suppose \( \epsilon_0 \) is a Case 1 equilibrium at stakes \( s_0 \): no two types are both willing to play the same pair of actions. We will show that as \( s \) varies locally, we can slide actions marginally up or down to maintain all indifferences without moving types across actions, and therefore without affecting the distribution of posterior beliefs. We work from left to right, the lowest equilibrium action to the highest. For all such actions we will check indifferences “to the left” – seeing whether any type that is willing to play the given action is also willing to play a lower action. Without loss, it suffices to check only indifferences to lower actions that are played in equilibrium with positive probability, and to action \( a = \eta \); by free downward deviations, other off-path actions can be taken to have sufficiently low beliefs that any agent willing to deviate to one of those would also deviate to a lower equilibrium action or to \( a = \eta \).

Base case: Start with the lowest equilibrium action, i.e., the lowest action played with positive probability by any type. If this action is \( \eta \), then keep it at \( \eta \) and move on to the next step. Otherwise, check if any type playing this action—in particular, the relevant one would just be the low type \((\eta, \gamma)\)—is willing to play \( a = \eta \) as well at the equilibrium beliefs. If not, then as we locally vary \( s \) no agent type wants to deviate down to \( \eta \), and so again keep this action fixed and move on to the next step. So suppose that there is a type playing this action which is willing to play \( \eta \); by assumption of Case 1 there can only be a single indifferent type. As \( s \) varies, slide this lowest equilibrium action up or down to keep this type indifferent at the given beliefs. In particular, when stakes \( s \) increase then the appeal of the current action relative to \( a = \eta \) increases, as there is now a larger benefit of taking a higher action to get higher beliefs, and so we slide the action up to recover the indifference by raising the costs of taking this. When stakes \( s \) decrease then the current action becomes less appealing relative to \( a = \eta \), and so we lower the action to recover the indifference by lowering costs. All the while we maintain the probability that each type chooses this action as it shifts around and therefore keep
indirectly implies which is such a binding indifference, then again there can only be a single indifferent type; this follows from the assumption that no two types are indifferent over the same pair of actions, combined with the characterization of Case (1) from Claim 4 that if two types are willing to play two different lower actions $a_1$ and $a_2$, then $a_2$ cannot be an equilibrium action. Proceed as above, sliding the action up or down to maintain the indifference to lower actions without changing beliefs or moving types across actions. (Here the indifferences are affected both by the direct change in the stakes, and also by potentially having moved the lower actions up or down in previous steps.)

Continue to proceed by induction for each next-higher equilibrium action until we are done. (Recall that there are only finitely many equilibrium actions.) This gives us a new equilibrium at the locally perturbed stakes: no type playing one action strictly wants to shift to any new action, because every previously optimal action remains optimal. This new equilibrium induces the same distribution over beliefs, and so information has not changed as we varied $s$.

**Case 2.** First we will move types across actions and/or slide locations of actions in order to maintain the appropriate indifferences across all actions actions at or below $a_2$. We treat each subcase separately and show that for an increase in $s$ these moves will decrease information, and for a decrease in $s$ these moves will increase information. Following that, without treating each subcase separately, we will slide around actions to maintain appropriate indifferences over actions above $a_2$ in a way that does not additionally change information.

**Subcase (a).** For a marginal decrease in $s$, types which were previously willing to play both $a_1$ and $a_2$ become more attracted to $a_1$. In this subcase there are no relevant binding incentive constraints attracting types playing $a_1$ to actions below $a_1$. Consider two possibilities. First, beliefs $\hat{\eta}$ at either $a_1$ or $a_2$ are in the range $(\eta, \bar{\eta})$, so either natural types play $a_1$ or gamer types play $a_2$ with positive probability. In that case, we follow the cross-type logic of Figure 2 and move either natural types up from $a_1$ to $a_2$, and/or gamer types down from $a_2$ to $a_1$, to increase beliefs at $a_2$ and decrease beliefs at $a_1$ until we recover the appropriate indifference of gamers and naturals between actions $a_1$ and $a_2$. By Claim 3, these moves increase information. The second possibility is that only types with $\eta = \bar{\eta}$ play $a_1$, and only types with $\eta = \hat{\eta}$ play $a_2$—we already have full separation. In that case the natural types at $a_2$ become more attracted to $a_1$ (which they were previously indifferent to); we can slide $a_2$ down to lower the cost of $a_2$ and recover the indifference of the natural type across $a_1$ and $a_2$. Because $a_2 > a_{\bar{\eta}}$, sliding $a_2$ down lowers the cost for the natural more than for the gamer type, and because the natural type is indifferent, the gamer type now strictly prefers $a_1$, the action it was playing, to $a_2$, the action it was not playing. In any event, this slide does not change information.

For a marginal increase in $s$, types which were previously willing to play both $a_1$ and $a_2$ become more attracted to $a_2$. Again, consider two possibilities. First, either natural types play $a_2$ or gamer types play $a_1$ with positive probability. In that case, we effectively reverse the direction of Figure 2: move gamers up from $a_1$ to $a_2$, and/or move naturals down from $a_2$ to $a_1$, to decrease beliefs at $a_2$ and increase beliefs at $a_1$ until we recover the indifferences. Such a move decreases information. Second, no natural
types play $a_2$ and no gamer types play $a_1$ (this can occur if enough high types play $a_2$, and enough low types play $a_1$, that beliefs at $a_2$ are above beliefs at $a_1$). In that case, slide $a_2$ up without moving types across actions until we recover the indifference of gamers across $a_1$ and $a_2$. Because $a_2$ was above $a_0$, sliding $a_2$ up imposes a higher cost increase on naturals than on gamers, and so naturals are no longer indifferent between $a_1$ and $a_2$; they now strictly prefer $a_1$, which they are already playing. In any event, this slide does not change information.

Subcase (b). For a marginal decrease in $s$, the argument proceeds as in subcase (a). Here we are in the “first possibility” where all natural and gamer types play $a_2$, and so moving gamer types down increases beliefs at $a_2$ while holding fixed beliefs of $\hat{\eta} = \eta$ at $a_1$.

For a marginal increase in $s$, we simply keep $a_2$ as it is: natural and gamer types now strictly prefer $a_2$ to $a_1$, and they were not previously playing $a_1$.\footnote{The argument of the “first possibility” of subcase (a) would fail because moving natural types down would discretely rather than continuously increase equilibrium beliefs at $a_1$ from $\eta$ to $\hat{\eta}$.}

Subcase (c). For a marginal decrease in $s$, types which were previously willing to play both $a_1$ and $a_2$ become more attracted to $a_1$, and the indifferent type between $a_0$ and $a_1$ becomes more attracted to $a_0$. Consider two possibilities. First, gamers play $a_2$ with a positive probability. In that case we follow the logic of Figure 3: move the indifferent type (which may be gamers or low types, but in either case have $\eta = \hat{\eta}$) left from $a_1$ to $a_0$ to increase beliefs at $a_1$ and recover the indifference between $a_0$ and $a_1$. Then move gamer types left from $a_2$ to $a_0$ to increase beliefs at $a_2$ and recover the indifference between $a_2$ and $a_1$. Both moves increase information. The second possibility is that no gamer types play $a_2$. In that case, we again start by moving the indifferent type from $a_1$ to $a_0$ to recover that indifference and increase information. Then we slide $a_2$ down to recover the indifference of natural types between $a_1$ and $a_2$; this leaves gamer types now strictly preferring $a_1$ over $a_2$, and does not affect information.

For a marginal increase in $s$, types which were previously willing to play both $a_1$ and $a_2$ become more attracted to $a_2$, and the indifferent type between $a_0$ and $a_1$ becomes more attracted to $a_1$. Now consider two possibilities. The first possibility is that gamers play $a_1$ with positive probability. In that case we do two steps: first, move the indifferent type (with $\eta = \hat{\eta}$) up from $a_0$ to $a_1$ to lower beliefs at $a_1$ and recover the indifference of that type across $a_1$ and $a_2$. This decreases information, and also makes $a_1$ less attractive relative to $a_2$. Second, move the indifferent type right from $a_0$ to $a_1$ while moving gamers right from $a_1$ to $a_2$ at exactly the same rate (or, if the indifferent type was the gamer type, we can move gamers directly from $a_0$ to $a_2$, essentially reversing the direction of Figure 3); this keeps beliefs at $a_1$ fixed, and also at $a_0$, because beliefs were already at $\hat{\eta} = \eta$. But it decreases beliefs at $a_2$, and so we can do this until we recover the indifference of naturals and gamers across $a_1$ and $a_2$. The second possibility is that gamers do not play $a_1$. In that case the indifferent type must be the low type, and gamers must not be willing to play $a_0$. Here, we first move the low type up from $a_0$ to $a_1$ to lower beliefs at $a_1$ and recover that indifference, just as before. This decreases information while further increasing the attractiveness of $a_2$ relative to $a_1$. Next, we slide action $a_2$ up to recover the indifference of natural types across $a_1$ and $a_2$; the gamer types now strictly prefer $a_2$ to $a_1$, since their costs of $a_2$ increase by less than those of the natural types. But they were not previously playing any action below $a_2$, so this does not affect their behavior.

Subcase (d). For a decrease in $s$, the low type which is indifferent over $a_0 = \eta$ and $a_1$ becomes more attracted to $a_0$ relative to $a_1$, since the benefit of higher beliefs has gone down while costs have not
changed. Likewise the gamer and natural type become more attracted to \( a_1 \) relative to \( a_2 \). Consider two possibilities. The first is that gamer types do not play \( a_2 \), so that beliefs at \( a_2 \) are at \( \hat{\eta} = \overline{\eta} \). In that case we can move natural types up from \( a_1 \) to \( a_2 \) to decrease beliefs at \( a_1 \) until we recover the indifference of the natural and gamer types between \( a_1 \) and \( a_2 \). This makes \( a_1 \) less appealing relative to \( a_0 \), so we do not have to worry about low types becoming attracted from \( a_0 \) to \( a_1 \). The second possibility is that gamer types play \( a_2 \). In that case we move gamer types down from \( a_2 \) to \( a_1 \) to reduce beliefs at \( a_1 \) and increase beliefs at \( a_2 \), recovering the indifference of naturals and gamers between \( a_1 \) and \( a_2 \) while increasing information. This again makes \( a_1 \) less appealing to the low types relative to \( a_0 \), so they continue to play \( a_0 \).

For a increase in \( s \), proceed as in subcase (c).

**Subcase (e).** For a decrease in \( s \), proceed as in subcase (c).

For an increase in \( s \) makes \( a_1 \) more appealing relative to \( a_0 \), and any moves of types across actions do not decrease beliefs at \( a_1 \), and so the previously indifferent type now strictly prefers \( a_1 \) to \( a_0 \).

As mentioned earlier, the above analysis corrects all of the incentives across actions less than or equal to \( a_2 \). We now turn to actions above \( a_2 \). If the equilibrium has no actions above \( a_2 \) (i.e., high types only play \( a_2 \)) then of course we are done. It may also be the case that the equilibrium has one action above \( a_2 \) (call it \( a_3 \)), or two actions above \( a_2 \) (\( a_3 \) and \( a_4 \)). When there is one action above \( a_2 \), it is played by the high type and possibly by the gamer type. When there are two actions above \( a_2 \), the lower one \( a_3 \) is played by the gamer type and the high type while the higher one \( a_4 \) is played only by the high type (the gamer and high type cannot both be indifferent over a pair of actions \( a_3 \) and \( a_4 \)). In any case, as we vary \( s \), we can slide actions around as in Case 1 to recover the appropriate equilibrium indifferences. Moving again from left to right, if a type that plays \( a_3 \) had been willing to play \( a_2 \), then after the above perturbations simply slide up or down as appropriate \( a_3 \) to maintain this indifference; if no such type had been indifferent, then keep \( a_3 \) where it was. Then do the same for \( a_4 \), maintaining any relevant indifferences with lower actions. There are no complications here because the two relevant types are ordered by single-crossing.

**Proof of Lemma 4.** We proceed similarly to the proof of Lemma 3 above, with three key differences. First, fixing an equilibrium \( e_0 \) at stakes \( s_0 \), we generate a path of equilibria only for \( s \geq s_0 \). Second, with the dimension of interest equal to gaming ability rather than natural action, the direction of the information effect is reversed: as stakes increase, we find perturbations that increase information. We do this by only ever moving low-\( \gamma \) types down, from actions with high beliefs to ones with low beliefs, and moving high-\( \gamma \) types up from low beliefs to high. Third, while in Lemma 3 all of the perturbations moved equilibria continuously, here we sometimes use a trick of looking for continuous perturbations about an equilibrium that induces the same distribution of beliefs as equilibrium \( e_0 \), but has different strategies for some types. This gives us a path of equilibria over \( s \in [s_0, \infty) \) in which distributions of beliefs vary continuously with the stakes, but strategies may jump discretely at \( s_0 \).

Using Claim 4, we can categorize all possible equilibria into a number of exhaustive cases (up to equivalence), and then address these cases separately.

**Case 1.** For any pair of distinct actions weakly above \( \underline{\eta} \), at most a single type is willing to play both of these.
Case 2. There exist actions $a_1$ and $a_2$ satisfying $\eta \leq a_1 < a_0 < a_2 \leq \gamma$ such that the gamer type $(\eta, \gamma)$ and the natural type $(\gamma, \gamma)$ are both willing to play $a_1$ and $a_2$. No actions in $(a_1, a_2)$ are played in equilibrium. Natural types $(\gamma, \gamma)$ are not willing to play any action strictly below $a_1$ or above $a_2$, low types $(\eta, \gamma)$ are not willing to play any action above $a_1$, and high types $(\eta, \gamma)$ are not willing to play any action below $a_2$.

Any equilibrium action above $a_2$ can only be played by the gamer and high type, and so it must have belief of $\gamma = \gamma$; thus, there can be at most one such equilibrium action, $a_3$. We now divide this case into subcases on two separate dimensions: each equilibrium is in one category (i)-(iii) characterizing its higher actions above $a_2$, and in one category (a)-(c) characterizing its lower actions below $a_1$.

(i) Some type plays both $a_2$ and $a_3 > a_2$. This type can be the high type or the gamer.

(ii) No type plays an action above $a_2$.

(iii) The action $a_3$ is played only by the high type, and the high type does not play $a_2$. Here the gamer must play $a_2$; otherwise no $\gamma = \gamma$ types would be playing $a_2$, so it would have to have beliefs $\hat{\gamma} = \gamma$ and so $a_1$ would be preferred by all types to $a_2$.

(a) Either there are no equilibrium actions below $a_1$; or, there are lower actions, but no type playing a lower action is willing to play $a_1$. Because only low and gamer types can be willing to play actions below $a_1$, and because gamer types are willing to play $a_1$, it means that only a low type could be playing a lower action without being willing to play $a_1$. Up to equivalence, this low type would play $a = \eta$.

(b) Some type plays an equilibrium action below $a_1$, call it $a_0$, and is also willing to play $a_1$. Moreover, natural types play $a_2$ in equilibrium. The indifferent type between $a_0$ and $a_1$ can be a low or a gamer type. There can only be one such indifference. In this case $a_1$ must be an equilibrium action.

(c) Some type plays an equilibrium action below $a_1$, call it $a_0$, and is also willing to play $a_1$. Moreover, natural types do not play $a_2$ in equilibrium. The indifferent type between $a_0$ and $a_1$ can be a low or a gamer type. There can only be one such indifference. In this case $a_1$ is an equilibrium action. Note that because natural types do not play $a_2$, beliefs are $\hat{\gamma} = \gamma$ at $a_2$ so we must be in subcase (ii) as well, where there are no actions above $a_2$.

In all cases, we assume without loss that all equilibrium actions are weakly above $\eta$.

Case 1. This case proceeds exactly as in the proof of Lemma 3. As stakes increase, we can perturb the equilibrium by sliding actions around in a way which has no impact on the distribution of beliefs.

Case 2. First, prior to varying the stakes, we tweak the equilibrium in the following way which does not affect the information. In subcase (i), do nothing. In subcase (ii), do not change any strategies, but define $a_3$ to be the action such that the highest type playing $a_2$ (the high type, if such types have positive measure; the gamer otherwise) would be just indifferent between $a_2$ at the current beliefs under $e_0$ and action $a_3$ under beliefs $\hat{\gamma} = \gamma$. Finally, in subcase (iii), slide $a_3$ down by a discrete amount until the gamer type is just indifferent to playing $a_3$. At this level the high type still strictly prefers $a_3$ to $a_2$ at the beliefs under equilibrium $e_0$. The key to these tweaks is that now the highest type playing $a_2$ (either gamer or high types) has become indifferent to deviating up to $a_3$, if it was not already indifferent.
Starting from this tweaked equilibrium, we now look for continuous perturbations that increase information as stakes $s$ increase. As stakes increase, types playing lower actions may become more attracted to higher actions at higher beliefs.

Now, consider any equilibrium actions strictly below $a_1$. Only low and gamer types can play actions below $a_1$, and there are either zero, one, or two of these actions. If there are no such actions, then types playing $a_1$ only become less inclined to deviate downwards as $s$ increases, so we can move on. If there is one such action, fix that action and move on. If there are two, then as stakes increase, fix the lowest action, and marginally slide the second-lowest action up as necessary in order to make sure that types playing the lowest action do not now want to deviate to the second one.

As stakes increase, types playing lower actions may become more attracted to $a_1$, but the reverse does not hold. Moreover, if in the previous step we slid a lower action up to maintain indifferences, that only makes the lower action even less appealing to a type playing $a_1$. So we now turn to maintaining indifferences between the lower actions and $a_1$, and also across higher actions. We treat subcases (a)-(c) separately.

**Subcase (a).** Here there are no relevant indifferences between lower actions and action $a_1$: as we marginally increase stakes, any types playing actions below $a_1$ do not become attracted to $a_1$. So we can move on to indifferences between actions $a_1$ and higher.

The increase in $s$ makes makes $a_2$ more appealing relative to $a_1$. So to recover the indifferece of naturals and gamers between $a_1$ and $a_2$, we move the highest type playing $a_2$ (either gamers or high types) up to $a_3$, as in Figure 4. (The equilibrium tweak from before guarantees that this type was previously indifferent to playing $a_2$ or $a_3$.) This move reduces beliefs at $a_2$ as desired, and makes the equilibrium more informative. Finally we slide $a_3$ as necessary to maintain the indifference of the type being moved between $a_2$ and $a_3$.

**Subcase (b).** As stakes increase, and as we potentially slide $a_0$ upwards to prevent deviations to $a_0$ from lower actions, action $a_1$ has become more attractive to the previously indifferent type. To recover the indifference between $a_0$ and $a_1$, we move natural types down from $a_2$ to $a_1$ to reduce beliefs at $a_1$. This move increases information, since it moves a low-$\gamma$ type from a high belief action to a low one.

The increase in $s$ makes makes $a_2$ more appealing relative to $a_1$, and the above move of natural types from $a_2$ to $a_1$—increasing beliefs at $a_2$ and decreasing beliefs at $a_1$—does the same. So to recover the indifference of naturals and gamers between $a_1$ and $a_2$, we proceed as in subcase (a) and move gamers or high types up from $a_2$ to $a_3$, then slide $a_3$ as necessary. These moves make the equilibrium more informative.

**Subcase (c).** As stakes increase, and as we potentially slide $a_0$ upwards to prevent deviations to $a_0$ from lower actions, action $a_1$ has become more attractive to the previously indifferent type. To recover the indifference between $a_0$ and $a_1$, we slide action $a_1$ up without moving types across actions. This has no effect on information. Next, slide action $a_2$ up to keep gamer types indifferent between $a_2$ and $a_1$; an increase in stakes has made $a_2$ relatively more attractive, and sliding $a_1$ up does the same, so we have to increase the costs of $a_2$ to keep the natural and gamer types from deviating to that. Notice that because $a_1 < a^0$, sliding $a_1$ up increases the cost of taking $a_1$ more for gamer types than for naturals; and because $a_2 > a^0$, sliding $a_2$ up increases the cost of taking $a_2$ less for gamer types than for naturals. So if gamers have been made indifferent between $a_1$ and $a_2$, naturals now strictly prefer $a_1$; but by assumption of subcase (c), the naturals had not been playing $a_2$, so we maintain the equilibrium. (In this subcase there is no need to address action $a_3$ because no types had been playing any action above $a_2$.)

Q.E.D.
D.2. Proofs for Subsection 4.2

Proof of Proposition 2. Consider the claim about small stakes. By hypothesis, \( \Theta_\eta \) is finite. We claim that for small enough \( s \) there is an equilibrium in which every type \( \theta = (\eta, \gamma) \) takes its natural action, \( a = \eta \); any off-path action \( a \notin \Theta_\eta \) is assigned the belief \( \hat{\tau} = \min \Theta_\eta \). Clearly, no type has a profitable deviation to any off-path action nor to any action below its natural action. So it suffices to show that there is no incentive for any type to deviate to any on-path action above its natural action (an “upward deviation”) when \( s \) is small enough. The proposition’s hypotheses about \( \Theta \) imply that there is some \( \epsilon > 0 \) such that \( C(a, \eta, \gamma) > \epsilon \) for any \((\eta, \gamma) \in \Theta \) and \( a \in \Theta_\eta \cap (\eta, \infty) \). For any type, the gain from deviating to any action is bounded above by \( V(\max \Theta_\eta; s) - V(\min \Theta_\eta; s) \), which, by Assumption 2, tends to 0 as \( s \to 0 \). It follows that for small enough \( s > 0 \), the cost of any upward deviation outweighs the benefit for all types.

The claim about large stakes follows from Lemma 5 and Part 3 of Assumption 2. Q.E.D.

Proof of Lemma 5. Fix an arbitrary type space containing \( \theta_1 \leftrightarrow \theta_2 \) and an arbitrary equilibrium in which each \( \theta_i \) \((i = 1, 2)\) uses action \( a_i \) inducing belief \( \hat{\eta}_i \). If \( \hat{\eta}_2 \leq \hat{\eta}_1 \) then the result is trivially true, so suppose \( \hat{\eta}_2 > \hat{\eta}_1 \). By belief monotonicity, \( a_1 < a_2 \). Incentive compatibility implies

\[
C(a_2, \theta_2) - C(a_1, \theta_2) \leq V(\hat{\eta}_2; s) - V(\hat{\eta}_1; s) \leq C(a_2, \theta_1) - C(a_1, \theta_1).
\]

Hence, \( V(\hat{\eta}_2; s) - V(\hat{\eta}_1; s) \) is bounded above by the maximum of \( C(a_2, \theta_1) - C(a_1, \theta_1) \) subject to \( a_2 \geq a_1 \) and \( C(a_2, \theta_1) - C(a_1, \theta_1) \geq C(a_2, \theta_2) - C(a_1, \theta_2) \). Lemma 1 implies that the constraint is violated if \( a_2 > a^{ce} \); hence the maximum is obtained when \( a_2 = a^{ce} \) and \( a_1 = \eta \). Q.E.D.

Proof of Proposition 3. Given an equilibrium, let \( \hat{\eta}(\theta) \) denote a belief induced by type \( \theta \). Given a sequence of equilibria as \( s \to \infty \), let \( \hat{\eta}^*(\theta) \) denote any limit point of such beliefs as \( s \to \infty \) (passing to sub-sequence if necessary). We first claim that in any sequence of equilibria, it holds for any \( \theta' = (\eta', \gamma') \) and \( \theta'' = (\eta'', \gamma'') \) with \( \gamma'' > \gamma' \) that \( \hat{\eta}^*(\theta'') \geq \hat{\eta}^*(\theta') \); in words, in the limit any type with a higher gaming ability induces a weakly higher belief about its natural action. If \( \eta'' \geq \eta' \), the claim follows from the fact that \( \eta'' \) and \( \eta' \) are ordered by single-crossing and hence \( \theta'' \) must induce a weakly larger belief than \( \theta' \) in any equilibrium; if \( \eta'' < \eta' \), the claim follows from Lemma 5 and Part 3 of Assumption 2.

Now fix an arbitrary sequence of equilibria as \( s \to \infty \). It suffices to prove that for any type \( \theta = (\eta, \gamma) \) with \( \gamma \in (\min \Theta_\gamma, \max \Theta_\gamma) \), \( \hat{\eta}^*(\theta) = E[\eta] \). To contradiction, suppose there is a type \( \theta' \) with \( \gamma' \in (\min \Theta_\gamma, \max \Theta_\gamma) \) and \( \hat{\eta}^*(\theta') = E[\eta] + \Delta \) for \( \Delta > 0 \). (A symmetric argument applies if \( \Delta < 0 \).) Let \( S_\Delta = \{ \theta : \hat{\eta}^*(\theta) \geq E[\eta] + \Delta \} \). The claim in the previous paragraph establishes that there exists some \( \hat{\gamma} < \max \Theta_\gamma \) such that \( S_\Delta \) contains all types with gaming ability strictly above \( \hat{\gamma} \) and no types with gaming ability strictly below \( \hat{\gamma} \). In other words, modulo “boundary types” with \( \gamma = \hat{\gamma} \)—a set that has probability zero, by the hypothesis of a continuous marginal distribution of \( \gamma \)—we can take \( S_\Delta = \{ \theta' = (\eta'', \gamma'') | \gamma'' \geq \hat{\gamma} \} \); note that \( S_\Delta \) has positive probability. Hence, \( E[\eta|\theta \in S_\Delta] = E[\eta|\gamma \geq \hat{\gamma}] \leq E[\eta] \), where the inequality owes to the hypothesis that \( E[\eta|\gamma] \) is non-increasing in \( \gamma \). But this contradicts the Bayesian consistency requirement that \( E[\eta|\theta \in S_\Delta] = E[\hat{\eta}^*(\theta)|\theta \in S_\Delta] \geq E[\eta] + \Delta \). Q.E.D.

Proof of Proposition 4. Let \( \eta_{max} := \sup \Theta_\eta \) and \( \eta_{min} := \inf \Theta_\eta \) and order the values of \( \gamma \in \Theta_\gamma \) as \( \gamma_1 < \gamma_2 < \cdots < \gamma_N \). Let \( a_i^{ce} \) be the cost-equalizing action between types \((\eta_{max}, \gamma_i)\) and \((\eta_{min}, \gamma_{i+1})\). At actions \( a'' > a_i^{ce} \) and \( a' < a'' \), it holds that \( C(a'', (\eta', \gamma_{i+1})) - C(a', (\eta', \gamma_i)) < C(a'', (\eta'', \gamma_i)) - C(a', (\eta'', \gamma_i)) \) for any \( \eta', \eta'' \) in \( \Theta_\eta \).
Now define \( a_i(s) \) as follows. Set \( a_1(s) = \eta_{\max} \). For \( i \geq 1 \), given \( a_i(s) \), inductively define \( a_{i+1}(s) \) to be the action such that \( C(a_{i+1}(s), (\eta_{\max}, \gamma_i)) - C(a_i(s), (\eta_{\max}, \gamma_i)) = V(\gamma_{i+1}; s) - V(\gamma_i; s) \). To help interpret, observe that this would be the sequence of least-cost separating actions at stakes \( s \) if the type space were \( \{\eta_{\max}\} \times \Theta_\gamma \) rather than \( \Theta \). Assumption 2 implies that for any \( i \geq 2 \), \( a_i(s) \to \infty \) as \( s \to \infty \). Hence, there exists \( \tilde{s} \) such that for any \( s > \tilde{s} \), \( a_{i+1}(s) > a^c_i \) for each \( i = 1, \ldots, N-1 \).

We claim that for any \( s > \tilde{s} \), there is an equilibrium in which (i) any type \( (\eta, \gamma_1) \in \Theta \) takes action \( \eta \) and (ii) any type \( (\eta, \gamma_i) \in \Theta \) with \( i > 1 \) takes action \( a_i(s) \). Plainly, this strategy is separating on \( \gamma \). To see that we have an equilibrium when \( s > \tilde{s} \), first consider local incentive constraints among the on-path actions.

Plainly, no type wants to deviate upwards, because \( (\eta_{\max}, \gamma_i) \) is by construction indifferent between playing \( a_i \) and \( a_{i+1} \), while all other \( (\eta, \gamma_i) \) types prefer \( a_i \) to \( a_{i+1} \). No type wants to deviate downwards because \( a_{i+1}(s) > a^c_i \), hence the indifference of \( (\eta_{\max}, \gamma_i) \) between \( a_i \) and \( a_{i+1} \) implies that for any \( \eta \), type \( (\eta, \gamma_{i+1}) \) prefers \( a_{i+1} \) to \( a_i \). Standard arguments using the single-crossing property on dimension \( \gamma \) then imply global incentive compatibility among on-path actions. Finally, off-path actions can be deterred by assigning them the lowest belief, \( \gamma_1 \).

Q.E.D.

**Proof of Lemma 6.** Suppose there is an equilibrium with a pair of actions \( a_1 \) and \( a_2 \), used respectively by \( \theta_1 \) and \( \theta_2 \), yielding beliefs \( \hat{\gamma}_1 > \hat{\gamma}_2 \). By belief monotonicity, \( a_1 > a_2 \). That neither type strictly prefers the other action over its own implies \( a_1 > a^\sigma \), or otherwise \( C(a_1, \theta_2) - C(a_2, \theta_2) < C(a_1, \theta_1) - C(a_2, \theta_1) \). That type \( \theta_1 \) is willing to play action \( a_1 \) rather than its natural action implies \( C(a_1, \theta_1) \leq V(\max \Theta_\gamma; s) - V(\min \Theta_\gamma; s) \), because the LHS of the inequality is the incremental cost while the RHS is an upper bound on the incremental benefit. It follows from the strict monotonicity of \( C(\cdot; \theta_1) \) on \( (a^\sigma, a_1) \) that \( C(a^\sigma, \theta_1) < V(\max \Theta_\gamma; s) - V(\min \Theta_\gamma; s) \).

Q.E.D.

**Proof of Proposition 5.** We omit the proof because the argument is analogous to that provided for Proposition 3, switching \( \gamma \) and \( \eta \), taking \( s \to 0 \) rather than \( s \to \infty \), and using Lemma 6 to conclude that in the limit of vanishing stakes, any type with a higher natural action induces a weakly higher belief about its gaming ability.

Q.E.D.

### E. Proofs for Section 5

Define the following notation: for any \( d \in \{\eta, \gamma\} \), let \( -d \) denote \( \eta \) if \( d = \gamma \) and \( \gamma \) if \( d = \eta \).

#### E.1. Characterizing linear equilibria

**Lemma 7.** Consider the LQE specification and fix \( \tau \in \{\eta, \gamma\} \). Suppose agents of type \( (\eta, \gamma) \) take action \( a = l_{\eta} \eta + l_{\gamma} \gamma + b \) for some constants \( l_{\eta} \neq 0 \), \( l_{\gamma} \neq 0 \), and \( b \). Then:

1. The vector \( (\tau, a) \) is distributed according to \( E \left( (\mu_\tau, \mu_a), \begin{pmatrix} \sigma_\tau^2 & \sigma_\tau a \\ \sigma_\tau a & \sigma_a^2 \end{pmatrix}, g_\tau a \right) \) with

\[
\begin{align*}
\mu_a &= l_{\eta} \mu_\eta + l_{\gamma} \mu_\gamma + b, \\
\sigma_a^2 &= 2l_{\eta} l_{\gamma} \rho \sigma_\eta \sigma_\gamma + l_{\eta}^2 \sigma_\eta^2 + l_{\gamma}^2 \sigma_\gamma^2, \\
\sigma_\tau a &= l_{\tau} \sigma_\tau^2 + l_{-\tau} \rho \sigma_\eta \sigma_\gamma, \\
g_\tau a(\cdot) &= g_\theta(\cdot).
\end{align*}
\]
2. $\hat{\tau}(a) \equiv \mathbb{E}[\tau|a] = \mu_\tau + \frac{\sigma_{\tau a}}{\sigma_a^2} (a - \mu_a)$.

3. $(\tau, a)$ has an $R^2$ of

$$R^2_{\tau a} = \frac{\left(l_\tau \sigma_\tau^2 + l_{-\tau} \rho \sigma_\eta \sigma_\gamma \right)^2}{\sigma_\tau^2 \left(2l_\eta l_{-\tau} \rho \sigma_\eta \sigma_\gamma + l_\eta^2 \sigma_\eta^2 + l_{-\tau}^2 \sigma_\gamma^2 \right)}.$$

**Proof of Lemma 7.** Parts 1 and 2 follow from the properties of elliptical distributions covered in Gómez et al. (2003); respectively, see Theorems 5 and 8 of that paper. Part 3 is immediate from Part 1 and that $R^2_{\tau a} \equiv \frac{\sigma_{\tau a}^2}{\sigma_\tau^2 \sigma_a^2}$.

Q.E.D.

For the strategy $a = \eta + sL\gamma$ (Equation 5) with $L > 0$, Lemma 7 implies

$$R^2_{\eta a} = \frac{\left(\sigma_\eta^2 + sL \rho \sigma_\eta \sigma_\gamma \right)^2}{\sigma_\eta^2 \left(2sL \rho \sigma_\eta \sigma_\gamma + \sigma_\eta^2 + sL^2 \sigma_\gamma^2 \right)},$$

$$R^2_{\gamma a} = \frac{\left(\rho \sigma_\eta \sigma_\gamma + sL \sigma_\gamma^2 \right)^2}{\sigma_\gamma^2 \left(2sL \rho \sigma_\eta \sigma_\gamma + \sigma_\eta^2 + sL^2 \sigma_\gamma^2 \right)}.$$

For the next result, recall that we use $\beta_\tau$ to denote the induced posterior beliefs on $\tau$ after observing the realization of $a$. These beliefs are determined in equilibrium by the joint distribution of $a$ and $\tau$.

**Lemma 8.** Fix $\tau \in \{\eta, \gamma\}$, as well as $\sigma_\tau$, $\mu_\tau$, and $g_{\tau a}$. If the equilibrium distribution of $(\tau, a)$ is

$$\mathcal{E} \left( (\mu_\tau, \mu_a), \left( \frac{\sigma_\tau^2}{\sigma_{\tau a}^2}, \frac{\sigma_{\tau a}}{\sigma_a^2} \right), g_{\tau a} \right),$$

then the ex ante distribution of $\beta_\tau$ (prior to the realization of the action) depends only on $R^2_{\tau a} \equiv \frac{\sigma_{\tau a}^2}{\sigma_\tau^2 \sigma_a^2}$. Beliefs are more informative about $\hat{\tau}$ if $R^2_{\tau a} \in [0, 1]$ is larger, with beliefs being uninformative about $\hat{\tau}$ if $R^2_{\tau a} = 0$ and fully informative about $\hat{\tau}$ (and $\tau$) if $R^2_{\tau a} = 1$.

**Proof of Lemma 8.** For any joint distribution of $\tau$ and $a$, the ex ante distribution of $\beta_\tau$ is clearly preserved under affine transformations of the action. Specifically, subtract $\mu_a$ from all actions; multiply actions by $\frac{1}{\sigma_a}$; and then multiply actions by $-1$ if $\sigma_{\tau a} < 0$. The transformation normalizes $\mu_a = 0$, $\sigma_a^2 = 1$, and $\sigma_{\tau a} \geq 0$ while preserving $R^2_{\tau a}$. Since $\sigma_{\tau a} \in [0, \sigma_\tau]$ is the only remaining parameter under the normalization, it is immediate that the ex ante distribution of $\beta_\tau$ depends only on $R^2_{\tau a}$. Moreover, there is a bijection from $\sigma_{\tau a} \in [0, \sigma_\tau]$ to $R^2_{\tau a} \in [0, 1]$ in which higher $\sigma_{\tau a}$ corresponds to higher $R^2_{\tau a}$.

Given the normalizations above of $\sigma_a^2 = 1$ and $\mu_a = 0$, Theorem 8 of Gómez et al. (2003) implies that $\hat{\tau}(a) = \mu_\tau + a\sigma_{\tau a}$. Since the marginal distribution of $a$ is independent of $\sigma_{\tau a}$, changes in $\sigma_{\tau a}$ simply correspond to scale shifts of the distribution of $\hat{\tau}$; hence, for any $0 \leq \xi_{\tau a} < \sigma_\tau \leq \sigma_\tau$, the distribution of $\hat{\tau}$ when $\sigma_{\tau a} = \xi_{\tau a}$ is a mean-preserving spread of that when $\sigma_{\tau a} = \sigma_\tau$.

Finally, beliefs are uninformative about $\hat{\tau}$ when $R^2_{\tau a} = 0$ because that corresponds to the case where $\sigma_{\tau a} = 0$, in which case $\hat{\tau}(a)$ is constant across actions. Beliefs are fully informative about $\hat{\tau}$ when $R^2_{\tau a} = 1$ because that corresponds to the case where $\sigma_{\tau a} = \sigma_\tau$, in which case the two variables are perfectly correlated and the conditional distribution of $\tau$ given $a$ is degenerate at a single point.

Q.E.D.
Lemma 9. Consider the LQE specification. Given linear market beliefs $\hat{\tau}(a) = La + K$ with $L > 0$ (Equation 4), the agent’s unique optimal action is given by $a = \eta + sL\gamma$ (Equation 5).

Conversely, given a linear strategy of the form $a = \eta + sL\gamma$, and denoting $\mathcal{L}(s, \hat{L}, \eta) \equiv 1$ and $\mathcal{L}(s, \hat{L}, \gamma) \equiv s\hat{L}$, the market belief on dimension $d \in \{\eta, \gamma\}$ is linear in the agent’s action, with slope coefficient

$$\frac{\mathcal{L}(s, \hat{L}, d)\sigma_d^2 + \mathcal{L}(s, \hat{L}, -d)\rho\sigma_d\sigma_g}{\sigma_\eta^2 + s^2L^2\sigma_\gamma^2 + 2sL\rho\sigma_\gamma\sigma_\eta}.$$ 

Therefore, an increasing linear equilibrium for dimension of interest $\tau \in \{\eta, \gamma\}$ is characterized by (4) and (5), where $L > 0$ solves

$$L = \frac{\mathcal{L}(s, L, \tau)\sigma_\tau^2 + \mathcal{L}(s, L, -\tau)\rho\sigma_\tau\sigma_g}{\sigma_\eta^2 + s^2L^2\sigma_\gamma^2 + 2sL\rho\sigma_\gamma\sigma_\eta}.$$ 

Proof. First consider the agent’s best response to any linear market belief, $\hat{\tau}(a) = La + K$ with $L > 0$. For any $\eta, \gamma$, the agent solves

$$\max_{a \in \mathbb{R}} s(La + K) - \frac{(\max\{a - \eta, 0\})^2}{2\gamma}.$$ 

This maximand is globally weakly concave in $a$, and strictly concave on $a \geq \eta$. Since $L > 0$, it is suboptimal for the agent to choose $a \leq \eta$, and so the first order condition yields the unique optimum $a = \eta + sL\gamma$.

Now consider the market posterior given a strategy $a = \eta + sL\gamma$. Plugging $l_\eta = 1$ and $l_\gamma = sL$ into Lemma 7 parts 1 and 2 gives the stated expression.

By Theorem 8 of Gómez et al. (2003), it follows that for any observed action $a$, the marginal distribution of the market posterior on dimension $d \in \{\eta, \gamma\}$ is elliptically distributed with mean $\mu_{d|a}$ given by

$$\mu_{d|a} = \mu_d + \frac{l_\eta\sigma_\eta^2 + l_\eta\rho\sigma_\eta\sigma_g}{l_\eta^2\sigma_\eta^2 + l_\eta^2\sigma_\gamma^2 + 2l_\eta\rho\sigma_\gamma\sigma_\eta}(a - k - l_\eta\mu_\eta - l_\gamma\mu_\gamma).$$

Plainly, $\mu_{d|a}$ is linear in $a$ with slope coefficient as stated in the lemma (with obvious notational adjustment).

Q.E.D.

E.2. Proof of Proposition 6

Part 1 of Proposition 6 is subsumed in Lemma 10 below; Part 2(a) and 2(b) of Proposition 6 are immediate consequences of expression (17) and Lemma 11 below; and Part 3 of Proposition 6 is Lemma 12 below. Given $\tau = \eta$, it will be convenient to rewrite Equation 14 as

$$f_\eta(L, s, \sigma_\eta, \sigma_\gamma, \rho) := s^2\sigma_\gamma^2L^3 + 2s\rho\sigma_\gamma\sigma_\eta L^2 + (\sigma_\eta^2 - s\rho\sigma_\eta\sigma_\gamma) L - \sigma_\eta^2 = 0.$$ 

Lemma 10. If $\rho \geq 0$ then there is unique solution to Equation 15 on the non-negative domain; this solution satisfies $L \in (0, 1)$ and $\frac{\partial f_\eta}{\partial L} > 0$ at the solution.

Proof. Assume $\rho \geq 0$. Differentiation shows that $f_\eta(\cdot)$ is strictly convex in $L$ on the domain $L > 0$. The result follows from the observation that $f_\eta(0, \cdot) < 0 < f_\eta(1, \cdot)$.

Q.E.D.

Lemma 11. Let $\rho \geq 0$. As $s \to \infty$, it holds for the solution $L$ to Equation 15 that $L \to 0$ and $L^2s \to \rho\sigma_\eta^2$. As $s \to 0$, it holds for the solution $L$ to Equation 15 that $L \to 1$. 

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Proof. The result for \( s \to 0 \) is immediate from Equation 15, so we only prove the limits as \( s \to \infty \). Assume \( \rho \geq 0 \). Divide Equation 15 by \( s^2 \) to get

\[
\sigma^2_s L^3 \frac{\sigma_s^2(1 - L)}{s^2} + \rho \sigma_s \gamma \frac{L}{s} (2L - 1) = 0.
\]

Let \( s \to \infty \). Since \( L \in (0, 1) \), all terms on the LHS above except the first one vanish as \( s \to \infty \), so it must be that \( L \to 0 \). Next rewrite Equation 15 by dividing by \( Ls \) as

\[
s \sigma^2_s L^2 + 2L \rho \sigma_s \gamma - \rho \sigma_s \gamma - \frac{\sigma_s^2}{Ls} (1 - L) = 0.
\]

Suppose, to contradiction, that \( Ls \) converges to something finite (which must be non-negative). Then \( L^2 s \to 0 \), and hence the first two terms on the LHS of (16) above vanish, which means the LHS of (16) goes to something strictly negative, a contradiction. Hence, \( Ls \to \infty \). This implies the LHS of (16) goes to \( s \sigma^2_s L^2 - \rho \sigma_s \gamma \), which implies \( L^2 s \to \rho \sigma^2_s \).

Q.E.D.

Lemma 12. If \( \rho \geq 0 \), then (i) \( \frac{d}{d\sigma_s} R^2_{\eta a} = 0 \), (ii) \( \frac{d}{d\sigma_s} R^2_{\eta a} < 0 \), (iii) \( \frac{d}{ds} R^2_{\eta a} < 0 \), and (iv) \( \frac{d}{d\rho} R^2_{\eta a} > 0 \).

Proof. First observe that Equation 15 implies

\[
2sL \rho \sigma_s \gamma + \sigma_s^2 + s^2 L^2 \sigma_s^2 = \frac{1}{L} (\sigma_s^2 + sL \rho \sigma_s \gamma).
\]

We now rewrite Equation 12 and plug in the above identity to get

\[
R^2_{\eta a} = \frac{(\sigma_s^2 + sL \rho \sigma_s \gamma)^2}{\sigma_s^2 (2sL \rho \sigma_s \gamma + \sigma_s^2 + s^2 L^2 \sigma_s^2)}
= \frac{(\sigma_s^2 + sL \rho \sigma_s \gamma)^2}{\sigma^2_s L (\sigma_s^2 + sL \rho \sigma_s \gamma)}
= L + sL^2 \rho \frac{\sigma_s}{\sigma_s}.
\]

For the first part, note from (17) that \( R^2_{\eta a} \) depends neither directly nor indirectly (through the solution \( L \) to Equation 15) on \( \mu_s \). For the other parts, we use the chain rule and the implicit function theorem (which Lemma 10 ensures we can apply), to derive that for any \( x = \sigma_s, s, \rho \),

\[
\frac{d}{dx} R^2_{\eta a} = \frac{\partial R^2_{\eta a}}{\partial L} \frac{\partial L}{\partial x} + \frac{\partial R^2_{\eta a}}{\partial s} \frac{\partial s}{\partial x} = \frac{1}{\frac{\partial f_s}{\partial L}} \left( \frac{\partial R^2_{\eta a}}{\partial L} \frac{\partial f_s}{\partial L} - \frac{\partial R^2_{\eta a}}{\partial s} \frac{\partial f_s}{\partial x} \right).
\]

Each partial derivative in (18) can be computed from either (15) or (17), and some manipulations then yield parts (ii), (iii), and (iv) of the result. In more detail:

\[
\frac{\partial R^2_{\eta a}}{\partial L} = 1 + 2s L \rho \frac{\sigma_s}{\sigma_s} \frac{\partial R^2_{\eta a}}{\partial \sigma_s} = sL^2 \rho \frac{\sigma_s}{\sigma_s}, \quad \frac{\partial R^2_{\eta a}}{\partial s} = L^2 \rho \frac{\sigma_s}{\sigma_s}, \quad \frac{\partial R^2_{\eta a}}{\partial \rho} = sL^2 \frac{\sigma_s}{\sigma_s},
\]

and

\[
\frac{\partial f_s}{\partial L} = \sigma_s^2 + 3L^2 s^2 \sigma_s^2 + (4L - 1) \rho s \sigma_s \sigma_s.
\]
\[
\frac{\partial f_\gamma}{\partial \sigma_\gamma} = Ls \left((2L^2s\sigma_\gamma + (2L - 1)\rho\sigma_\eta)\right) , \\
\frac{\partial f_\gamma}{\partial s} = L\sigma_\gamma \left(2L^2s\sigma_\gamma + (2L - 1)\rho\sigma_\eta\right) , \\
\frac{\partial f_\gamma}{\partial \rho} = L(2L - 1)s\sigma_\gamma\sigma_\eta .
\]

Since \(\frac{\partial f_\gamma}{\partial L} > 0\) at the solution to \(L\) (Lemma 10), we drop the \(\frac{1}{\sigma_\gamma}\) term in (18) and compute

\[
\begin{align*}
\frac{d}{d\sigma_\gamma} R_{\eta a}^2 &\propto \frac{Ls}{\sigma_\eta} \left(-L^3\rho s^2 \sigma_\gamma^2 + (1 - L)\rho s^2 \sigma_\eta^2 + Ls\sigma_\gamma\sigma_\eta (\rho^2 - 2L)\right) \\
&= \frac{Ls}{\sigma_\eta} \left(\rho \left(2L^2\rho s\sigma_\gamma\sigma_\eta + L \left(\sigma_\eta^2 - \rho s\sigma_\gamma\sigma_\eta\right) - \sigma_\eta^2\right) - (L - 1)\rho \sigma_\eta^2 + Ls\sigma_\gamma\sigma_\eta (\rho^2 - 2L)\right) \\
&= -2L^3 \left(1 - \rho^2\right) s^2 \sigma_\eta < 0 ,
\end{align*}
\]

where the first line obtains from plugging in the formulae for partial derivatives into (18) and some algebraic manipulation, the second line obtains from substituting in for \(-L^3\rho s^2 \sigma_\gamma^2\) from Equation 15, and third line is algebraic simplification.

Analogous steps prove \(\frac{dR_{\eta a}^2}{ds} < 0\). Finally, plugging in partial derivatives into (18) and simplifying,

\[
\begin{align*}
\frac{dR_{\eta a}^2}{d\rho} &\propto \frac{Ls\sigma_\gamma}{\sigma_\eta} \left(3L^3s^2 \sigma_\gamma^2 + (1 - L)\sigma_\eta^2 + L\rho s\sigma_\gamma\sigma_\eta\right) > 0 ,
\end{align*}
\]

where the inequality uses the fact that \(L < 1\), as was established in Lemma 10. \(Q.E.D.\)

**E.3. Proof of Proposition 7**

Part 1 of Proposition 7 is subsumed in Lemma 13 below; Part 2(a) and 2(b) of Proposition 7 are immediate consequences of expression (20) and Lemma 14 below; and Part 3 of Proposition 7 is Lemma 15 below. Given \(\tau = \gamma\), it will be convenient to rewrite Equation 14 as

\[
f_\gamma(L, s, \sigma_\gamma, \sigma_\eta, \rho) := s^2 \sigma_\gamma^2 L^3 + 2s\rho \sigma_\gamma\sigma_\eta L^2 + (\sigma_\eta^2 - s\sigma_\gamma^2) L - \rho \sigma_\eta \sigma_\gamma = 0 .
\]

**Lemma 13.** If \(\rho > 0\) there is a unique solution to Equation 19 on the non-negative domain; it is strictly positive. If \(\rho = 0\), (i) \(L = 0\) is always a solution, and (ii) there is a positive solution if and only if \(\sigma_\eta^2 < s\sigma_\gamma^2\); when a positive solution exists, it is unique. For any \(\rho \geq 0\) and at any solution \(L > 0\) to Equation 19, it holds that \(\frac{\partial f_\gamma}{\partial L} > 0\).

**Proof.** Assume \(\rho \geq 0\). First, there is at most one strictly positive solution to Equation 19 because \(f_\gamma(\cdot)\) intersects 0 from below at any strictly positive solution:

\[
\left.\frac{\partial f_\gamma}{\partial L}\right|_{f_\gamma = 0} = \sigma_\eta^2 + 3L^2 s^2 \sigma_\gamma^2 + 4s(L\rho)\sigma_\gamma\sigma_\eta - s\sigma_\gamma^2 > \sigma_\eta^2 + s^2 \sigma_\gamma^2 L^2 + 2L s\rho \sigma_\gamma\sigma_\eta - s\sigma_\gamma^2 = s\sigma_\gamma^2 + \frac{b\sigma_\gamma\sigma_\eta}{L} - s\sigma_\gamma^2 \geq 0 ,
\]

where the second equality uses Equation 19.
Next, observe that because $f_{\gamma}(0, \cdot) \leq 0$ while $f_{\gamma}(L, \cdot) \to \infty$ as $L \to \infty$, there is always at least one non-negative solution to Equation 19. If $\rho > 0$, then $f_{\gamma}(0, \cdot) < 0$, so any non-negative solution is strictly positive. If $\rho = 0$, $f_{\gamma}(0, \cdot) = 0$, so $L = 0$ is always a solution; that there is a strictly positive solution if and only if $\sigma_{\eta}^2 < s\sigma_{\gamma}^2$ follows from the observations that $\frac{\partial f_{\gamma}(0, \cdot)}{\partial L} = \sigma_{\eta}^2 - s\sigma_{\gamma}^2$ and $\frac{\partial f_{\gamma}(\cdot, \cdot)}{\partial L} > 0$ for all $L > 0$ if $\sigma_{\eta}^2 \geq s\sigma_{\gamma}^2$. Q.E.D.

**Lemma 14.** Let $\rho \geq 0$. As $s \to \infty$, it holds for the unique strictly positive solution $L$ to Equation 19 that $L \to 0$ and $L^2 s \to 1$. If $\rho > 0$ then as $s \to 0$, it holds for the unique strictly positive solution $L$ to Equation 19 that $L \to \rho \sigma_{\gamma}$.\(^{48}\)

**Proof.** The proof is omitted as the argument is analogous to that in the proof of Lemma 11, but applied to Equation 19.

**Lemma 15.** If $\rho \geq 0$, then in an increasing equilibrium (when it exists, i.e. when there is a strictly positive solution to Equation 19): (i) $\frac{d}{d\rho} R_{\gamma a}^2 = 0$, (ii) $\frac{d}{d\sigma_{\eta}} R_{\gamma a}^2 < 0$, (iii) $\frac{d}{ds} R_{\gamma a}^2 > 0$, and (iv) $\frac{d}{d\rho} R_{\gamma a}^2 > 0$.

**Proof.** First observe that Equation 19 implies

$$2sL \rho \sigma_{\eta} \sigma_{\gamma} + \sigma_{\eta}^2 + s^2 L^2 \sigma_{\gamma}^2 = \frac{1}{L}(\rho \sigma_{\eta} \sigma_{\gamma} + sL \sigma_{\gamma}^2).$$

We now rewrite Equation 13 and plug in the above identity to get

$$R_{\gamma a}^2 = \frac{(\rho \sigma_{\eta} \sigma_{\gamma} + sL \sigma_{\gamma}^2)^2}{\sigma_{\gamma}^2 (2sL \rho \sigma_{\eta} \sigma_{\gamma} + \sigma_{\eta}^2 + s^2 L^2 \sigma_{\gamma}^2)} = \frac{(\rho \sigma_{\eta} \sigma_{\gamma} + sL \sigma_{\gamma}^2)^2}{\sigma_{\gamma}^2 L (\rho \sigma_{\eta} \sigma_{\gamma} + sL \sigma_{\gamma}^2)} = sL^2 + L \rho \frac{\sigma_{\eta}}{\sigma_{\gamma}}. \quad (20)$$

The rest of the argument is analogous to that in the proof of Lemma 12, but applied to expressions (20) and (19). The algebraic details are available on request. Q.E.D.

**E.4. Mixed dimensions of interest**

Here we provide formal results for the LQE specification with mixed dimensions of interest discussed in Subsection 5.3. As discussed in the text, a linear equilibrium under mixed dimensions of interest is characterized by a solution $(L_{\eta}, L_{\gamma})$ to the following simultaneous equations:

$$L_{\eta} = \frac{\sigma_{\eta}^2 + sL \rho \sigma_{\eta} \sigma_{\gamma}}{\sigma_{\eta}^2 + s^2 L^2 \sigma_{\gamma}^2 + 2sL \rho \sigma_{\eta} \sigma_{\gamma}}, \quad (21)$$

$$L_{\gamma} = \frac{sL \sigma_{\gamma}^2 + \rho \sigma_{\eta} \sigma_{\gamma}}{\sigma_{\eta}^2 + s^2 L^2 \sigma_{\gamma}^2 + 2sL \rho \sigma_{\eta} \sigma_{\gamma}}. \quad (22)$$

\(^{48}\)When $\rho = 0$, Lemma 13 assures that there is a strictly positive solution if and only if $s$ is large enough.
where we use the shorthand \( L \equiv \kappa L_\gamma + (1 - \kappa)L_\eta \). An equilibrium is said to be increasing if both \( L_\eta > 0 \) and \( L_\gamma > 0 \). From Equation 12,

\[
R^2_{\eta a} = \frac{(\sigma^2_\eta + sL_\rho \sigma_\gamma)^2}{\sigma^2_\eta (2sL\rho \sigma_\eta \sigma_\gamma + \sigma^2_\eta + s^2 L^2 \sigma^2_\gamma)} = L_\eta L_\rho \sigma_\eta^2 + L_\eta,
\]

(23)

where the second equality can be verified by substituting for \( L_\eta \) on the right-hand side using Equation 21. Similarly, Equation 13 gives

\[
R^2_{\gamma a} = \frac{(\rho \sigma_\gamma^2 + sL_\sigma_\gamma^2)^2}{\sigma^2_\gamma (2sL\rho \sigma_\eta \sigma_\gamma + \sigma^2_\eta + s^2 L^2 \sigma^2_\gamma)} = L_\gamma \rho \sigma_\gamma^2 + L_\gamma sL,
\]

(24)

where the second equality substitutes for \( L_\gamma \) on the right-hand side using Equation 22.

**Proposition 11.** Consider the LQE specification with mixed dimensions of interest and \( \rho = 0 \). Then, for any \( \kappa \in (0, 1) \), an increasing linear equilibrium exists; moreover:

1. \( \forall \varepsilon > 0, \exists \bar{s} > 0 \) such that if \( s < \bar{s} \) then in any such equilibrium, \( R^2_{\eta a} \in [1 - \varepsilon, 1] \) and \( R^2_{\gamma a} \in [0, \varepsilon] \).
2. \( \forall \varepsilon > 0, \exists \bar{s} > 0 \) such that if \( s > \bar{s} \) then in any such equilibrium, \( R^2_{\eta a} \in [0, \varepsilon] \) and \( R^2_{\gamma a} \in [1 - \varepsilon, 1] \).
3. Any such equilibrium has the following local comparative statics: \( \frac{\partial}{\partial s} R^2_{\eta a} < 0 \) and \( \frac{\partial}{\partial s} R^2_{\gamma a} > 0 \).

**Remark 6.** Subject to technical qualifiers, the above result can be generalized to \( \rho \geq 0 \); the only modifications are that the limiting \( R^2_{\eta a} \) as \( s \to \infty \) becomes \( \rho^2 \) (cf. Proposition 6 Part 2(b)) while the limiting \( R^2_{\gamma a} \) as \( s \to 0 \) becomes \( \rho^2 \) (cf. Proposition 7 Part 2(a)).

**Proof of Proposition 11.** Assume \( \rho = 0 \) and \( \kappa \in (0, 1) \). With \( \rho = 0 \), Equations 21 and 22 can be rewritten as

\[
f_{\kappa, \eta}(L_\eta, L_\gamma, s, \sigma_\eta, \sigma_\gamma) := L_\eta \sigma_\eta^2 + s^2 L_\eta L^2 \sigma_\gamma^2 - \sigma_\eta^2 = 0, \tag{25}
\]

\[
f_{\kappa, \gamma}(L_\eta, L_\gamma, s, \sigma_\eta, \sigma_\gamma) := L_\gamma \sigma_\gamma^2 + s^2 L_\gamma L^2 \sigma_\gamma^2 - sL_\sigma_\gamma^2 = 0, \tag{26}
\]

and (23) and (24) simplify to

\[
R^2_{\eta a} = L_\eta, \tag{27}
\]

\[
R^2_{\gamma a} = sL_\gamma, \tag{28}
\]

It is straightforward from Equation 25 and Equation 26 that there is a positive solution (i.e. both \( L_\eta > 0 \) and \( L_\gamma > 0 \));\(^{49}\) moreover, in any solution, \( L_\eta \in (0, 1) \).

Manipulating (25) and (26) along similar lines to the proof of Lemma 11, it can be established that:

1. \( \forall \varepsilon > 0, \exists \bar{s} > 0 \) such that if \( s < \bar{s} \) then in any positive solution, \( L_\eta \in (1 - \varepsilon, 1) \) and \( L_\gamma < \varepsilon \).
2. \( \forall \varepsilon > 0, \exists \bar{s} > 0 \) such that if \( s > \bar{s} \) then in any positive solution, \( L_\eta < \varepsilon, L_\gamma < \varepsilon \), and \( sL_\gamma L \in (1 - \varepsilon, 1) \).

\(^{49}\)Note that this uses \( \kappa < 1 \) (cf. Proposition 7).
Part 1 and part 2 of the proposition follow applying these two facts to Equation 27 and Equation 28. To prove part 3, first note that by the implicit function theorem,

\[
\frac{\partial L_\eta}{\partial s} = -\frac{\det \begin{bmatrix}
\frac{\partial f_{\kappa,\eta}}{\partial s} & \frac{\partial f_{\kappa,\gamma}}{\partial L_\eta} \\
\frac{\partial f_{\kappa,\eta}}{\partial L_\eta} & \frac{\partial f_{\kappa,\gamma}}{\partial s}
\end{bmatrix}}{\det \begin{bmatrix}
\frac{\partial f_{\kappa,\eta}}{\partial s} & \frac{\partial f_{\kappa,\gamma}}{\partial L_\eta} \\
\frac{\partial f_{\kappa,\eta}}{\partial L_\eta} & \frac{\partial f_{\kappa,\gamma}}{\partial s}
\end{bmatrix}}
\]

and

\[
\frac{\partial L_\gamma}{\partial s} = -\frac{\det \begin{bmatrix}
\frac{\partial f_{\kappa,\eta}}{\partial s} & \frac{\partial f_{\kappa,\gamma}}{\partial L_\eta} \\
\frac{\partial f_{\kappa,\eta}}{\partial L_\eta} & \frac{\partial f_{\kappa,\gamma}}{\partial s}
\end{bmatrix}}{\det \begin{bmatrix}
\frac{\partial f_{\kappa,\eta}}{\partial s} & \frac{\partial f_{\kappa,\gamma}}{\partial L_\eta} \\
\frac{\partial f_{\kappa,\eta}}{\partial L_\eta} & \frac{\partial f_{\kappa,\gamma}}{\partial s}
\end{bmatrix}}
\]

Computing all the relevant partial derivatives of \(f_{\kappa,\eta}\) and \(f_{\kappa,\gamma}\) and simplifying yields

\[
\frac{\partial L_\eta}{\partial s} = -\frac{2s\sigma_\gamma^2 L_\eta L^2}{\sigma_\gamma^2 (3sL^2 - \kappa) + \sigma_\eta^2} \quad \text{and} \quad \frac{\partial L_\gamma}{\partial s} = \frac{\sigma_\gamma^2 L(2sL_\gamma L - 1)}{\sigma_\gamma^2 (3sL^2 - \kappa) + \sigma_\eta^2}.
\]  \(29\)

Using (28) and (29), some algebra yields

\[
\frac{d}{ds} R_{\gamma a}^2 = \frac{\partial L_\gamma}{\partial s} + \frac{\partial L_\eta}{\partial s} = \frac{L}{s^2 \sigma_\gamma^2 (3sL^2 - \kappa) + \sigma_\eta^2}
\]

\(30\)

We manipulate (30) as follows:

Numerator of (30) = \(L \left[-s^2 L^2 \sigma_\gamma^2 L_\gamma + \sigma_\eta^2 L_\gamma + sL_\eta \sigma_\eta^2 \right] \) using \(L = \kappa L_\gamma + (1 - \kappa) L_\eta \)

\(> 0\).

Denominator of (30) = \(s\sigma_\gamma^2 \left( sL^2 - L_\gamma \right) + \sigma_\gamma^2 + s\sigma_\gamma^2 \left( L_\eta \sigma_\eta^2 + sL_\gamma \right) \)

\(= s\sigma_\gamma^2 \left( sL^2 - L_\gamma \right) + \sigma_\gamma^2 \left( L_\eta - \kappa + 2sL^2 \right) \) as Equation 26 implies \(s\sigma_\gamma^2 \left( sL^2 - L_\gamma \right) + \sigma_\gamma^2 = 0 \)

\(= \frac{s\sigma_\gamma^2 \left( (1 - \kappa) L_\eta + 2L_\gamma sL^2 \right)}{L_\gamma} > 0\).

Consequently, \(\frac{d}{ds} R_{\gamma a}^2 > 0\). Finally, observe from (27) and (29) that

\[
\frac{d}{ds} R_{\eta a}^2 = \frac{d}{ds} L_\eta \propto -s\sigma_\gamma^2 \left( 3sL^2 - \kappa \right) - \sigma_\eta^2 < 0.
\]  \(Q.E.D.\)

**F. Proofs for Section 7**

The proof of Proposition 8 follows immediately from the results of Proposition 1.
F.1. Proof of Proposition 9

Denote the prior mean of \( \eta \) by \( P \equiv \mathbb{E}[\eta] \). Without loss, we will assume that no actions below \( \eta \) are played in an equilibrium.\(^{50}\)

Step 1: For sufficiently large \( s \), any informative equilibrium has two on-path actions. This follows because the two types can simultaneously be indifferent only over a single pair of actions. Suppose, to contradiction, that an informative equilibrium involves three (or more) actions. Then one of these actions is only played by a single type, and so has degenerate beliefs of \( \hat{\eta} = \eta \) or \( \hat{\eta} = \bar{\eta} \). If one action induces beliefs \( \hat{\eta}_1 = \eta \), then there must be another action inducing beliefs \( \hat{\eta}_2 > \mathbb{E}[\eta] \). If one action induces beliefs \( \hat{\eta}_2 = \bar{\eta} \), then there must be another action inducing beliefs \( \hat{\eta}_1 < \mathbb{E}[\eta] \). In either case, the benefit of increasing from the low to the high belief, \( sv(\hat{\eta}_2) - sv(\hat{\eta}_1) \to \infty \) as \( s \to \infty \). But by Lemma 5, \( sv(\hat{\eta}_2) - sv(\hat{\eta}_1) \leq C(a^{ce}, \theta_1) \) for any \( s \); a contradiction.

Step 2: A program that bounds allocative efficiency in two-action equilibria. Fix some stakes \( s \). Say that in a two-action equilibrium with actions \( a_1 < a_2 \), a probability mass of \( q_i \) agents take action \( a_i \) and induce corresponding belief \( \hat{\eta}_i \), for \( i = 1, 2 \). Using the identities \( q_2\hat{\eta}_2 + q_1\hat{\eta}_1 = P \) and \( q_1 + q_2 = 1 \), we can solve for \( q_1 \) and \( q_2 \) as

\[
q_1 = \frac{\hat{\eta}_2 - P}{\hat{\eta}_2 - \hat{\eta}_1}, \quad q_2 = \frac{P - \hat{\eta}_1}{\hat{\eta}_2 - \hat{\eta}_1}.
\]

Thus, allocative efficiency is

\[
s\mathbb{E}[v(\hat{\eta})] = s\mathbb{E}[w(\hat{\eta}) - w(P)] = s\left(q_2w(\hat{\eta}_2) + q_1w(\hat{\eta}_1) - w(P)\right)
= s\left(\frac{P - \hat{\eta}_1}{\hat{\eta}_2 - \hat{\eta}_1}w(\hat{\eta}_2) + \frac{\hat{\eta}_2 - P}{\hat{\eta}_2 - \hat{\eta}_1}w(\hat{\eta}_1) - w(P)\right).
\]

Lemma 5 implies that \( sw(\hat{\eta}_2) - sw(\hat{\eta}_1) \leq C(a^{ce}, \theta_1) \). The following program therefore gives us an upper bound on allocative efficiency across all two-action equilibria:

\[
\max_{\hat{\eta}_1, \hat{\eta}_2} s\left(\frac{P - \hat{\eta}_1}{\hat{\eta}_2 - \hat{\eta}_1}w(\hat{\eta}_2) + \frac{\hat{\eta}_2 - P}{\hat{\eta}_2 - \hat{\eta}_1}w(\hat{\eta}_1) - w(P)\right)
\] (31)

subject to \( \hat{\eta}_1 \leq P \leq \hat{\eta}_2 \) and

\[
w(\hat{\eta}_2) - w(\hat{\eta}_1) \leq \frac{C(a^{ce}, \theta_1)}{s}.
\] (32)

We will show that the value of the above program tends to zero as \( s \to \infty \); this establishes the desired conclusion that allocative efficiency is maximized at some interior \( s \), because for any \( s > 0 \) there is an equilibrium with strictly positive allocative efficiency while allocative efficiency is obviously zero in any equilibrium when \( s = 0 \).

Step 3: The value of the program asymptotes to 0. In the solution to program (31), constraint (32) must be satisfied with equality; otherwise there is a mean-preserving spread of \( \hat{\eta} \) that strictly increases

\(^{50}\)If they are, they must all have the same market belief (as these actions are costless to both types), in which case there is an outcome-equivalent equilibrium that collapses all such actions to single one at \( \eta \).
allocative efficiency. Notice also that as \( s \to \infty \), the constraints jointly imply \( \hat{\eta}_2, \hat{\eta}_1 \to P \), and hence, for \( s \) large, \( w(\hat{\eta}_2) - w(\hat{\eta}_1) \approx w'(P) \cdot (\hat{\eta}_2 - \hat{\eta}_1) \). Thus, for any \( \alpha < w'(P) \),

\[
\alpha(\hat{\eta}_2 - \hat{\eta}_1) \leq w(\hat{\eta}_2) - w(\hat{\eta}_1) \leq \frac{C(a^\ast, \theta_1)}{s}
\]
at any solution at large enough \( s \).

Fixing any \( k > \frac{C(a^\ast, \theta_1)}{w'(P)} \) and noting that relaxing constraint (32) can only increase the value of the program, it follows that the value of the program (31) for large enough \( s \) is no larger than the value of the following program:

\[
\max_{\eta_2, \eta_1} s \left( \frac{P - \hat{\eta}_1}{\eta_2 - \eta_1} w(\eta_2) + \frac{\hat{\eta}_2 - P}{\eta_2 - \eta_1} w(\hat{\eta}_1) - w(P) \right)
\]
subject to \( \hat{\eta}_1 \leq P \leq \hat{\eta}_2 \) and

\[
\hat{\eta}_2 - \hat{\eta}_1 \leq \frac{k}{s}.
\]

(33)

Since (33) will again be satisfied with equality, we substitute \( \hat{\eta}_2 - \hat{\eta}_1 = \frac{k}{s} \) into the objective to simplify the above program to:

\[
\max_{\hat{\eta}_2 \in [P, P + \frac{k}{s}]} \varphi(\hat{\eta}_2; s) \equiv s \left( \frac{P - \hat{\eta}_2}{\frac{k}{s}} w(\hat{\eta}_2) + \frac{\hat{\eta}_2 - P}{\frac{k}{s}} w(\hat{\eta}_1) - w(P) \right).
\]

(34)

**Claim 5.** The function \( \varphi \) from (34) satisfies \( s \varphi(P + \frac{ks}{s}; s) \xrightarrow{s \to \infty} \frac{k^2}{2} x(1 - x) w''(P) \) for \( x \in [0, 1] \).

**Proof.** Using the definition of \( \varphi(\cdot) \),

\[
 s \varphi \left( P + \frac{ks}{s}; s \right) = s^2 \left[ (1 - x)w \left( P + \frac{ks}{s} \right) + xw \left( P - \frac{ks}{s} \right) - w(P) \right].
\]

The second-order Taylor expansion, \( w(P + \varepsilon) = w(P) + \varepsilon w'(P) + \frac{\varepsilon^2}{2} w''(P) + o(\varepsilon^3) \), yields

\[
 s \varphi \left( P + \frac{ks}{s}; s \right) = s^2 \left[ (1 - x) \left( w(P) + \frac{ks}{s} w'(P) + \frac{k^2 s^2}{2s^2} w''(P) + o(s^{-3}) \right) \right.

+ \left. x \left( w(P) - \frac{k(1 - x)}{s} w'(P) + \frac{k^2 (1 - x)^2}{2s^2} w''(P) + o(s^{-3}) \right) - w(P) \right]
\]

\[
= s^2 \left[ (1 - x) \frac{k^2 s^2}{2s^2} w''(P) + x \frac{k^2 (1 - x)^2}{2s^2} w''(P) + o(s^{-3}) \right]
\]

\[
= \frac{k^2}{2} x(1 - x) w''(P) + o(s^{-1})
\]

\[
\to \frac{k^2}{2} x(1 - x) w''(P) \text{ as } s \to \infty.
\]

Q.E.D.

Claim 5 implies that \( s \max_{\hat{\eta}_2 \in [P, P + \frac{k}{s}]} \varphi(\hat{\eta}_2; s) \) asymptotes to a constant. Hence, allocative efficiency, which was shown to be bounded above for large \( s \) by \( \max_{\hat{\eta}_2 \in [P, P + \frac{k}{s}]} \varphi(\hat{\eta}_2; s) \), asymptotes to 0.
F.2. Proof of Proposition 10

We prove a result that is slightly more general than Proposition 10. We maintain the assumption of binary \( \eta \) and continuous \( \gamma \), but we generalize from independent types. (Allowing for discrete distributions of \( \gamma \) or mixed distributions with atoms would not materially change any conclusions but would complicate notation.)

From Proposition 9, we have an example in which the distribution of high type gaming abilities is below the low type gaming abilities, and in which allocative efficiency has an interior maximum in \( s \). To guarantee that there do exist equilibria where allocative efficiency diverges to infinity, we will invoke a condition that the distribution of high type gaming abilities is in some sense not below that of the low types.

**Condition 1.** The natural actions are in \( \Theta_\eta = \{ \eta, \bar{\eta} \} \), with probability mass \( p \in (0, 1) \) on \( \bar{\eta} \) and \( 1 - p \) on \( \eta < \bar{\eta} \). Conditional on \( \eta \in \Theta_\eta \), gaming ability \( \gamma \) is continuously distributed with cdf \( G_\eta \), pdf \( g_\eta \), and compact support in \( \mathbb{R}_{++} \). Moreover, either

1. \( \max \text{Supp} G_\eta \leq \min \text{Supp} G_{\bar{\eta}} \); or
2. there exists \( \gamma^* \) such that \( 0 < G_{\bar{\eta}}(\gamma^*) < G_\eta(\gamma^*) < 1 \); or
3. there exists \( \gamma^* \) such that \( 0 < G_{\bar{\eta}}(\gamma^*) = G_\eta(\gamma^*) < 1 \), and \( \lim_{\gamma \uparrow \gamma^*} g_{\bar{\eta}}(\gamma) > 0 \).

Part 1 says that all high-\( \eta \) types have gaming ability above that of any low types; it implies the type space is ordered by single-crossing as there are no cross types. Part 2 says that the support of gaming ability overlaps for the two values of \( \eta \) and that the gaming-ability distribution for high-\( \eta \) types is strictly above that of low-\( \eta \) types at some point (in the sense that the value of the cdf is below). Part 3 requires that the distributions are equal at some point where the left-neighborhood contains some high \( \eta \) types. When \( \gamma \) and \( \eta \) are independent and \( \gamma \) is continuously distributed as in Proposition 10, Condition 1 part 3 is satisfied.

**Proposition 12** (Generalization of Proposition 10). Assume the joint distribution of types satisfies **Condition 1**. Let \( \tau = \eta \); let \( V(\hat{\eta}; s) = sv(\hat{\eta}) \) with \( v(\hat{\eta}) = (w(\hat{\eta}) - w(E[\eta])) \) for some strictly convex \( w \); and let \( C(a, \eta, \gamma) = (a - \eta)^r \) on \( a \geq \eta \), for some \( r > 1 \). As \( s \to \infty \), there exists a sequence of equilibria with allocative efficiency \( sE[v(\hat{\eta})] \to \infty \). In particular, if **Condition 1** parts 1 or 2 are satisfied, then there exists a sequence of equilibria with \( sE[v(\hat{\eta})] \) increasing at a linear rate in \( s \). Under **Condition 1** part 3, there exists a sequence of equilibria with \( sE[v(\hat{\eta})] \) increasing at rate \( s^{\frac{r}{r+1}} \) (or faster).

If there are no cross types—that is, **Condition 1** part 1 holds—then for all \( s \) there exists a separating equilibrium on \( \eta \). The result is then trivial because \( E[v(\hat{\eta})] = pv(\bar{\eta}) + (1-p)v(\eta) \) in any separating equilibrium, and hence allocative efficiency \( sE[v(\hat{\eta})] \) increases linearly in \( s \) along a sequence of separating equilibria.

The more interesting cases are when parts 2 or 3 of **Condition 1** hold. In these cases, we can construct a two-action equilibrium of the following form, illustrated in Figure 5.

**Lemma 16.** Suppose \( \tau = \eta \), **Condition 1** holds with either part 2 or 3 satisfied, and \( C(a, \eta, \gamma) = (a - \eta)^r \) on \( a \geq \eta \), for \( r > 1 \). Given any \( \gamma^* \) satisfying \( 0 < G_\eta(\gamma^*) \leq G_{\bar{\eta}}(\gamma^*) < 1 \), there exists an action \( \bar{a} > \eta \) and a gaming ability \( \tilde{\gamma} < \gamma^* \) such that there is a two-action equilibrium in which agents of type \((\eta, \gamma)\) play action \( a_2(\gamma) \) defined by

\[
a_2(\gamma) = \begin{cases} \eta & \text{if } \gamma \leq \gamma^* \\ \bar{a} & \text{if } \gamma > \gamma^* \end{cases}
\]
\[ a_{\eta}(\gamma) = \begin{cases} \eta & \text{if } \gamma \leq \tilde{\gamma} \\ \tilde{a} & \text{if } \gamma > \tilde{\gamma}. \end{cases} \]

Figure 5 – Strategies under the equilibrium of Lemma 16.

The following conditions are sufficient for the strategy in Lemma 16 to constitute an equilibrium:

\[ \begin{align*}
sv(\hat{\eta}(\overline{\eta})) &= sv(\hat{\eta}(\tilde{a})) - \frac{(\tilde{a} - \eta)}{\tilde{\gamma}} & \text{if } \tilde{a} > \overline{\eta} \\
\tilde{\gamma} &= 0 & \text{if } \tilde{a} \leq \overline{\eta},
\end{align*} \tag{35} \]

\[ sv(\hat{\eta}(\eta)) = sv(\hat{\eta}(\tilde{a})) - \frac{(\tilde{a} - \eta)}{\gamma^*}. \tag{36} \]

Equation 36 requires type \((\eta, \gamma^*)\) to be indifferent between actions \(\eta\) and \(\tilde{a}\). Condition 35 requires indifference for type \((\overline{\eta}, \tilde{\gamma})\) if \(\tilde{a} > \overline{\eta}\); on the other hand, if \(\tilde{a} \leq \overline{\eta}\), then all high-\(\eta\) types strictly prefer \(\tilde{a}\) (because both actions are costless but \(\tilde{a}\) induces a higher belief, as confirmed below) and we take \(\tilde{\gamma} = 0\). Note that deviations to off-path actions can be deterred by assigning any off-path action the belief \(\hat{\eta} = \eta\).

Given the strategy of Lemma 16, the observer’s posterior probability \(\Pi(a)\) that the agent’s type is \(\eta = \overline{\eta}\) conditional on action \(a \in \{\eta, \tilde{a}\}\) is given by

\[ \Pi(\eta) = \frac{pG_{\overline{\eta}}(\tilde{\gamma})}{pG_{\overline{\eta}}(\tilde{\gamma}) + (1 - p)G_{\eta}(\gamma^*)}, \tag{37} \]

\[ \Pi(\tilde{a}) = \frac{p(1 - G_{\overline{\eta}}(\tilde{\gamma}))}{p(1 - G_{\overline{\eta}}(\tilde{\gamma})) + (1 - p)(1 - G_{\eta}(\gamma^*)}. \tag{38} \]

For \(a \in \{\eta, \tilde{a}\}\), \(\hat{\eta}(a) = \Pi(a)\overline{\eta} + (1 - \Pi(a))\eta\); that \(\hat{\eta}(\tilde{a}) > \hat{\eta}(\eta)\), or equivalently that \(\Pi(\tilde{a}) > \Pi(\eta)\), follows from \(G_{\overline{\eta}}(\gamma^*) \leq G_{\eta}(\gamma^*)\) and \(\tilde{\gamma} < \gamma^*\).

\[ ^{51} \text{Modulo inessential multiplicities in } \tilde{\gamma}, \text{the conditions are also necessary.} \]
If $G_{\pi}(\gamma^*) < G_{\eta}(\gamma^*)$, as it is under Condition 1 part 2, it is straightforward to compute that even if $\tilde{\gamma} \to \gamma^*$ as $s \to \infty$ (and a fortiori if $\tilde{\gamma} \to \gamma^*$), $\hat{\eta}(\tilde{a})$ remains bounded away from $\hat{\eta}(\eta)$. In this case, $E[v]$ stays bounded away from 0, and hence allocative efficiency $sE[v] \to \infty$ at a linear rate as $s \to \infty$. This proves Proposition 12 for Condition 1 part 2.

For Condition 1 part 3, the conclusion of Proposition 12 follows from the following two claims.

**Claim 6.** In a sequence of equilibria of Lemma 16, if $\gamma^* - \tilde{\gamma} \to 0$ at a rate of $f(s)$, then $E[v] \to 0$ at a rate of $(f(s))^2$.

**Claim 7.** Assume Condition 1 holds with part 3 satisfied. In a sequence of equilibria of Lemma 16, $\gamma^* - \tilde{\gamma} \to 0$ at a rate of $s^{-1/\gamma^*}$.

Accordingly, it remains to prove Lemma 16, Claim 6, and Claim 7.

**Proof of Lemma 16.** The strategy has two free parameters: $\tilde{a}$ and $\tilde{\gamma}$, with the constraints that $\tilde{a} > \eta$ and $\tilde{\gamma} < \gamma^*$, and the equilibrium conditions (35) and (36). To ease notation going forward, let

$$\hat{\eta}(\tilde{\gamma}) \equiv \frac{p(1 - G_{\pi}(\tilde{\gamma}))}{p(1 - G_{\pi}(\tilde{\gamma})) + (1 - p)(1 - G_{\eta}(\gamma^*))} \eta + \frac{(1 - p)(1 - G_{\eta}(\gamma^*))}{p(1 - G_{\pi}(\tilde{\gamma})) + (1 - p)(1 - G_{\eta}(\gamma^*))} \gamma$$

denote the observer’s belief about $\eta$ when action $\tilde{a}$ is observed given a value of $\tilde{\gamma}$.

**Case 1:** $s[v(\hat{\eta}(0)) - v(\eta)] \leq \frac{(\eta - \eta)^r}{\gamma^*}$.

For this case we look for a solution with $\tilde{\gamma} = 0$ and $\tilde{a} \in (\eta, \tilde{\eta}]$. The induced on-path beliefs will be $\hat{\eta}(\eta) = \eta$ and $\hat{\eta}(\tilde{a}) = \hat{\eta}(0)$. It suffices to check that there exists $\tilde{a} \in (\eta, \tilde{\eta}]$ to satisfy the $(\eta, \gamma^*)$ indifference at these beliefs, condition (36). A solution exists precisely under the hypothesis of the case being considered.

**Case 2:** $s[v(\hat{\eta}(0)) - v(\eta)] > \frac{(\eta - \eta)^r}{\gamma^*}$.

For this case, we look for solutions with $\tilde{a} > \eta$ and $\tilde{\gamma} > 0$. Supposing that $\tilde{a} > \eta$, combine the indifferences in (35) and (36) to get $\frac{(\tilde{a} - \eta)^r}{\tilde{\gamma}} = \frac{(\tilde{a} - \eta)^r}{\gamma^*}$. Fixing any $\tilde{\gamma} \in (0, \gamma^*)$, this equality uniquely pins down a corresponding $\tilde{a}$, which we write as a function

$$\tilde{a}(\tilde{\gamma}) = \eta + \frac{\eta - \eta}{(\gamma^*/\tilde{\gamma})^2 - 1}.$$ 

(39)

The function $\tilde{a}(\tilde{\gamma})$ is continuous and increasing in $\tilde{\gamma}$, with $\tilde{a}(\tilde{\gamma}) \to \eta$ as $\tilde{\gamma} \to 0^+$ and $\tilde{a}(\tilde{\gamma}) \to \infty$ as $\tilde{\gamma} \to \gamma^*$. We seek to find a $\tilde{\gamma} > 0$ which, along with the corresponding action $\tilde{a} = \tilde{a}(\tilde{\gamma})$, constitutes an equilibrium. It is enough to find $\tilde{\gamma}$ satisfying the $(\eta, \gamma^*)$ indifference, condition (35), as the $(\tilde{\eta}, \tilde{\gamma})$ indifference is assured by $\tilde{a} = \tilde{a}(\tilde{\gamma})$. Rearranging (35) gives

$$s(v(\hat{\eta}(\tilde{a})) - v(\hat{\eta}(\eta))) = \frac{(\tilde{a}(\tilde{\gamma}) - \eta)^r}{\gamma^*}.$$ 

(40)

For any sufficiently small but positive $\tilde{\gamma}$, $\hat{\eta}(\tilde{a}) = \hat{\eta}(0)$ and $\hat{\eta}(\eta) = \eta$ because $\min \text{Supp} G_{\pi} > 0$ and $G_{\eta}(\gamma^*) > 0$. Hence, as $\tilde{\gamma} \to 0^+$, the LHS of (40) goes to $s(v(\hat{\eta}(0)) - v(\hat{\eta}(\eta)))$, while the RHS goes to $\frac{(\eta - \eta)^r}{\gamma^*}$. From the hypothesis of Case being considered, it follows that for sufficiently small $\tilde{\gamma}$, the LHS of (40) is greater than the RHS. On the other hand, as $\tilde{\gamma} \to \gamma^*$, the LHS of (40) converges to a constant while the RHS diverges to
(because \( \tilde{a}(\tilde{\gamma}) \to \infty \)). Hence, for sufficiently large \( \tilde{\gamma} \), the LHS of (40) is less than the RHS. By continuity, there exists \( \tilde{\gamma} > 0 \) for which (40) holds; this value of \( \tilde{\gamma} \) together with \( \tilde{a}(\tilde{\gamma}) \) constitutes an equilibrium. \( \text{Q.E.D.} \)

**Proof of Claim 6.** To ease notation, let \( G^* \equiv G^*(\gamma^*) = G^*(\gamma^*) \in (0,1) \) and \( g^* \equiv \lim_{\gamma \uparrow \gamma^*} g(\gamma) > 0 \).

For \( \tilde{\gamma} = \gamma^* - \varepsilon \) with \( \varepsilon > 0 \) small, \( G^*(\tilde{\gamma}) \) is approximately linear in \( \varepsilon \); to a first order Taylor approximation, \( G^*(\tilde{\gamma}) \approx G^* - \varepsilon g^* \). As \( \varepsilon \to 0^+ \), this yields beliefs at the two actions approaching \( E[\eta]\eta \) at a linear rate, with \( \hat{\eta}(\cdot) \approx E[\eta] - p(1-p)g^* \eta \gamma \gamma^*\varepsilon, \hat{\eta}(\tilde{a}) \approx E[\eta] + p(1-p)g^* \eta \gamma \gamma^*\varepsilon \). Since \( E[v] = (pG^*(\tilde{\gamma}) + (1-p)G^*)v(\hat{\eta}(\eta)) + (p(1-G^*(\tilde{\gamma})) + (1-p)(1-G^*))v(\hat{\eta}(\tilde{a})) \), substituting in the approximations for \( \hat{\eta}(\cdot) \) at small \( \varepsilon \) gives \( E[v] \) approximately quadratic in \( \varepsilon \). \( \text{Q.E.D.} \)

**Proof of Claim 7.** First, it is routine to verify using conditions (35) and (36) that \( \tilde{\gamma} \uparrow \gamma^* \) and \( \tilde{a} = \tilde{a}(\tilde{\gamma}) \to \infty \) as \( s \to \infty \).

Second, while \( \tilde{a}(\tilde{\gamma}) \to \infty \) as \( \tilde{\gamma} \to \gamma^* \), it holds that \( \varepsilon \cdot \tilde{a}(\gamma^* - \varepsilon) \) goes to a constant as \( \varepsilon \to 0^+ \). In particular, Equation 39 implies

\[
\varepsilon \cdot \tilde{a}(\gamma^* - \varepsilon) = \eta\varepsilon + (\eta - \eta) \frac{\varepsilon}{(\gamma^* - \varepsilon)^2 - 1} \xrightarrow{\varepsilon \to 0^+} (\eta - \eta)\gamma^*\varepsilon
\]

where the limit follows from L'Hopital’s Rule. In other words, for large \( s \), \( \tilde{a} \) is of order \( s^{-1} \).

Finally, we observe that for \( \tilde{\gamma} = \gamma^* - \varepsilon \), for \( \varepsilon \) small we have \( v(\eta(\tilde{a})) - v(\hat{\eta}(\eta)) \) approximately linear in \( \varepsilon \) (following the proof of Claim 6), and therefore Equation 36 implies \( (\tilde{a} - \eta)^r \)—and therefore \( \tilde{a}^r \)—diverges to \( \infty \) at a rate that is linear in \( s \cdot \varepsilon \). Applying the previous result, we have \( \varepsilon^{-r} \) approximately linear in \( s \cdot \varepsilon \), or equivalently \( \varepsilon = \gamma^* - \tilde{\gamma} \) is of order \( s^{-1} \).

\( \text{Q.E.D.} \)