Effective Communication in Cheap-Talk Games*

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Abstract

This paper presents arguments based on weak dominance and learning for selecting informative equilibria in a model of cheap-talk communication where players must use monotonic strategies. Under a standard regularity condition, only one equilibrium survives iterated deletion of interim dominated strategies. Under the same condition, we establish that best-response dynamics converges to this outcome.

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1 Introduction

Talk is a useful way to communicate private information in strategic situations. At least since the formal models of Crawford and Sobel [9] and Green and Stokey [15], the possibility that cheap talk can be influential in games has been understood. However, these early papers recognized that equilibrium analysis is generally indeterminate. Models of cheap-talk communication have multiple equilibria and, in particular, they typically have an uninformative equilibrium in which players do not take advantage of opportunities to communicate. A central concern of this research has been finding conditions under which communication is effective, that is in which the predicted outcome involves non-trivial information transmission. We hope this paper advances the literature on effective communication.

We present arguments that lead to equilibrium selection results in simple cheap-talk games. We establish that iterated deletion of interim weakly dominated strategies selects an outcome with effective communication when such an outcome exists, for example, when the regularity condition from Crawford and Sobel [9] (hereafter CS) is satisfied. We also show that best-response dynamic leads to the same selection. These arguments are subtle because they require a reformulation of the strategic situation in order to work. The paper presents two different ways to think about the reformulation: one reformulation studies a game in which the players are restricted to monotonic strategies; the other reformulation looks at learning processes. In the first case, our solution concept involves iterated deletion of weakly dominated strategies. In the second case, it is a form of dynamic stability.

Underlying our analysis is a basic model of strategic communication. An informed Sender sends a message to an uninformed Receiver. The Receiver responds to the message by making a decision that is payoff relevant to both players. Talk is cheap because the payoffs of the players do not depend directly on the Sender’s message. CS characterize the set of equilibrium outcomes in a one-dimensional model of cheap talk with a particular kind of conflict of interest. CS demonstrate that there is a finite upper bound, $N^*$, to the number of distinct actions that the Receiver takes in equilibrium, and that for each $N = 1, \ldots, N^*$, there is an equilibrium in which the Receiver takes $N$ actions. In addition, when a technical condition holds, CS demonstrate that for all $N = 1, \ldots, N^*$, there is a unique equilibrium outcome in which the Receiver takes $N$ distinct actions, and the ex-ante expected payoff for both Sender and Receiver is strictly increasing in $N$. The equilibrium with $N^*$ actions is often the outcome selected for analysis in applications.

The multiple-equilibria problem arises in three different ways in cheap-talk games. Typically, some messages are sent with zero probability in equilibrium. There will often be multiple ways to specify behavior off the path of play. This first kind of multiplicity, off-path indeterminacy, is familiar in games with incomplete information and need not be essential. Changing off-path behavior does not change what we observe. If cheap-talk games had a unique equilibrium outcome, then we would be able to confidently classify situations in which communication is effective even if many different equilibria supported that outcome. The second kind of multiplicity, message indeterminacy, is that the meaning of messages is arbitrary. Given any equilibrium, one can generate another equilibrium by changing the use and interpretation of messages. This kind of
problem identifies a way in which language is arbitrary. The word used to describe
the color of a white house in Paris is blanche and in Warsaw is biały. Predictions are
still possible with this kind of indeterminacy when the different equilibria induce the
same relationship between types and actions. What matters is that French speakers and
Polish speakers classify the same set of houses as “white” (and their audiences understand
that) rather than the particular word they use to describe the color. The third type of
multiplicity, type-action indeterminacy, is fundamental. Cheap-talk games typically have
an uninformative equilibrium\(^1\) and may have qualitatively different equilibria in which
the Receiver takes at least two different actions with positive probability. It is this type
of multiplicity that we wish to examine, but our approach shows how eliminating the
problem of message indeterminacy can resolve type-action indeterminacy.

Our analysis depends on the critical assumption that players in the basic cheap-talk
game use monotonic strategies. We assume that there is an exogenous order on messages
and restrict players to strategies that are monotonic with respect to this order. We
view this as a way to incorporate “exogenous meaning” into the communication game.
Players enter a strategic setting with a shared ordering and it is common knowledge
that they will behave in a way that is consistent with the order in the communication
game. The resulting monotonic cheap-talk game has all three kinds of multiplicity, but
the restriction to monotonic strategies eliminates some message indeterminacy. Our
main result is that combining the restriction to monotonic strategies with an equilibrium
refinement solves the problem of type-action indeterminacy. The analysis has a bonus.
Imposing monotonicity actually selects the messages that are used in equilibrium. That
is, the monotonicity condition (combined with the rest of the analysis) eliminates message
indeterminacy. We find the selection intuitive. When the Sender has an upward bias (for
each type, the ideal action of the Sender is strictly greater than the ideal action of the
Receiver), the selected equilibrium uses only the highest messages. That is, upward bias
leads to exaggeration in equilibrium.

Our results on dynamics and iterated dominance hinge on identifying two sequences
of strategy profiles. The sequences are defined by specifying an initial condition and
deriving subsequent elements of the sequence by iterating best responses. One sequence
starts with the highest strategy profile; the other sequence starts with the lowest strategy
profile.\(^2\) We can show that these sequences are monotonic and converge to equilibria.
Furthermore, when a regularity condition introduced in CS holds, the high and low
sequence have a common limit and that common limit is equal to the CS equilibrium
that has the largest number of actions induced. Our results on limits of best-response
dynamics follow because any sequence of best responses must be sandwiched between the
highest and lowest sequence. Our results on iterated deletion follow because we can show
that strategies larger than the higher limit or lower than the lower limit must eventually
be deleted.

This paper collects the main ideas common for several working papers by various
subsets of the authors, namely, Gordon [13], Gordon [14], Kartik and Sobel [18], Lo
and Olszewski [21], Lo [22], and Lo [23]. We decided to formulate the results based on

\(^1\)To be precise, they have many uninformative equilibria when one takes into account the first two
kinds of multiplicity.

\(^2\)The restriction to monotonic cheap-talk games makes it possible to partially order strategies.
these ideas within the basic communication model. We also generalized, extended and modified some of the results from these working papers. The working papers still contain other results that are more general, or simply different, in certain directions. This paper concentrates on what we view as the most significant findings from the working papers.

The paper proceeds as follows. Section 2 introduces the basic cheap-talk game. Section 3 contains a simple example that illustrates how the restriction to monotonic strategies and removal of weakly dominated strategies has the power to select an equilibrium when neither the restriction to monotonic strategies or the refinement alone suffice to do so. Section 4 contains some preliminary results. Section 5 states the main results. Section 6 presents two examples that illustrate the proof technique. Section 7 briefly describes results when the bias of the Sender is not upward. Section 8 interprets the main result and connects it to the literature. The proofs are in the appendices.\(^3\)

## 2 The Basic Cheap-Talk Model

### 2.1 The cheap-talk game

Recall the basic cheap-talk model: It has two players, the Sender S and the Receiver R. The Sender and Receiver have utility functions \(u^S\) and \(u^R\), respectively. They are defined on \(\mathbb{R} \times [0, 1]\). We assume that for \(j = S\) and \(R\), \(w^j : \mathbb{R} \times [0, 1] \to \mathbb{R}\) is twice continuously differentiable, strictly concave in its first argument, and has strictly positive mixed partial derivative. The first argument of \(w^j(\cdot)\) is an action, \(a\). The second argument is the type of the Sender, \(t\). The type is drawn from a strictly positive, continuous density \(f\) on \([0, 1]\).

We assume that \(\max_{a \in \mathbb{R}} w^j(a, t)\) exists and let \(a^j(t) = \arg\max_{a \in \mathbb{R}} w^j(a, t)\) for \(j = S\) and \(R\). (Strict concavity guarantees that the maximizer of \(w^j(\cdot, t)\) is unique.) For \(t' < t''\), let

\[
a^R(t', t'') = \arg\max_{a \in \mathbb{R}} \int_{t'}^{t''} u^R(a, t) f(t) dt
\]

and, for \(t' = t'' = t\), \(a^R(t', t'') = a^R(t)\). If \(a^R(t) < (>) a^S(t)\) for all \(t\), we say that there is upward (downward) bias. Unless otherwise noted, we assume an upward bias. We normalize preferences so that \(a^R(0) = 0\) and \(a^R(1) = 1\). With this normalization, actions outside of the unit interval are dominated for \(R\) and would not be part of any equilibrium outcome. We therefore restrict the action set to be \([0, 1]\).

In the basic cheap-talk game, the Sender learns \(t\), sends the Receiver a message about \(t\), and the Receiver takes an action \(a\). More precisely, let \(M\) be a finite set of available messages. A (pure) strategy for \(S\) is a mapping \(s : [0, 1] \to M\) that associates with every type \(t\) the message \(s(t)\). A pure strategy for \(R\) is a mapping \(a : M \to \mathbb{R}\) that associates with every message \(m\) an action \(a(m)\). Given a strategy profile \((s, a)\) the payoff of player \(j\)

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\(^3\)Appendix A contains details of the main argument. Appendix B presents an example that demonstrates that limit of best-response dynamics may depend on the initial condition if our regularity condition fails. Supplementary Appendix C (not for publication) contains a spreadsheet associated with an example in Section 6.
is \( f_0^1 u^1(a(s(t)), t)f(t)dt \). We denote R’s (pure) strategy set by \( A \) and S’s (pure) strategy set by \( S \).

We impose the following monotonicity restriction. Assume \( M \) is linearly ordered, and denote the order by \( \leq \). A strategy \( a(\cdot) \) for R is monotonic if \( m < m' \) implies that \( a(m) \leq a(m') \), and a strategy \( s(\cdot) \) for S is on-path monotonic if \( t < t' \) implies that \( s(t) \leq s(t') \). The monotonicity restriction permits us to describe strategies in a convenient way. Let \( N \) be the number of available messages, and let the messages be ordered as \( m_1 < \cdots < m_N \). Every monotonic strategy of the Receiver can be uniquely represented by actions \( a_1 \leq \cdots \leq a_N \) such that \( a_i \) for \( i = 1, \ldots, N \) is induced by message \( m_i \). We will identify a monotonic strategy of the Sender with cutoffs \( t_0 \leq t_1 \leq \cdots \leq t_N \) such that \( t_0 = 0 \) and \( t_N = 1 \), with the interpretation that types in the interval \((t_{i-1}, t_i)\) send the message \( m_i \). To complete the interpretation we impose an additional restriction on strategies that have off-path messages, that is, messages that are sent with probability zero. For example, suppose that \( N = 3 \) and the types from \([0, 1/2)\) send message \( m_1 \) and the types from \([1/2, 1]\) send message \( m_3 \).\(^4\) We represent this strategy with cutoffs \((0, 1/2, 1/2, 1)\). \( m_2 \) lies off-path, because either it is sent by no type (if \( 1/2 \) sends \( m_1 \) or \( m_3 \)) or it is sent only by type \( 1/2 \) whose probability is equal to zero. A strategy for S is off-path monotonic if any off-path message \( m \) is sent (possibly only with some probability) by a type that is greater than or equal to the supremum of types \( t \) that send a lower message and less than or equal to the the infimum of types that send a higher message. Off-path monotonicity requires that the off-path message \( m_2 \) from our example must be sent with positive probability by type \( 1/2 \). An important property of off-path monotonicity is that every message is sent by some type, although some messages are sent only with probability zero. Therefore the Receiver’s beliefs are determined by the Sender’s strategy. A strategy for S is monotonic if it is on-path and off-path monotonic.

There is a natural order on monotonic strategies. For R, \((a_1, \ldots, a_N) \leq (a'_1, \ldots, a'_N)\) if and only if \( a_i \leq a'_i \) for all \( i \). For S, when strategies are represented by cutoffs, \((t_0, \ldots, t_N) \leq (t'_0, \ldots, t'_N)\) if and only if \( t_i \leq t'_i \) for all \( i \). These orders are partial, but there exists a largest and smallest strategy for both S and R.

Given a strategy profile \((s, a)\), the associated type-action mapping \( \gamma \) is defined as \( \gamma(t) = a(s(t)) \). Every monotonic strategy profile uniquely determines the associated type-action mapping up to the actions induced by the cutoffs.

A monotonic cheap-talk game is derived from the basic cheap-talk game by assuming that players are allowed to play only monotonic strategies. We denote R’s (pure) monotonic strategy set by \( A_0 \) and S’s (pure) monotonic strategy set by \( S_0 \).

The monotonicity assumption is restrictive in the trivial sense that non-monotonic strategies exist. It is also restrictive in the stronger sense that non-monotonic strategies may be best responses even if the opponent is restricted to playing monotonic strategies. Finally, if we allowed for playing mixed strategies, then there would also exist mixtures of monotonic pure strategies with only non-monotonic best responses. However, the Sender’s best response to any strictly monotonic strategy of the Receiver must be on-path monotonic, and the Sender always has a monotonic best response to a monotonic strategy of the Receiver. Similarly, if the Sender plays a monotonic strategy, then the

\(^4\)Note that we distinguish off-path messages from messages that are never used. In this example, \( m_2 \) is an off-path message even if it is sent by (only) type \( t = 1/2 \).
Receiver has a monotonic best response. In addition, any best response of the Receiver must be monotonic on the path, but the Receiver may also have a non-monotonic best response off the path.

Any equilibrium type-action mapping for the original game can be derived from monotonic strategies. To see this, use the fact (described in Section 2.2) that any equilibrium involves a finite partition of the Sender’s types into adjacent intervals. Construct a strategy for the Sender in which the types from higher partition elements send higher messages. Any best response to this strategy will be monotonic on the equilibrium path. One can define specific off-the-path actions to preserve monotonicity and support the equilibrium.

2.2 The structure of equilibria

Using the assumption of upward bias, CS demonstrate that there exists a positive integer $N^*$ such that for every integer $1 \leq N \leq N^*$, there exists at least one equilibrium in which there are $N$ induced actions, and moreover, there is no equilibrium that induces strictly more than $N^*$ actions. Any equilibrium can be characterized by cutoffs $0 = t_0 < t_1 < \cdots < t_N = 1$, and actions $a_1 \leq \cdots \leq a_N$ such that

$$u^S(a_{i+1}, t_i) - u^S(a_i, t_i) = 0$$  \hspace{1cm} (1)

for $i = 1, \ldots, N - 1$, and

$$a_i = a^R(t_{i-1}, t_i)$$  \hspace{1cm} (2)

for $i = 1, \ldots, N$. In such an equilibrium, adjacent types pool and send a common message. Condition (1) states that the cutoff types are indifferent between pooling with types immediately below or immediately above. Condition (2) states that R best responds to the information in S’s message. Except for the specification of the Sender’s behavior at cutpoints, Conditions (1) and (2) uniquely describe an equilibrium relationship between types and actions.

CS make another assumption that permits them to strengthen the characterization of equilibria. For $t_{i-1} \leq t_i \leq t_{i+1}$, let

$$V(t_{i-1}, t_i, t_{i+1}) = u^S(a^R(t_i, t_{i+1}), t_i) - u^S(a^R(t_{i-1}, t_i), t_i).$$

A (forward) solution to (1) of length $L$ is a sequence $t_0, \ldots, t_L$ such that $V(t_{i-1}, t_i, t_{i+1}) = 0$ for $0 < i < L$ and $t_0 < t_1$.

**Definition 1** (Regularity Condition). The cheap-talk game satisfies the Regularity Condition (RC) if for any two solutions to (1) of length $L$, $(t_0, \ldots, t_L)$ and $(t'_0, \ldots, t'_L)$ with $t_0 = t'_0$ and $t_1 < t'_1$, we have that $t_i < t'_i$ for all $i \geq 2$.

(RC) is satisfied by the leading “uniform-quadratic” example in CS, which has been the focus of many applications. CS prove that if (RC) holds, then there is exactly one equilibrium type-action mapping (up to the Sender’s cutoffs) for each $N = 1, \ldots, N^*$, and the ex-ante equilibrium expected utility for both S and R is increasing in $N$. These results provide an argument for the salience of the $N^*$ equilibrium.
Another argument in support of this equilibrium outcome requires a definition.

**Definition 2 (NITS).** An equilibrium \((a^*, s^*)\) satisfies the No Incentive to Separate (NITS) Condition if \(u^S(a^*(m_1), 0) \geq u^S(a^R(0), 0)\).

NITS states that the lowest type of the Sender prefers her equilibrium payoff to the payoff she would receive if the Receiver knew her type (and responded optimally). Chen, Kartik, and Sobel [7] show that only the (essentially unique) equilibrium type-action mapping with \(N^*\) actions induced satisfies NITS when (RC) holds. Uniqueness fails only because type \(t_i\) is indifferent between messages \(i\) and \(i+1\) for \(i = 1, \ldots, N-1\). From now on, we drop the word “essentially,” and call such mappings unique.

### 3 Coordination Game Example

Although our results mainly concern the basic cheap-talk model, our first example is a coordination game. This will enable us to present the main ideas in the simplest manner. Examples in Section 6 are conducted within the basic cheap-talk model, but are more involved.

There are two types (high and low), two actions (high and low), and two messages (high and low). The types are equally likely. The Sender and Receiver have common interests. The players receive a payoff of two if the action matches the type and a payoff of zero otherwise. Limiting attention to binary types and actions and identical preferences makes the analysis transparent. The strategic form of the game is given in the following table.

A strategy for the Sender is a pair \((i, j)\) where the Sender sends message \(i\) when her type is low and \(j\) when her type is high. Similarly, the first component in the Receiver’s strategy is his response to a low message and the second is his response to a high messages. Hence \((l, h)\) is the Sender strategy that reports a message that matches the state, \((h, l)\) is the strategy that sends the high message when the state is low and the low message when the state is high. Similarly, \((H, L)\) is the strategy of the Receiver that responds to the low message with the high action and the high message with the low action.

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<thead>
<tr>
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<td>((h,h))</td>
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<tr>
<td>((h,l))</td>
<td>1, 1</td>
<td>0, 0</td>
<td>2, 2</td>
<td>1, 1</td>
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<td>((l,h))</td>
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The game has an uninformative equilibrium in which S mixes equally between \((h, l)\) and \((l, h)\) and R mixes equally between \((H, L)\) and \((L, H)\). (There are also inefficient pure-strategy equilibria and other inefficient mixed equilibria.) There are also two efficient equilibria in which the Sender distinguishes between the states and the Receiver correctly interprets this information. The mixed-strategy equilibrium satisfies standard refinements (from perfection to strategic stability) although it is, intuitively, implausible.
Our approach is to replace the original game by a game in which non-monotonic strategies are not available. The strategic form of the monotonic game is given in the following table.

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<tr>
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<tr>
<td>(l,h)</td>
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<td>(l,l)</td>
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</table>

We obtain this game by deleting the non-monotonic strategies from the original game. Doing so eliminates some inefficient equilibria, but it does not eliminate any equilibrium payoffs. The monotonic game has multiple equilibria, but weak dominance selects the efficient one. We emphasize that the example demonstrates why our approach requires both a restriction to monotonic strategies and an equilibrium refinement. On one hand, if we do not limit attention to the $3 \times 3$ game, then weak dominance arguments have no power to select an equilibrium. On the other hand, the $3 \times 3$ game has multiple (Nash) equilibria.

Our subsequent analysis of the basic cheap-talk model requires multiple rounds of deletion, but the example captures an essential feature of the construction. Restricting attention to monotonic strategies eliminates some coordination problems. Without the restriction, every informative equilibrium type-action distribution can be supported in multiple ways by permuting the assignment of types to messages. Imposing an order on messages removes this indeterminacy.

4 Concepts

4.1 Iterated deletion of weakly dominated strategies

Let $U^j(s,a)$ for $j = S, R$ be the payoff of player $j$ given the strategy profile $(s,a)$. Consider a game with strategy sets $\tilde{S} \subset S$ for $S$ and $\tilde{A} \subset A$ for $R$. A strategy $s \in \tilde{S}$ weakly dominates a strategy $s' \in \tilde{S}$ if $U^S(s,a) \geq U^S(s',a)$ for every strategy $a \in \tilde{A}$ and $U^S(s,\tilde{a}) > U^S(s',\tilde{a})$ for some strategy $\tilde{a} \in \tilde{A}$; a strategy $a \in \tilde{A}$ weakly dominates a strategy $a' \in \tilde{A}$ if $U^R(s,a) \geq U^R(s,a')$ for every strategy $s \in \tilde{S}$ and $U^R(\tilde{s},a) > U^R(\tilde{s},a')$ for some strategy $\tilde{s} \in \tilde{S}$. A general procedure of deleting strategies produces a sequence of sets $\tilde{S}_k$ and $\tilde{A}_k$ such that:

1. $\tilde{S}_0 = S_0$, $\tilde{A}_0 = A_0$;
2. $\tilde{S}_k$ is a subset of $\tilde{S}_{k-1}$ obtained by deleting a (possibly empty) subset of $S$'s weakly dominated strategies in the game with strategy sets ($\tilde{S}_{k-1}, \tilde{A}_{k-1}$);
3. $\tilde{A}_k$ is a subset of $\tilde{A}_{k-1}$ obtained by deleting a (possibly empty) subset of $R$'s weakly dominated strategies in the game with strategy sets ($\tilde{S}_{k-1}, \tilde{A}_{k-1}$);
4. The sets $\tilde{S}^* = \bigcap_{k=0}^{\infty} \tilde{S}_k$ and $\tilde{A}^* = \bigcap_{k=0}^{\infty} \tilde{A}_k$ are non-empty.
5. There are no weakly dominated strategies in either \( \tilde{S}^* \) or \( \tilde{A}^* \) in the game with strategy sets \( (\tilde{S}^*, \tilde{A}^*) \).

The second and third conditions permit the deletion of only some weakly dominated strategies (and for deletions to be simultaneous). The fourth condition guarantees that the limit of the process exists. The fifth condition guarantees that the process continues as long as weakly dominated strategies remain. In general games (beyond our setting), there may exist no procedure satisfying our conditions (see Lipman [20]), and \( (\tilde{S}^*, \tilde{A}^*) \) may depend on the order of deletion (see Dufwenberg and Stegeman [11]).

Our extension of IDWDS to games with infinite sets of strategies coincides with that of Dufwenberg and Stegeman [11]. Alternatively, one may consider procedures that produce transfinite sequences of sets \( \tilde{S}_\kappa \) and \( \tilde{A}_\kappa \) (\( \kappa \) stands here for an ordinal number), or define sets \( \tilde{S}^* \) or \( \tilde{A}^* \) in terms of stable sets (see Chen, Long, and Luo [5]).

4.2 Interim dominance

Our arguments sometimes require a different notion of dominance.

Consider a game with strategy sets \( \tilde{S} \subseteq S \) for \( S \) and \( \tilde{A} \subseteq A \) for \( R \). A strategy \( a \in \tilde{A} \) interim weakly dominates a strategy \( a' \in \tilde{A} \) if for every strategy \( s \in \tilde{S} \) and message \( m \), strategy \( a \) yields a weakly higher payoff than that of strategy \( a' \), both contingent on the Sender’s type sending message \( m \); in addition, the payoff of \( a \) is strictly higher than that of \( a' \) for at least one strategy of the Sender in \( \tilde{S} \) and at least one message \( m \). We emphasize that for some strategies of the Sender some messages are sent by only a single type of the Sender; such messages correspond to (degenerate) intervals \([t_{i-1}, t_i]\) with \( t_{i-1} = t_i \). Our definition of interim weak dominance does not disregard such messages (intervals); strategy \( a \) must prescribe a better action than that of strategy \( a' \) contingent on each such message (the Sender’s type belonging to such an interval, that is, contingent on this type being equal to \( t_{i-1} = t_i \)). For the Sender, a strategy \( s \) interim dominates a strategy \( s' \) if every type of the Sender weakly prefers \( s \) to \( s' \) for every strategy of the Receiver, and some type strictly prefers \( s \) to \( s' \) for some strategy of the Receiver.

On one hand, a strategy may interim dominate another strategy without weakly dominating the other strategy. This possibility arises if the interim domination is the result of strictly better performance in response to a message sent with probability zero for all remaining strategies of the Sender. On the other hand, a strategy may weakly dominate another strategy without dominating it in the interim sense. This possibility arises if the weakly dominating strategy fails to respond optimally to a message sent with probability zero for all remaining strategies of the Sender. In addition, weak dominance allows for compensating for an inferior response to one message by a superior response to another message.

In Theorem 3, we will rule out the Sender’s strategies that satisfy another dominance condition. Consider a game with strategy sets \( \tilde{S} \subseteq S \) for \( S \) and \( \tilde{A} \subseteq A \) for \( R \). A strategy \( s \in \tilde{S} \) allowable weakly dominates a strategy \( s' \in \tilde{S} \) if \( U^S(s, a) \geq U^S(s', a) \) for every strategy \( a \in \tilde{A} \) and, in addition, there exists \( \varepsilon > 0 \) and \( \tilde{a} \in \tilde{A} \) such that for all strictly increasing \( \tilde{a}' \) such that \( |\tilde{a}_i - \tilde{a}'_i| < \varepsilon \) for all \( i \), \( U^S(s, \tilde{a}') > U^S(s', \tilde{a}') \). For the Receiver, allowable interim weak dominance is defined as interim weak dominance.
The notion of allowable weak dominance differs from weak dominance in situations where all strategies \( a \in A \) satisfy \( a_i = a_{i+1} \) for some \( i \). In this case, the Sender has multiple best responses. For example, suppose that \( 1 < i < N - 1 \) and \( a_{i-1} < a_i = a_{i+1} < a_{i+2} \); let \( t_{i-1} \) be the type that is indifferent between actions \( a_{i-1} \) and \( a_i \), and let \( t_{i+1} \) be the type that is indifferent between actions \( a_{i+1} \) and \( a_{i+2} \). Then, the choice of \( t_i \) from \([t_{i-1}, t_{i+1}]\) does not affect the type-action mapping. We use the behavior of Sender’s strategies against strictly monotonic strategies of the Receiver close to \( a \) to require \( t_i \) to be the type for whom \( a_i = a_{i+1} \) is the most preferred action.

4.3 Iterated deletion of interim weakly dominated strategies

We modify IDWDS by replacing weak dominance by interim weak dominance. That is, we study Iterated Deletion of Interim Weakly Dominated Strategy (IDIWDS), which is defined as a procedure that satisfies the following properties. A general procedure of deleting strategies produces a sequence of sets \( \tilde{S}_k \) and \( \tilde{A}_k \) such that:

1. \( \tilde{S}_k = S_0, \tilde{A}_k = A_0; \)

2. \( \tilde{S}_k \) is a subset of \( \tilde{S}_{k-1} \) obtained by deleting a (possibly empty) subset of S’s interim weakly dominated strategies in the game with strategy sets \( (\tilde{S}_{k-1}, \tilde{A}_{k-1}) \);

3. \( \tilde{A}_k \) is a subset of \( \tilde{A}_{k-1} \) obtained by deleting a (possibly empty) subset of R’s interim weakly dominated strategies in the game with strategy sets \( (\tilde{S}_{k-1}, \tilde{A}_{k-1}) \);

4. The sets \( \tilde{S}^* = \bigcap_{k=0}^{\infty} \tilde{S}_k \) and \( \tilde{A}^* = \bigcap_{k=0}^{\infty} \tilde{A}_k \) are non-empty.

5. There are no interim weakly dominated strategies in either \( \tilde{S}^* \) or \( \tilde{A}^* \) in the game with strategy sets \( (\tilde{S}^*, \tilde{A}^*) \).

We can further modify the definition of IDWDS and IDIWDS by replacing weakly dominated by allowable weakly dominated for the Sender to obtain the concepts of allowable IDWDS and allowable IDIWDS.

One of our main conclusions, Corollary 2, holds for both IDWDS and IDIWDS. Our proof of the other main conclusion, Corollary 3, requires allowable IDIWDS.

4.4 Equilibrium Bounds

Our construction uses two sequences of strategy profiles. The sequences consist of best responses to an initial specification of strategies for S and R. One specification begins with the lowest possible strategies; the other specification begins with the highest possible strategies. Given initial conditions, the best response property does not uniquely define the sequences because best responses need not be unique. We will show, however, that there is a way to select best responses that guarantee that the lower sequence is increasing and converges to an equilibrium and the upper sequence is decreasing and converges to an equilibrium. Furthermore, when (RC) holds, the limit of the lower sequence is equal
to the limit of the upper sequence. To show convergence of the best-response dynamic we show that any sequence of strategy profiles generated by interim best responses remains sandwiched between the lower and upper sequence. Hence, whenever the lower and upper sequences have a common limit, any sequence converges to the same limit. To show that iterated deletion of strategies selects a particular outcome, we show that strategies that are less than the lower limit and strategies that are greater than the upper limit are eventually deleted.

We now describe the lower and upper sequences consistent with the best-response dynamic. Let

\[ 0 = t_0^0 = \cdots = t_{N-1}^0 < t_N^0 = 1 \text{ and } 0 = t_0^1 < t_1^1 = \cdots = t_N^1 = 1, \]

\[ a_1^0 = \cdots = a_N^0 = 0 \text{ and } a_1^1 = \cdots = a_N^1 = 1. \]

By induction, let \((a_1^{k+1}, \ldots, a_N^{k+1})\) be defined as the interim best response of the Receiver to strategy \((t_0^k, t_1^k, \ldots, t_N^k)\) of the Sender, and let \((\bar{a}_1^{k+1}, \ldots, \bar{a}_N^{k+1})\) be the interim best response of the Receiver to strategy \((\bar{t}_0^k, \bar{t}_1^k, \ldots, \bar{t}_N^k)\) of the Sender. These best responses specify optimal actions contingent on all messages, even the messages that correspond to degenerate intervals of the Sender’s strategy. Because the interim best responses of the Receiver are unique, actions \((a_1^{k+1}, \ldots, a_N^{k+1})\) and \((\bar{a}_1^{k+1}, \ldots, \bar{a}_N^{k+1})\) are uniquely defined.

Let \((t_0^{k+1}, t_1^{k+1}, \ldots, t_N^{k+1})\) be a best response of the Sender to strategy \((a_1^k, \ldots, a_N^k)\) of the Receiver, and let \((\bar{t}_0^{k+1}, \bar{t}_1^{k+1}, \ldots, \bar{t}_N^{k+1})\) be a best response of the Sender to strategy \((\bar{a}_1^k, \ldots, \bar{a}_N^k)\) of the Receiver. Because the Sender has more than one best response to any strategy of the Receiver such that \(a_i = a_{i+1}\) for some \(i\), we must pick among them. For \((t_0^{k+1}, t_1^{k+1}, \ldots, t_N^{k+1})\), we pick the smallest best response to \((a_1^k, \ldots, a_N^k)\), that is, the best response such that if \((t_0, t_1, \ldots, t_N)\) is another best response of the Sender to \((a_1^k, \ldots, a_N^k)\), then \(t_i \geq t_i^{k+1}\) for \(i = 0, 1, \ldots, N\). This requires picking for \(t_i^{k+1}\) the lowest type that weakly prefers \(a_i\) to all strictly lower actions in the profile \((a_1^k, \ldots, a_N^k)\). For \((\bar{t}_0^{k+1}, \bar{t}_1^{k+1}, \ldots, \bar{t}_N^{k+1})\), we pick greatest best response, that is, the best response such that if \((t_0, t_1, \ldots, t_N)\) is another best response of the Sender to \((\bar{a}_1^k, \ldots, \bar{a}_N^k)\), then \(t_i \leq \bar{t}_i^{k+1}\) for \(i = 0, 1, \ldots, N\), except for all \(i\) such that \(a_i = a_{i+1} = 0\) when we pick \(t_i = 0\). This requires picking for \(t_i^{k+1}\) the highest type that weakly prefers \(a_i\) to all strictly higher actions in the profile \((\bar{a}_1^k, \ldots, \bar{a}_N^k)\).

We will now describe the construction of the sequence of upper bounds in more detail. First, notice that \(\bar{t}_1^1 < 1\) and \(\bar{a}_1^1 = \cdots = \bar{a}_N^1 = 1\); in turn,

\[ \bar{t}_i^1 = \bar{t}_{i-1}^0 = 1 \quad (3) \]

for \(i = 1, \ldots, N\) and therefore

\[ \bar{a}_i^2 = \bar{a}_i^1 \quad (4) \]

for \(i = 1, \ldots, N\). Continuing, \(\bar{t}_i^2\) is the type that is indifferent between actions \(\bar{a}_1^1\) and \(\bar{a}_2^1\), and \(\bar{t}_2^2 = \cdots = \bar{t}_N^2 = 1\). If all types prefer \(\bar{a}_2^1\) to \(\bar{a}_1^1\), then \(\bar{t}_1^2 = 0\). Further, \(\bar{a}_1^2 < \bar{a}_2^3 < \bar{a}_3^3 = \cdots = \bar{a}_N^3 = 1\), and \(\bar{t}_3^3\) is the type that is indifferent between actions \(\bar{a}_1^3\) and \(\bar{a}_2^3\), \(\bar{t}_4^3\) is the type that is indifferent between actions \(\bar{a}_1^3\) and \(\bar{a}_2^3\), and \(\bar{t}_3^4 = \cdots = \bar{t}_N^4 = 1\). If all
types prefer $\tilde{a}^0_i$ to $\tilde{a}^1_i$, then $\tilde{t}^4_i = 0$. And if all types prefer $\tilde{a}^0_i$ to $\tilde{a}^2_i$, then also $\tilde{t}^4_i = 0$. We continue in this fashion. Note that equations (3) and (4) imply that $\tilde{t}^k_{i+1} = \tilde{t}^k_i$ when $k$ is even and $\tilde{t}^k_{i+1} = \tilde{a}^k_i$ when $k$ is odd. After $N - 1$ changes to R’s strategy, we reach a stage $k^*$ such that the strategy $(\tilde{a}^{k^*}_1, \ldots, \tilde{a}^{k^*}_N)$ has all positive actions different, and from that moment (that is, for $k \geq k^* = 2N - 3$), the Sender’s best responses to $(\tilde{a}^k_1, \ldots, \tilde{a}^k_N)$ with the property that $t_i = 0$ when $a_i = a_{i+1} = 0$ are unique.\footnote{Because we assume upward bias, $t^k_i \leq t^k_{i+1}$ and $\tilde{t}^k_i \geq \tilde{t}^k_{i+1}$ for all $i$ and $k$. Monotonicity implies convergence, so $t^k_i \rightarrow t^*_{i+1}, \tilde{t}^k_i \rightarrow \tilde{t}^*_{i+1}, a^k_i \rightarrow a^*_{i}$ and $\tilde{a}^k_i \rightarrow \tilde{a}^*_{i}$. The limit actions and cutoff strategies are an equilibrium.}

Thus, sequences $(t^k_i)_{k=1}^\infty$, $(\tilde{t}^k_i)_{k=1}^\infty$, $(a^k_i)_{k=1}^\infty$ and $(\tilde{a}^k_i)_{k=1}^\infty$ are well-defined by induction. Recall now two properties of the basic cheap-talk model:

(i) the Receiver’s optimal action $a^R(t_i, t_h)$ given the belief that the Sender’s types belong to an interval $[t_i, t_h]$ strictly increases in $t_i$ and in $t_h$;

(ii) for each pair of actions $a_l \leq a_h$, if there is a type that is indifferent between actions $a_l$ and $a_h$, then this type strictly increases in $a_l$ and in $a_h$.

An inductive proof referring to (i) and (ii) shows that sequences $(t^k_i)_{k=1}^\infty$, $(\tilde{t}^k_i)_{k=1}^\infty$, $(a^k_i)_{k=1}^\infty$ and $(\tilde{a}^k_i)_{k=1}^\infty$ are monotonic:

$$t^k_i \leq \tilde{t}^{k+1}_i \quad \text{and} \quad \tilde{t}^k_i \geq \tilde{t}^{k+1}_i$$

and

$$a^k_i \leq a^{k+1}_i \quad \text{and} \quad \tilde{a}^k_i \geq \tilde{a}^{k+1}_i$$

for all $i$ and $k$. Monotonicity implies convergence, so $t^k_i \rightarrow t^*_{i+1}, \tilde{t}^k_i \rightarrow \tilde{t}^*_{i+1}, a^k_i \rightarrow a^*_{i}$ and $\tilde{a}^k_i \rightarrow \tilde{a}^*_{i}$. The limit actions and cutoff strategies are an equilibrium.

### 5 Results

We now state the three main results of our paper.

#### 5.1 Convergence of best-response sequences

**Definition 3.** Sequences of strategies $(t^k_i)_{k=0}^\infty$ and $(a^k_i)_{k=0}^\infty$, where $t^k = (t^k_0, \ldots, t^k_N)$ and $a^k = (a^k_1, \ldots, a^k_N)$, are called interim best-response sequences if for $k = 0, 1, \ldots$, strategy $a^{k+1}$ is the Receiver’s interim best response to the strategy $t^k$ of the Sender, and strategy $t^{k+1}$ is the Sender’s best response, to the strategy $a^k$ of the Receiver.

The sequences of interim best responses $(t^k_i)_{k=0}^\infty$ and $(a^k_i)_{k=0}^\infty$ starting from any initial conditions $(t^0_i, \ldots, t^N_i)$ and $(a^0_1, \ldots, a^0_N)$ are sandwiched between $(\tilde{t}^k_i)_{k=0}^\infty$ and $(\tilde{t}^k_i)_{k=0}^\infty$, and $(\tilde{a}^k_i)_{k=0}^\infty$ and $(\tilde{a}^k_i)_{k=0}^\infty$, respectively. That is,\footnote{For $a_l = a_h$, we take this to be the type for whom $a_l = a_h$ is the most-preferred action.}

$$t^k_i \leq \tilde{t}^k_i \leq \tilde{t}^k_i \leq \tilde{a}^k_i$$

\footnote{Because we assume upward bias, $t^k_i \leq t^k_{i+1} = 1$ and $\tilde{t}^k_i \leq \tilde{t}^k_{i+1} = \tilde{a}^k_i$. Therefore, positive actions will be distinct.}

$$a^k_i \leq a^{k+1}_i \quad \text{and} \quad \tilde{a}^k_i \geq \tilde{a}^{k+1}_i$$

\footnote{Actually, they are eventually strictly monotonic, except at the lower end, but we will not need this property.}
for all \( k \) and \( i \). These inequalities follow by induction from the monotonicity of the greatest best responses and the smallest best responses with respect to the opponent’s strategy. This yields the following result.

**Theorem 1.** For any interim best-response sequences \( (t^k)_{k=0}^\infty \) and \( (a^k)_{k=0}^\infty \) we have that

\[
t^*_{i} \leq \liminf_{k} t^k_{i} \leq \limsup_{k} t^k_{i} \leq \bar{t}^*_{i}
\]

for \( i = 1, \ldots, N - 1 \), and

\[
a^*_{i} \leq \liminf_{k} a^k_{i} \leq \limsup_{k} a^k_{i} \leq \bar{a}^*_{i}
\]

for \( i = 1, \ldots, N \).

In Appendix A, we show that Theorem 1 implies the following corollary.

**Corollary 1.** Assume \( N \geq N^* \). If there exists a unique equilibrium type-action mapping that satisfies NITS, then any interim best-response sequences converge to an equilibrium with this type-action mapping.

Appendix B contains an example in which there exist two equilibrium type-action mappings that satisfy NITS. The example illustrates a general property. When there are multiple type-action distributions that satisfy NITS, then the limit of an interim best-response sequence will depend on the initial condition. In particular, if the initial conditions specify that the highest messages induce the on-path actions of a NITS equilibrium and all lower messages induce the action 0, then the sequences of best responses are constant.

### 5.2 Characterization of Iterated Undominated Strategies

We present results parallel to those from the previous section for iterated deletion of weakly dominated strategies. The first result will refer to a specific procedure of eliminating dominated strategies. Our second result will ensure that this finding is independent of the order of eliminating dominated strategies.

**Theorem 2.** There exists a procedure of iterated deletion of interim weakly dominated strategies such that the sets \( \bar{S}^* \) and \( \bar{A}^* \) consist of the strategies \( (t_0, \ldots, t_N) \) and \( (a_1, \ldots, a_N) \) such that

\[
t^*_{i} \leq t_i \leq \bar{t}^*_{i} \text{ for } i = 0, \ldots, N \text{ and } a^*_{i} \leq a_i \leq \bar{a}^*_{i} \text{ for } i = 1, \ldots, N.
\]

In every round of deletion in the procedure we use to prove Theorem 2 we eliminate only strategies that are also weakly dominated (in addition to being interim weakly dominated). However, the strategies that satisfy condition (5) may also be weakly dominated (although they are not interim weakly dominated) in the game with strategy sets described by (5). Therefore, we cannot offer an analogue of Theorem 2 for weakly dominated strategies. However, the following corollary holds true for both interim dominance and dominance if there exists a unique equilibrium type-action mapping that satisfies NITS. The corollary follows from Theorem 2 because the uniqueness of NITS type-action mapping implies that \( t^*_{i} = \bar{t}^*_{i} \) for \( i = 0, \ldots, N \) and \( a^*_{i} = \bar{a}^*_{i} \) for \( i = 1, \ldots, N \).
Corollary 2. Assume \( N \geq N^* \). If there exists a unique equilibrium type-action mapping that satisfies NITS, then there is a procedure of iterated deletion of weakly dominated and interim weakly dominated strategies that retains only this type-action mapping. Furthermore, the surviving strategy uses only the highest \( N^* \) messages with positive probability.

Corollary 2 states that IDWDS selects a unique type-action mapping when there is a unique equilibrium type-action mapping that satisfies NITS. In this mapping, the Sender “exaggerates” by using only the highest messages. Consequently the combination of restriction to the monotonic cheap-talk game and elimination of weakly dominated strategies resolves message indeterminacy in addition to type-action indeterminacy.

For arbitrary procedures, we can show that:

Theorem 3. For any procedure of iterated deletion of allowable interim weakly dominated strategies, the sets \( \tilde{S}^* \) and \( \tilde{A}^* \) consist of the strategies that satisfy (5).

Theorem 2 does not follow from Theorem 3, because in some infinite games there is no procedure that satisfies our definition of procedures that iteratively eliminate weakly dominated strategies or that iteratively eliminate interim weakly dominated strategies; more precisely, there are games such that no procedure stops after \( \omega \) rounds of deletion, and we consider only procedures with this property.\(^8\) In addition, the proof of Theorem 3 refers to allowable dominance.

It follows from Theorem 3 that:

Corollary 3. Assume \( N \geq N^* \). If there exists a unique equilibrium type-action mapping that satisfies NITS, then every procedure of iterated deletion of allowable interim weakly dominated strategies retains only this type-action mapping.

We conjecture, but we have not managed to prove that a result similar to Corollary 3 holds for IDWDS procedures. Such a result holds true in some special cases, for example, the case discussed in Section 6.3, but in general it is difficult to show that an arbitrary procedure eliminates some strategies such that some actions coincide or such that some cutoffs coincide. An inspection of the proof of Corollary 3 shows that the cases in which agents have multiple best responses are the only obstacles to obtaining an analogous result for IDWDS.

6 Examples

6.1 Convergence of Best-Response Dynamic

The goal of this subsection is to illustrate the proof of Theorem 1 and Corollary 1 using an example. The proof involves constructing sequences of best responses starting from two extreme initial conditions. If the initial condition is high, then the resulting sequence monotonically decreases to an outcome that satisfies NITS. If the initial condition is low, then the resulting sequence monotonically increases to an outcome that satisfies NITS. In the example, there is only one equilibrium that satisfies NITS. Hence the two sequences

\(^8\)Symbol \( \omega \) stands for the smallest infinite ordinal.
have a common limit. Our result follows because a sequence of best responses starting from an arbitrary initial condition is sandwiched between the two extreme sequences.

Suppose that the Sender’s type is distributed uniformly on interval \([0, 1]\), and the players’ utilities are: \(u^S(a, t) = -(a - t - b)^2\) and \(u^R(a, t) = -(a - t)^2\), where \(b > 0\). In this case, there is an \(N^* \geq 1\) such that for every \(N \leq N^*\) there exists a unique equilibrium type-action mapping with \(N\) partition intervals (i.e., with \(N\) equilibrium actions). There exist no other equilibrium type-action mapping. For \(b = 0.05\), we have that \(N^* = 3\), i.e., the game has three equilibrium type-action mappings. In the largest of them, the types from \([t_0^*, t_1^*]\) = \([0, 4/30]\) induce action \(a_1^* = 2/30\), the types from \((t_1^*, t_2^*)\) = \((4/30, 14/30)\) induce action \(a_2^* = 9/30\), and the types from \((t_2^*, t_3^*)\) = \((14/30, 1]\) induce action \(a_3^* = 22/30\).

Assume that there exist three messages, \(m_1 < m_2 < m_3\). Let \(0 = t_0^0 \leq t_0^1 \leq t_0^2 \leq t_3^0 = 1\) denote the cutoffs of a strategy of the Sender in period 0. That is, the types from interval \((t_{i-1}^0, t_i^0)\) send message \(m_i\) in period 0. Let \((a_1^0, a_2^0, a_3^0)\) denote a strategy of the Receiver in period 0. Suppose that the Sender’s cutoffs and the Receiver’s actions in period \(k + 1\) are determined by the following equations:

\[
t_{i}^{k+1} + 0.05 - a_i^k = a_{i+1}^k - t_{i+1}^{k+1} - 0.05, \tag{6}
\]

for \(i = 1, 2, 9\) and

\[
a_{i+1}^k = \frac{t_{i-1}^k + t_{i+1}^k}{2}, \tag{7}
\]

for \(i = 1, 2, 3\). Recall that we always fix \(t_0^k\) at 0 and \(t_3^k\) at 1.

Formulas (6)–(7) define best-responses of the players to the strategies of their opponents from period \(k\). The best responses of the Sender are defined by imposing a specific tie-breaking rule. For example, if \(a_1^k = a_2^k = a_3^k = 0.5\), then any strategy of the Sender is a best response, and action 0.5 best responds to the Sender’s strategy in which all types send the same message. Our formulas imply that the Sender chooses the strategy \(t_{1}^{k+1} = t_{2}^{k+1} = 0.45\). Our results do not require this tie-breaking rule. The convergence of best-response dynamics follows from Corollary 1, which only assumes that players play monotonic strategies.

The reader may wonder why the uninformative equilibrium from the previous paragraph is not the limit of a constant sequence of strategy profiles. The reason is that any monotonic strategy of the Sender restricts off-path beliefs. For example, if the Sender sends \(m_2\) with probability 1, then \(t_1 = 0\) and \(t_2 = 1\), and since the Sender is supposed to play off-path monotonic strategies, the Receiver believes that message \(m_1\) is sent only by type 0, and message \(m_3\) is sent only by type 1. This unravels the uninformative equilibrium.

We show that \((t_0^k)_{k=0}^\infty\) and \((t_2^k)_{k=0}^\infty\) converge by considering two specific initial conditions: (i) \(t_1^0 = t_0^0 = 0\) and \(a_1^0 = a_2^0 = a_3^0 = 0\); (ii) \(t_1^0 = t_0^0 = 1\) and \(a_1^0 = a_2^0 = a_3^0 = 1\). Note these are not the bounding sequences we defined in Section 4.4. The bounding sequences must be defined more carefully, taking into account the multiplicity of the Sender’s best responses when the Receiver’s responses to some messages coincide. However, to obtain convergence of sequences defined by formulas (6)–(7), it is enough to consider bounding sequences that are also defined by these formulas. In case (i), since \(t_0^k\) and \(a_i^k\) take the

\[a_i^{k+1} < .1\] so that the equation defining \(t_i^{k+1}\) has no solution in \([0, 1]\), we set \(t_i^{k+1} = 0, i = 1, 2.\]
lowest possible values, so \( t_i^0 \leq t_i^1 \) for \( i = 1, 2 \) and \( a_i^0 \leq a_i^1 \) for \( i = 1, 2, 3 \). This implies that the sequences \( (t_i^k)_{k=0}^\infty \) and \( (a_i^k)_{k=0}^\infty \) for \( i = 1, 2, 3 \) are increasing. We obtain this from (6)–(7) by induction. So, \( (t_i^k)_{k=0}^\infty \) and \( (t_i^k)_{k=0}^\infty \) must converge to some \( t_i^* \) and \( t_i^* \), and the sequences \( (a_i^k)_{k=0}^\infty \), \( i = 1, 2, 3 \), must converge to some \( a_i^* \). In addition, \( t_i^{k+1} = t_i^k = t_i^* \), \( \bar{t}_i^{k+1} = \bar{t}_i^k = \bar{t}_i^* \), and \( a_i^{k+1} = a_i^k = a_i^* \), \( i = 1, 2, 3 \), must satisfy (6)–(7). It follows that \( t_1^* = 4/30, t_2^* = 14/30, a_1^* = 9/30, a_2^* = 9/30, \) and \( a_3^* = 22/30 \).

In case (ii), the sequences \( (t_i^k)_{k=0}^\infty \) and \( (t_i^k)_{k=0}^\infty \) for \( i = 1, 2, 3 \) are decreasing, but they also converge to \( t_i^1 = 4/30, t_i^2 = 14/30, a_i^1 = 2/30, a_i^2 = 9/30, \) and \( a_i^3 = 22/30 \). Therefore, \( (t_i^k)_{k=0}^\infty \) and \( (t_i^k)_{k=0}^\infty \) for arbitrary initial conditions \( (t_i^0, t_i^0) \) and \( (a_i^0, a_i^0, a_i^0) \) converge to \( t_i^1 = 4/30, t_i^2 = 14/30, a_i^1 = 2/30, a_i^2 = 9/30, \) and \( a_i^3 = 22/30 \), because they are “sandwiched” between the sequences from case (i) and case (ii). The sequences \( (t_i^k)_{k=0}^\infty \) and \( (t_i^k)_{k=0}^\infty \) may not be monotonic in general. For example, if \( t_1^0 = 0.25 \) and \( t_2^0 = 0.75 \), and \( (a_1^0, a_2^0, a_3^0) \) is the Receiver’s best response to this strategy of the Sender, then \( t_1^0 < t_1^1 \) but \( t_1^1 > t_2^1 > t_1^2 > \cdots \), while \( t_2^0 > t_2^1 > t_2^2 > t_3^2 > \cdots \). Therefore, showing their convergence requires our slightly more subtle argument.

For the general case with possibly type-dependent biases, sequences \( (t_i^k)_{k=0}^\infty \), \( (t_i^k)_{k=0}^\infty \), and \( (a_i^k)_{k=0}^\infty \) defined as in cases (i) and (ii) are monotonic, and it is easy to see that their limits induce equilibrium type-action mappings. Also, the sandwich argument applies. This does not guarantee the convergence of sequences \( (t_i^k)_{k=0}^\infty \), \( (t_i^k)_{k=0}^\infty \), and \( (a_i^k)_{k=0}^\infty \) for arbitrary initial conditions, because the two limit equilibrium type-action mappings: that from case (i) and that from case (ii), may not coincide.\(^{10}\) However, if the number of available messages is \( N^\ast \) or higher, and there is only one equilibrium type-action mapping that satisfies NITS (e.g., (RC) is satisfied),\(^ {11}\) then these limit mappings must coincide, and so \( (t_i^k)_{k=0}^\infty \), \( (t_i^k)_{k=0}^\infty \), and \( (a_i^k)_{k=0}^\infty \) for arbitrary initial conditions converge to the cutoffs and actions of this equilibrium.\(^ {12,13}\)

### 6.2 Iterated Deletion of Weakly Dominated Strategies

We will now use the uniform-quadratic example to illustrate the proofs of Theorem 2 and Corollary 2. The central idea is that the strategies such that the cutoff \( t_i \) is smaller than \( t_i^k \) or such that the cutoff \( t_i \) is greater than \( t_i^k \) defined in Section 4.4 are weakly dominated and are eventually deleted. So are the strategies such that some actions are smaller than \( a_i^k \) or such that some actions are greater than \( a_i^k \). In the example, all strategies that we delete will be both weakly dominated and interim dominated. For simplicity, we will refer to them as dominated.

The cutoffs \( t_i^0 = t_i^1 = t_i^2 = 0 \) and \( t_i^3 = 1 \) determine a monotonic strategy of the Sender. The strategy that responds with actions \( a_i^1 = a_i^2 = 0 \) and \( a_i^3 = 0.5 \) is a best response of the Receiver to this strategy of the Sender. Any strategy such that \( a_3 < 0.5 \) (i.e., such

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\(^{10}\)Olszewski [27] shows that \( (t_i^k)_{k=0}^\infty \), \( (t_i^k)_{k=0}^\infty \), and \( (a_i^k)_{k=0}^\infty \) actually converge for an arbitrary set of initial conditions. However, the initial conditions may affect the limit equilibrium type-action mapping.

\(^{11}\)These conditions are satisfied in the present example.

\(^{12}\)In Appendix B, we give an example in which two equilibria satisfy NITS, and our result no longer holds true.

\(^{13}\)By Proposition 1 in Chen et al. (2008), all equilibria with \( N^\ast \)-interval partition satisfy NITS, but there may exist more than one such equilibrium.
that \(a_3 < \overline{a_3}^1\) is weakly dominated by the strategy \((a_1, a_2, \overline{a_3}^1)\). This follows because no matter what the strategy \((t_0, t_1, t_2, t_3)\) of the Sender, the Receiver weakly prefers playing action 0.5 to playing any action \(a < 0.5\) in response to message \(m_3\) (i.e., the message sent by the types from \((t_2, t_3) = (t_2, 1)\)), and she strictly prefers playing 0.5 to any \(a < 0.5\) if \(t_2 < 1\). Denote the set of dominated strategies described in this paragraph as \(\overline{D}^R_1\).

The cutoffs \(\tau_1^0 = 0\) and \(\tau_1^1 = \tau_2^0 = \tau_3^0 = 1\) also determine a monotonic strategy of the Sender. The strategy that responds with actions \(\overline{a}_1^1 = 0.5\) and \(\overline{a}_3^1 = 1\) is a best response of the Receiver to this strategy of the Sender. Any strategy such that \(a_1 > 0.5\) (i.e., such that \(a_1 > \overline{a}_1^1\)) is weakly dominated by the strategy \((\overline{a}_1^1, a_2, a_3)\). Indeed, no matter what the strategy \((t_0, t_1, t_2, t_3)\) of the Sender, the Receiver weakly prefers playing action 0.5 to playing any action \(a > 0.5\) in response to message \(m_1\), and she strictly prefers playing 0.5 to any \(a > 0.5\) if \(t_1 > 0\). Denote the set of dominated strategies described in this paragraph as \(\overline{D}^R_1\). Note that there may exist other strategies of the Receiver (i.e., not belonging to \(\overline{D}^R_1 \cup \overline{D}^R_1\)) that are weakly dominated, about which we make no claim.\(^{14}\)

At this point, we have illustrated how IDWDS may remove strategies of the Receiver. We have established a non-trivial lower bound \((a_1^1, a_2^1, a_3^1)\) and a non-trivial upper bound \((\overline{a}_1^1, \overline{a}_2^1, \overline{a}_3^1)\) on the retained strategies of the Receiver. It is plausible to conjecture that iterating the process will eliminate more strategies. In fact, we use the fact that we have deleted some of the Receiver’s strategies to impose non-trivial lower and upper bounds on the Sender’s strategies. Specifically, we can delete (as weakly dominated) all strategies of the Sender with some coordinate lower than the corresponding coordinate of \((t_0^3, t_1^3, t_2^3, t_3^3)\) or some coordinate higher than the corresponding coordinate of \((\tau_0^2, \tau_1^2, \tau_2^2, \tau_3^2)\). That is, we can delete any strategy of the Sender with a coordinate that is lower (respectively, higher) that the corresponding coordinate of the best response to the lower (respectively, upper) bound on the Receiver’s strategies. These bounds on the Sender’s strategies make tighter bounds on the Receiver’s strategies. We can delete (as weakly dominated) all strategies of the Receiver with a coordinate that specifies an action less than the best response to \((t_0^3, t_1^3, t_2^3, t_3^3)\) or with a coordinate that specifies an action greater than the best response to \((\tau_0^2, \tau_1^2, \tau_2^2, \tau_3^2)\). And we can continue in this fashion to obtain tighter and tighter bounds.

Formally, \(t_1^2 = 0\) and \(t_2^2 = 0.2\) (with \(t_0^2 = 0\) and \(t_3^2 = 1\)) is a best response of the Sender to \(a_1^1 = a_2^1 = 0\) and \(a_3^1 = 0.5\), and if the Receiver plays only strategies \((a_1, a_2, a_3)\) from the complement of \(\overline{D}^R_2\), any strategy \((t_0, t_1, t_2, t_3)\) such that \(t_2 < t_2^2\) is weakly dominated by the strategy \((t_0, t_1, t_2^2, t_3)\). This follows because types \(t < 0.2\) weakly prefer actions \(a_2\) to \(a_3 \geq \max\{0.5, a_2\}\), and strictly so if \(a_2 < 0.5\). Thus, types \(t < 0.2\) will never induce action \(a_3\). Denote the set of dominated strategies described in this paragraph as \(\overline{D}^S_2\).

Similarly, \(t_1^2 = 0.7\) and \(t_2^2 = 1\) is a best response of the Sender to \(\overline{a}_1^2 = 0.5\) and \(\overline{a}_2^2 = \overline{a}_3^2 = 1\), and if the Receiver plays only strategies \((a_1, a_2, a_3)\) from the complement of \(\overline{D}^R_2\), any strategy \((t_0, t_1^2, t_2, t_3)\) such that \(t_1 > t_1^2\) is weakly dominated by the strategy \((t_0, t_1^2, t_2, t_3)\). Denote the set of dominated strategies described in this paragraph as \(\overline{D}^S_3\). Again, there are other weakly dominated strategies of the Sender (which are not in

\(^{14}\)Indeed, it can be checked that \(a_1 = a_2 = 0, a_3 = 1\) is dominated by \(a_1 = a_2 = 0.1, a_3 = 0.9\). However, the argument is not as obvious as for the strategies such that \(a_3 < 0.5\) or \(a_1 > 0.5\).
Hence we do not delete each equilibrium action for $k = \infty$ (and if the Receiver plays only strategies analogous to that for restricted to playing strategies such that by the strategy Section 4.4.

At stage $k$ of the process, (i) every strategy $(a_1, a_2, a_3)$ such that $a_i < a_i^k$ for at least one $i$ is weakly dominated by the strategy $(\max\{a_1, a_1^k\}, \max\{a_2, a_2^k\}, \max\{a_3, a_3^k\})$, provided that the Sender is restricted to playing strategies such that $t_i \geq t_i^{k-1}$ for all $i$. Similarly, (ii) every strategy $(a_1, a_2, a_3)$ such that $a_i > a_i^k$ for at least one $i$ is weakly dominated by the strategy $(\min\{a_1, a_i^k\}, \min\{a_2, a_i^k\}, \min\{a_3, a_i^k\})$, provided that the Sender is restricted to playing strategies such that $t_i \leq t_i^{k-1}$ for all $i$. The inductive argument is analogous to that for $k = 1$. Denote the set of strategies described in (i) and (ii) as $D^R_k$ and $D^R_k$, respectively. Note that $D^R_{k-2} \subset D^R_k$ and $D^R_{k-2} \subset D^R_k$, because sequences $(a_i^k)_{k=1}^\infty$, $i = 1, 2, 3$, are increasing, and sequences $(\bar{a}_i^k)_{k=1}^\infty$, $i = 1, 2, 3$, are decreasing.

The strategy of the Sender given by $t_1^k$ and $t_2^k$ is a best response of the Sender to $(a_1^{k-1}, a_2^{k-1}, a_3^{k-1})$, and if the Receiver plays only strategies $(a_1, a_2, a_3)$ from the complement of $D_{k-1}^R$, every strategy $(t_0, t_1, t_2, t_3)$ such that $t_i < t_i^k$ for at least one $i$ is weakly dominated by the strategy given by $(t_0, \max\{t_1, t_1^k\}, \max\{t_2, t_2^k\}, t_3)$. Similarly, the strategy of the Sender given by $t_1^k$ and $t_2^k$ is a best response of the Sender to $(a_1^{k-1}, a_2^{k-1}, a_3^{k-1})$, and if the Receiver plays only strategies $(a_1, a_2, a_3)$ from the complement of $D_{k-1}^R$, every strategy $(t_0, t_1, t_2, t_3)$ such that $t_i > t_i^k$ for at least one $i$ is weakly dominated by the strategy $(t_0, \min\{t_1, t_1^k\}, \min\{t_2, t_2^k\}, t_3)$. The inductive argument is analogous to that for $k = 2$. Denote the set of dominated strategies described in this paragraph as $D^S_k$ and $D^S_k$. Again, $D^S_{k-2} \subset D^S_k$ and $D^S_{k-2} \subset D^S_k$ by the monotonicity of sequences $(t_i^k)_{k=1}^\infty$ and $(\bar{t}_i^k)_{k=1}^\infty$.

Because $(t_i^k)_{k=0}^\infty$ and $(\bar{t}_i^k)_{k=0}^\infty$ converge to $t_i^*$, and $(a_i^k)_{k=0}^\infty$ and $(\bar{a}_i^k)_{k=0}^\infty$ converge to $a_i^*$ for $i = 1, 2, 3$, only the largest equilibrium belongs to the complement of the sets $\bigcup_{k \text{ odd}} (D^R_k \cup D^S_k)$ and $\bigcup_{k \text{ even}} (D^R_k \cup D^S_k)$. The largest equilibrium cannot be deleted under this or any other procedure of iterated deletion of weakly dominated strategies, because each equilibrium action $a_i^*$, $i = 1, 2, 3$, is the Receiver’s unique best response to message $m_i$ given the Sender’s equilibrium strategy, and the Sender’s equilibrium strategy is the unique best response to the Receiver’s equilibrium strategy.

The discussion thus far leaves several issues unresolved. What forces lead the one- and two-interval equilibria to be deleted when there are three messages? What happens when there are more than three messages? What happens when there are only two messages? (How) do the arguments depend on the order of deletion of weakly dominated strategies?

Continue to think in terms of the deletion process that we have outlined. Assume that there are more than three messages. Arguments analogous to those for three messages

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15Recall that $t_i^{k+1} = t_i^k$ and $\bar{a}_i^{k+1} = \bar{a}_i^k$ when $k$ is even and $\bar{a}_i^{k+1} = \bar{a}_i^k$ and $a_i^{k+1} = a_i^k$ when $k$ is odd. Hence we do not delete $R$’s strategies when $k$ is even or $S$’s strategies when $k$ is odd.

16Note that $(\max\{a_1, a_1^k\}, \max\{a_2, a_1^k\}, \max\{a_3, a_1^k\}) \notin D^R_{k-2} \cup D^R_k$.

17Note that $(\min\{a_1, a_3^k\}, \min\{a_2, a_3^k\}, \min\{a_3, a_3^k\}) \notin D^R_{k-2} \cup D^R_k$. 

18Note that $(\max\{a_1, a_1^k\}, \max\{a_2, a_1^k\}, \max\{a_3, a_1^k\}) \notin D^R_{k-2} \cup D^R_k$.
imply that the only type-action mapping left under our IDWDS procedure would have $a_i^* = 0$ and $t_i^* = 0$ except the three highest $i$’s. No type of the Sender would choose a message that induces action 0, because the three-interval equilibrium satisfies the NITS condition. In contrast, the one- and two-interval equilibria do not satisfy NITS. In particular, if a two-interval equilibrium were to survive with three (or more) available messages, then the Receiver’s equilibrium responses would have to be 0, 2/10 and 7/10. Because NITS fails for the two-interval equilibrium, type 0 prefers action 0 to action 0.2. This property is key to our argument. We can show that if there is an unused message, then all lower messages are unused. But then any surviving equilibrium must satisfy NITS. Therefore, provided that there are at least $N^*$ messages, only equilibria that satisfy NITS can survive.

Naturally, the process could not converge to the largest equilibrium if $N < N^*$. In this case, we can show that when the (RC) holds, the only type-action mapping left under our IDWDS procedure would be the (unique) equilibrium type-action mapping with $N$ distinct actions.

We devote the next subsection to a discussion of different orders of deletion.

### 6.3 Different Orders of Deletion of Weakly Dominated Strategies

In Section 6.2, we have focused on a specific procedure of deleting weakly dominated strategies that eliminate the strategies with $a_i > a_i^k$ or $a_i < a_i^k$, and the strategies with $t_i > t_i^k$ or $t_i < t_i^k$. An arbitrary procedure need not eliminate these strategies as soon as they become dominated. If they are not deleted immediately, then they may not be dominated later, and prevent other strategies from being dominated. One potentially possible scenario is that a strategy $x_0$ ($x$ is $s$ or $a$) is dominated by $x_1$ but $x_0$ is not eliminated in round 1. In turn, $x_1$ is dominated by $x^2$ and $x_1$ is eliminated in round 2; $x^2$ is dominated by $x^3$ and $x^2$ is eliminated in round 3; and so on. A problem may arise if, at some stage, $x^0$ is no longer weakly dominated.

Intuitively, we show that if some strategies higher than $(t_0^*, \ldots, t_N^*)$ (or lower than $(t_0^*, \ldots, t_N^*)$) are not eliminated in rounds $k = 1, 2, \ldots$, then the best response to the supremum (the infimum) over them is the highest strategy (the lowest strategy) of the Receiver that may not be eliminated. Due to upward bias of the Sender, this will imply that the supremum (the infimum) is dominated in the game with the limit strategy sets $\hat{S}^*$ and $\hat{A}^*$.

To make this intuition more precise, consider any parameters such that the basic cheap-talk game has only two equilibrium type-action mappings: an uninformative equilibrium in which all types induce the same action, and the one in which the types from $[0, t_1^*)$ induce action $a_1^*$ and the types from $(t_1^*, 1]$ induce action $a_2^*$. For example, the uniform-quadratic case with the bias $b = 0.1$ (instead of 0.05 from Examples 2)\(^{18}\) satisfies our condition: in the uninformative equilibrium action 0.5 is induced by all types, and in the other equilibrium the types from $[0, 0.3)$ induce action 0.15 and the types from $(0.3, 1]$ induce action 0.65. Suppose further that there are only two messages $m_1 < m_2$.

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\(^{18}\)We chose a setting simpler than that in Section 6.2, because the argument is somewhat more involved for $b = 0.05$. 

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So, the Sender’s strategy is fully described by the cutoff $t_1$ such that the types lower than $t_1$ send message $m_1$, and the types higher than $t_1$ send message $m_2$. The Receiver’s strategy can be described by two actions: action $a_i$ responds to message $m_i$ for $i = 1, 2$.

We will argue that an arbitrary IDIWDS procedure eliminates all strategies $t_1$ of the Sender such that $t_1 > t^*_1$. The argument will also imply that all strategies of the Receiver such that $a_1 > a^*_1$ or $a_2 > a^*_2$ must also be eliminated. Let $\tilde{t}_1$ be the greatest strategy of the Sender that survives all rounds of the procedure, more precisely, $\tilde{t}_1 = \sup \tilde{S}^*$. Because the equilibrium strategies are unique interim best responses to each other, none of them can be eliminated. Therefore, $t^*_1 \in \tilde{S}^*$. Thus, $\tilde{S}^* \neq \emptyset$, and $\tilde{t}_1 \geq t^*_1$.

Let $(\tilde{a}_1, \tilde{a}_2)$ be the unique strategy of the Receiver that interim best responds to the strategy $t_1$. Observe that $(\tilde{a}_1, \tilde{a}_2) \in \tilde{A}^*$. Indeed, suppose that some $(a'_1, a'_2)$ weakly interim dominates $(\tilde{a}_1, \tilde{a}_2)$. If $a'_1 \neq \tilde{a}_1$, then $\tilde{a}_1$ is a better action than $a'_1$ in response to $[0, t_1)$ for any strategy $t_1 \in \tilde{S}^*$ close enough to $\tilde{t}_1$. So, it must be that $a'_1 = \tilde{a}_1$. Similarly, $a'_2 \neq \tilde{a}_2$, then $\tilde{a}_2$ is a better action than $a'_2$ in response to $(t_1, 1]$ for any strategy $t_1 \in \tilde{S}^*$ close enough to $\tilde{t}_1$. So, it must also be that $a'_2 = \tilde{a}_2$, but then $(a'_1, a'_2)$ cannot weakly interim dominate $(\tilde{a}_1, \tilde{a}_2)$. Thus, $(\tilde{a}_1, \tilde{a}_2) \in \tilde{A}^*$.

We will now show that any strategy $(a_1, a_2)$ such that $a_1 > \tilde{a}_1$ or $a_2 > \tilde{a}_2$ must be eliminated in some round of the procedure, that is,

$$(a_1, a_2) \in \tilde{A}^* \text{ implies } a_1 \leq \tilde{a}_1 \text{ and } a_2 \leq \tilde{a}_2.$$  \hfill (8)

We establish (8) by studying three cases.

**Case 1.**

$$a_1 \geq \tilde{a}_1, \ a_2 \geq \tilde{a}_2 \text{ and } (a_1, a_2) \neq (\tilde{a}_1, \tilde{a}_2).$$  \hfill (9)

If (9) holds and $(a_1, a_2) \in \tilde{A}^*$, then $(\tilde{a}_1, \tilde{a}_2)$ weakly interim dominates $(a_1, a_2)$ given that the sender is allowed to play only the strategies from $\tilde{S}^*$ because $\tilde{t}_1 = \sup \tilde{S}^*$. Because $(\tilde{a}_1, \tilde{a}_2) \in \tilde{A}^*$ and weakly interim dominates $(a_1, a_2)$, we obtain a contradiction with Condition 5 from the definition of IDIWDS procedures.

**Case 2.**

$$a_1 < \tilde{a}_1 \text{ and } a_2 > \tilde{a}_2.$$  \hfill (10)

We consider two subcases:

(i) The best response to $[0, \inf \tilde{S}^*]$ is no greater than $a_1$.

In this case, we claim that the strategy $(a_1, \tilde{a}_2)$ belongs to $\tilde{A}^*$ and weakly dominates $(a_1, a_2)$. First we show $(a_1, \tilde{a}_2) \in \tilde{A}^*$. Any strategy that would weakly interim dominate it must have $\tilde{a}_2$ as the second action, because $\tilde{a}_2$ is the unique best response to $\tilde{t}_1 = \sup \tilde{S}^*$. If a putative dominating strategy had the first action smaller than $a_1$, it would do strictly worse than $a_1$ against $[0, t_1]$ for any $t_1 \in \tilde{S}^*$ that is close enough to $\tilde{t}_1$, because $a_1 < \tilde{a}_1$. If a putative dominating strategy had the first action greater than $a_1$, it would do strictly worse than $a_1$ against $[0, t_1]$ for any $t_1 \in \tilde{S}^*$ that is close enough to $\inf \tilde{S}^*$. So, $(a_1, \tilde{a}_2) \in \tilde{A}^*$.
Second, \((a_1, \tilde{a}_2)\) weakly interim dominates \((a_1, a_2)\) given that the Sender plays only the strategies from \(\tilde{S}^*\), because \(a_2 > \tilde{a}_2\). This contradicts Condition 5 from the definition of IDIWDS procedures.

(iii) The best response to \([0, \inf \tilde{S}^*]\) is strictly greater than \(a_1\).

In this case, consider the strategy \((a'_1, \tilde{a}_2)\), where \(a'_1\) is the (unique) best response to \([0, \inf \tilde{S}^*]\). This strategy belongs to \(\tilde{A}^*\), because \(a'_1\) (uniquely) best responds to \([0, \inf \tilde{S}^*]\), and \(\tilde{a}_2\) (uniquely) best responds to \([\sup \tilde{S}^*, 1]\). And \((a'_1, \tilde{a}_2)\) weakly interim dominates \((a_1, a_2)\) given that the sender plays only the strategies from \(\tilde{S}^*\), because \(a'_1 > a_1\) in this case. This contradicts Condition 5 from the definition of IDIWDS procedures.

\[ a_1 > \tilde{a}_1 \text{ and } a_2 < \tilde{a}_2. \quad (11) \]

Analogous arguments to Case 2 exclude (11).

Consequently, (8) has now been established.

Suppose that \(\tilde{t}_1 > t'_1\). Let \(t_1\) be the best response of the Sender to \((\tilde{a}_1, \tilde{a}_2)\). Because \(t'_1 < \tilde{t}_1\), \(\tilde{t}_1\) cannot be an equilibrium cutoff. That is, \(\tilde{t}_1\) cannot be the best response to \((\tilde{a}_1, \tilde{a}_2)\). Therefore, because of the upward bias, it must be that \(t_1 < \tilde{t}_1\). Observe that \(t_1 \in \tilde{S}^*\). Indeed, it cannot be weakly interim dominated, because \(t_1\) is the best response to \((\tilde{a}_1, \tilde{a}_2)\) that belong to \(\tilde{A}^*\). Finally, \(t_1\) weakly interim dominates \(\tilde{t}_1\) given that the Receiver plays only the strategies from \(\tilde{A}^*\), because \(t_1 < \tilde{t}_1\) is a best response of the Sender to \((\tilde{a}_1, \tilde{a}_2)\), and \(a_1 \leq \tilde{a}_1\) and \(a_2 \leq \tilde{a}_2\) for all strategies \((a_1, a_2) \in \tilde{A}^*\), a contradiction with Condition 5 from the definition of IDIWDS procedures, which requires that \(\tilde{t}_1\) is not dominated.

Thus, \(\tilde{t}_1 = t'_1\) and \((\tilde{a}_1, \tilde{a}_2) = (a'_1, a'_2)\). Analogous arguments show that an arbitrary IDIWDS procedure eliminates all strategies \(t_1\) of the Sender such that \(t_1 < t'_1\), as well as all strategies of the Receiver such that \(a_1 < a'_1\) or \(a_2 < a'_2\).

Remark. The presented arguments also show that in this special case only the largest type-action mapping survives any IDWDS procedure.

7 Other Biases

Thus far we have maintained the assumption that \(a^S(\theta) > a^R(\theta)\) for all \(\theta\). Our analysis contains the elements needed to handle other cases. In particular, the construction of bounding sequences does not depend on any assumption about bias. The case in which \(a^S(\theta) < a^R(\theta)\) for all \(\theta\) is completely symmetric to the case of upward bias. In some other cases, when the bias of some types is upward and the bias of some other types is downward, there exist equilibria with \(K\) distinct actions induced for all positive integers \(K\) (see Gordon (2010)). Consequently the size \(N\) of the message space will constrain the set of equilibria. When the bias is outward \((a^S(1) > a^R(1)\) and \(a^S(0) < a^R(0)\)), there is
a procedure of iterated deletion of weakly dominated strategies in which all messages are used with positive probability. Hence the outcomes of the monotonic cheap-talk game that survive IDWDS exhibit effective communication in the sense that the uninformative outcome is eliminated.

8 Discussion

The literature contains different theoretical arguments that suggest why, under upward bias, the equilibrium with $N^*$ actions is salient. Under their regularity condition, CS demonstrate that there is an essentially unique equilibrium type-action mapping with $N^*$ actions and that, under some conditions, this equilibrium is ex ante preferred to all other equilibria by both the Sender and the Receiver.

Mensch [24] notes that monotonicity restrictions in cheap-talk games can lead to the kind of selection that we describe. Mensch imposes a monotonicity condition on off-path beliefs and argues that this leads to a selection of equilibria that satisfy NITS.

Milgrom and Roberts [25] and Vives [31] study the class of supermodular games introduced by Topkis [29]. In a supermodular game, each player’s strategy set is partially ordered and there are strategic complementarities that cause a player’s best response to be increasing in opponents’ strategies. Milgrom and Roberts [25] demonstrate that supermodular games have a largest and smallest equilibrium and that these extreme equilibria can be obtained by iterating the best-response correspondence. Our argument uses similar techniques. There are two differences. Our game is not a supermodular game. In particular, it does not satisfy the increasing difference condition of Milgrom and Roberts. In addition, Milgrom and Roberts study the implications of deletion of strictly dominated strategies. Our analysis uses weak dominance. Sobel [28] shows how Milgrom and Roberts’s general arguments extend to a broader class of games and a more restrictive solution concept. He points out that the monotonic cheap-talk game satisfies a weak form of supermodularity that makes it possible to bound the set of strategies that survive deletion of weakly dominated strategies using arguments similar to ours. Sobel does not provide conditions under which the process leads to a unique prediction.

Words have commonly accepted meanings. When there are no conflicts of interest, it is natural to assume that agents will use words in conventional ways. In strategic situations, however, sophisticated agents will not take words at face value. Standard models of cheap talk abstract from the conventional meaning of words in order to focus on strategic problems. A limitation of this approach is that meaning is determined completely endogenously. An equilibrium type-action mapping determines the minimum number of distinct messages that the Sender must use, but does not specify which message is associated with which action. If there is to be a connection between the equilibrium use of messages and exogenous meaning, then we must impose additional assumptions. The literature has approached this issue in several ways.

Farrell [12] introduced the first attempt to refine the equilibrium set in cheap-talk games. Farrell’s notion of neologism-proof equilibrium models the idea that messages have commonly accepted meanings and that players are able to use these statements provided that they were consistent with strategy constraints. This general idea does
refine the set of equilibria in cheap-talk games, but lacks general existence properties.\textsuperscript{19}

Chen [6] and Kartik [17] assume that the message space is equal to the type space, which suggests a natural correspondence between types and messages. They make this connection operational by modifying the game. Chen assumes that with positive probability the Sender sends a message equal to her type (and with positive probability the Receiver interprets the message literally). Kartik assumes that the Sender has a cost of “lying.” These perturbations create an exogenous meaning for messages. In these models, the limits of equilibria in monotonic strategies as the perturbations vanish converge to an equilibrium that satisfies NITS.\textsuperscript{20} Furthermore, the limit equilibrium involves the use of “inflated” messages. Hence these arguments are alternative ways to make the same selection that we make. Our result imposes the monotonicity condition directly on the game and makes a selection without perturbations.

Dilmé [10] also provides an argument that selects equilibrium outcomes with communication. Dilmé studies cheap-talk games in which payoffs are perturbed. He then looks for equilibria of the underlying game that are robust, where a robust equilibrium is close to some equilibrium in every nearby game. He shows that in games with a upward bias satisfying the standard regularity condition, only the equilibrium with the maximal number of actions induced is robust. He extends this result to more general cheap-talk games. Dilmé’s selection generally coincides with the outcomes we select.\textsuperscript{21} His approach has a superficial similarity to Chen, Kartik, and Sobel [7, Section 4.4], in that both operate by perturbing payoffs. Chen, Kartik, and Sobel study a particular kind of signaling cost introduced in Kartik [17] and impose an equilibrium refinement (restriction to monotonic strategies), while Dilmé uses the freedom to specify signaling costs to attain a selection result. Dilmé’s approach is also related to solution concepts like strategic stability (Kohlberg and Mertens [19]) or truly perfect equilibria (Van Damme [30]) that require robustness with respect to a large family of perturbations. In addition to reaching similar conclusions, the source of Dilmé’s results is similar to ours. Both approaches exploit the fact that there are a limited number of specifications of off-path behavior that are consistent with equilibrium. For example, in a cheap-talk model in which the Sender is upward biased, equilibrium requires that off-path actions either agree with on-path actions or are strictly lower than the lowest on-path action. Furthermore, when a regularity condition holds, only the equilibrium with the maximal number of actions induced can be supported using low off-path responses. Dilmé’s argument, like ours, operates by showing that some messages must lead to low off-path actions.

In the context of CS games where the sender has an upward bias, Gordon [14] studies a selection from the composed best response dynamics, defined directly on the set of interval partitions outcomes with any finite number of intervals, without keeping track of messages. Under regularity conditions, the only equilibrium that is stable for this dynamics is the one with the maximal number of intervals. Theorem 1 and Corollary 1 generalize this result by showing that convergence is in fact global. Gordon [14] also obtains results on local stability of equilibria for other biases.

\textsuperscript{19}In particular, typically no equilibrium is neologism-proof in the uniform-quadratic special case of the CS model.

\textsuperscript{20}Chen, Kartik, and Sobel [7] introduce the NITS criterion, which we described in Section 2.

\textsuperscript{21}Dilmé does not resolve message indeterminacy.
Antić and Persico [1] study a game in which the players make a costly investment that can alter ideal points prior to playing a cheap-talk game. They study equilibria of the two-stage game that satisfy a forward-induction refinement. A fixed cheap-talk game can be viewed as a two-stage game in which players face infinite costs associated with changing their biases. Antić and Persico identify conditions on the underlying cheap-talk game and the investment-cost function that imply that only an outcome that satisfies NITS is the limit of refined equilibria of the two-stage game as the investment costs grow to infinity. This argument selects the same type-action mapping as Chen [6] and Kartik [17] by examining limits of equilibria, but the logic of the arguments appears to be different. Chen and Kartik perturb payoffs, while Antić and Persico perturb strategy spaces. Furthermore, Chen and Kartik’s selection, like ours, resolves the message-indeterminacy problem while Antić and Persico do not.

Clark and Fudenberg [8] introduce an equilibrium refinement (justified communication equilibrium) for signaling games with both cheap and costly signals. Justified communication equilibria are stable outcomes of learning processes. Assuming that Receivers trust cheap-talk messages initially, they show that these messages must satisfy off-path credibility conditions in stable outcomes. They provide conditions under which cheap-talk messages influence equilibrium outcomes in interesting classes of signaling games. Justified communication equilibria coincide with perfect Bayesian equilibria in cheap-talk games.

Blume [2] and [3] propose refinements for finite cheap-talk games based on Kalai and Samet’s [16] concept of persistent equilibrium. Blume [2] introduces perturbations to Sender strategies. The perturbations guarantee that there are no off-path messages. Unused messages cannot be interpreted arbitrarily. Rather, they take on exogenous meanings (messages are associated with subsets of types) independent of the strategic context. Blume [3] demonstrates that these perturbations to the Sender’s messages determine the relationship between types and messages in the equilibria selected by his refinement. These perturbations, like the initial conditions in our dynamic arguments, solve the message-indeterminacy problem.

Blume [4] introduces a concept of “language equilibrium” in cheap-talk games. He takes as given a distinguished Receiver strategy, called a pre-existing language. Given a cheap-talk game and a pre-existing language, he constructs games that use a subset of the strategy set of the original game by iterating best replies to the pre-existing language and then judiciously adding certain strategies to the resulting limit. He calls equilibria of these games language equilibria and establishes their existence. Similar to our approach, language equilibria resolve message indeterminacy. Blume also shows that in finite versions of the CS setting, language equilibria feature language inflation.

Olszewski [26] investigates the stability of equilibria in cheap-talk game with respect to the introduction of new messages and shows through examples that this idea destabilizes “implausible” equilibria. The initial conditions of our adaptive processes act like new messages do in Olszewski’s paper. Hence the approaches share the feature of investigating conditions under which the introduction of novel interpretations of messages (either through the addition of new message that the Receiver interprets randomly as in Olszewski or a rich initial condition that the Receiver responds to optimally as in our paper) and adaptive dynamics can select equilibria.
Lo [22] imposes restrictions on the set of strategies available to agents in a discrete cheap-talk game and then studies the outcomes that survive deletion of weakly dominated strategies. Like Lo, we impose restrictions on strategies and study the implications of IDWDS. Our results differ from hers because we impose only the restrictions that messages are linearly ordered, that higher sender types send weakly higher signals and that the receiver takes weakly higher actions for higher signals. These restrictions do not eliminate any equilibrium outcomes of the original game. Lo makes further restrictions on the strategy space and shows that these can actually lead to outcomes that are not equilibria of the original game.

There are several criticisms of IDWDS. It is well known that, unlike deletion of strongly dominated strategies, the order of deletion may matter. In some games with large strategy spaces, equilibrium may fail to exist in weakly undominated strategies. The process of eliminating weakly dominated strategies may introduce new equilibria. It also leads to strong predictions that are not behaviorally accurate in common settings (like the centipede game).

Some of the technical problems with IDWDS may hold in our setting, but we obtain a selection result independent of the order of deleting strategies that are interim weakly dominated.

Appendix A (Proofs)

**Corollary 1.** Assume \( N \geq N^* \). If there exists a unique equilibrium type-action mapping that satisfies NITS, then any interim best-response sequences converge to an equilibrium with this type-action mapping.

**Proof.** We first state a property of the limit equilibria \( t^* = (t^*_0, \ldots, t^*_N) \), \( a^* = (a^*_1, \ldots, a^*_N) \), and \( \bar{t}^* = (\bar{t}^*_0, \ldots, \bar{t}^*_N) \), \( \bar{a}^* = (\bar{a}^*_1, \ldots, \bar{a}^*_N) \) that were defined in Section 4.4.

**Claim 1.** No two messages induce the same action \( a > 0 \) in equilibrium \( (t^*, a^*) \), and no two messages induce the same action \( a > 0 \) in equilibrium \( (\bar{t}^*, \bar{a}^*) \).

**Proof.** If a strictly positive action was induced by two messages, then for sufficiently large \( k \) the intervals of the Sender’s types sending each of these messages would have to be arbitrarily short (possibly degenerate). However, for sufficiently large \( k \) any message that induces a strictly positive action is used by an interval \( I \) of types of length bounded away from zero. Otherwise, if \( I \) was arbitrarily short, upward bias implies that some types of the Sender placed to the left of \( I \) and bounded away from \( I \) would prefer the action induced by \( I \) to the action induced by the interval to which they belong. \( \square \)

When \( N \geq N^* \), Claim 1 implies that the equilibria \( t^* = (t^*_0, \ldots, t^*_N) \), \( a^* = (a^*_1, \ldots, a^*_N) \), and \( \bar{t}^* = (\bar{t}^*_0, \ldots, \bar{t}^*_N) \), \( \bar{a}^* = (\bar{a}^*_1, \ldots, \bar{a}^*_N) \) must satisfy NITS. This is so because either some

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22 Lo [23] applies similar arguments to study cheap-talk extensions of games with complete information.

23 A simple example is a first-price auction in which two surplus-maximizing agents bid for an item with known, common value. The only equilibrium of the game involves both players bidding the common value, but this strategy is weakly dominated by bidding less.

24 If \( N > N^* \), then there exist multiple messages that induce action \( a = 0 \). No message can induce the action \( a = 1 \) because we assume an upward bias.
messages induce action zero, or all messages induce positive actions. In the former case, NITS is satisfied because the Sender has the option of inducing action zero, but no type of the Sender chooses this option. In the latter case, \( N = N^* \) and NITS is satisfied by Proposition 1 in Chen, Kartik, and Sobel [7].

Thus, if there is a unique equilibrium that satisfies NITS, then \( t^* = \overline{t}^* \) and \( a^* = \overline{a}^* \). This yields Corollary 1 by Theorem 1, because the sequences of best responses \((t^k_i)_{k=0}^\infty\) and \((a^k_i)_{k=0}^\infty\) starting from any initial conditions \((t^0_i, \ldots, t^0_N)\) and \((a^0_i, \ldots, a^0_N)\) are sandwiched between \((\overline{t}^k_i)_{k=0}^\infty\) and \((\overline{t}^k_i)_{k=0}^\infty\), and \((\overline{a}^k_i)_{k=0}^\infty\) and \((\overline{a}^k_i)_{k=0}^\infty\), respectively. □

We note a consequence of Claim 1 that we use in the proof of Corollary 2.

Claim 2. If \( i \leq N - N^* \), then \( t^*_i = \overline{t}^*_i = 0 \).

Proof. If R plays strategy \( a \) in an equilibrium that satisfies NITS, it must be the case that either \( a_i \) is equal to an action that is induced with positive probability or is strictly less than all actions induced with positive probability. We know that \( \overline{a}^* \) and \( a^* \) are actions from equilibria that satisfy NITS. Furthermore, no two messages induce the same positive action in equilibrium. Also \( N^* \) actions are induced. The remaining messages must therefore induce actions lower than the action induced with positive probability. By monotonicity, the lowest \( N - N^* \) actions are induced with probability zero. □

Theorem 2. There exists a procedure of iterated deletion of interim weakly dominated strategies such that the sets \( \overline{S}^* \) and \( \overline{A}^* \) consist of the strategies \((t_0, \ldots, t_N)\) and \((a_1, \ldots, a_N)\) such that

\[
    t^*_i \leq t_i \leq \overline{t}^*_i \text{ for } i = 0, \ldots, N \text{ and } a^*_i \leq a_i \leq \overline{a}^*_i \text{ for } i = 1, \ldots, N. \tag{5}
\]

Proof. We will first describe a procedure that in every round eliminates interim weakly dominated strategies and retains only the strategies between the equilibrium bounds defined in Section 4.4. The procedure will eliminate all strategies other than the strategies such that:

\[
    t_i \leq \overline{t}^*_i \text{ and } a_i \leq \overline{a}^*_i, \ i = 1, \ldots, N. \tag{12}
\]

An analogous procedure eliminates all strategies other than the strategies such that:

\[
    t_i \geq t^*_i \text{ and } a_i \geq a^*_i, \ i = 1, \ldots, N. \tag{13}
\]

By applying the two procedures simultaneously, we obtain a procedure that retains only the strategies between the equilibrium bounds.

This procedure will eliminate some strategies that are not weakly dominated. We will point out the place in which strategies that are not weakly dominated are eliminated, and then we will describe a more involved procedure that in every round eliminates weakly dominated and interim weakly dominated strategies, and retains only the strategies between the equilibrium bounds defined in Section 4.4.

Lemma 1. No strategy of the Sender that satisfies the first parts of (12) and (13), and no strategy of the Receiver that satisfies the second parts of (12) and (13) can be interim dominated when the opponent can use all strategies satisfying these two conditions.
This will show that the strategies satisfying (12) and (13) are the only elements of sets $S^*$ and $A^*$ for our IDIWDS procedure.

Proof. We will prove Lemma 1 for the Sender’s strategies; the argument is analogous for the Receiver’s strategies. Take any $(t_0, \ldots, t_N)$ that satisfies the first parts of (12) and (13). If $t_i^* > 0$ for some $i$, then $t_i$ is indifferent between actions $a_i < a_{i+1}$ for some $(a_1, \ldots, a_N)$ that satisfies the second parts of (12) and (13). Indeed, $t_i^* \leq t_i$ weakly prefers $a_{i+1}^*$ to $a_i^*$, and $t_i^*$ is indifferent between $a_i^*$ and $\bar{a}_{i+1}^*$. By increasing differences, this implies that $t_i$ weakly prefers $a_{i+1}^*$ to $a_i^*$ and $\bar{a}_i^*$ to $\bar{a}_{i+1}^*$. Thus, by the intermediate value theorem, there exist a convex combination of $(a_i^*, a_{i+1}^*)$ and $(\bar{a}_i^*, \bar{a}_{i+1}^*)$, denoted by $(\tilde{a}_i, a_{i+1})$ such that $t_i$ is indifferent between $a_i$ and $a_{i+1}$. Thus, if $(t_0^*, \ldots, t_N^*)$ weakly interm dominates $(t_0, \ldots, t_N)$, then $t_i' = t_i$ for any $i$ such that $t_i^* > 0$. If $t_i^* = 0$, then $t_i' = t_i = 0$ as well for any $(t_0^*, \ldots, t_N^*)$ that satisfies the first parts of (12) and (13). Thus $t_i' = t_i$ for all $i$, and $(t_0^*, \ldots, t_N^*)$ satisfying these two conditions cannot weakly interm dominate $(t_0, \ldots, t_N)$. □

The procedure that achieves (12) deletes in round $k + 1$ the strategies of the Receiver such that $a_i > \bar{a}_i^*$ for some $i = 1, \ldots, N$, and deletes the strategies of the Sender such that $t_i > t_i^*$ for some $i = 1, \ldots, N$. This procedure retains only the strategies satisfying (12), because $t_i^* \rightarrow_k t_i^*$ and $\bar{a}_i^* \rightarrow_k \bar{a}_i^*$ for all $i$. It remains to show that in each round we eliminate weakly dominated and interim weakly dominated strategies, provided that players can use in that round only the strategies not eliminated in the previous rounds. We will show this by induction.

For $k = 1$, it can be that $a_i > \bar{a}_i^*$ only when $i = 1$, because $\bar{a}_i^* = 1$ for $i > 1$. Any strategy $(a_1, a_2, \ldots, a_N)$ with $a_1 > \bar{a}_1^*$ is weakly dominated and interim weakly dominated by the strategy $(\bar{a}_1^*, a_2, \ldots, a_N)$. Indeed, the comparison of the two strategies reduces to the payoff generated by the lowest action against the lowest interval $[0, t_1]$ of the Sender’s strategy $(t_0, t_1, \ldots, t_N)$. This payoff is strictly greater for the latter strategy than for the former strategy when $t_1 > 0$, because the Receiver’s best response to $[0, t_1]$ is always no higher than $\bar{a}_1^*$, which is the Receiver’s best response to $[0, 1]$. When $t_1 = 0$, the payoffs are equal.

No strategy of the Sender is deleted for $k = 1$. This completes the first inductive step. However, to present our arguments for the Sender’s strategies in the simplest non-trivial case, consider $k = 2$. It can be that $t_i > t_i^2$ only when $i = 1$, because $t_i^2 = 1$ for $i > 1$. The comparison of strategies $(t_0, t_1, \ldots, t_N)$ and $(t_0, t_1^2, \ldots, t_N)$ reduces to comparing which cutoff $t_1$ or $t_1^2$ is better against the Receiver’s strategies $(a_1, a_2, \ldots, a_N)$ not eliminated in the first round. Because $a_1 \leq \bar{a}_1^*$ and type $t_1^2$ is by definition indifferent between $\bar{a}_1^*$ and 1, cutoff $t_1 > t_1^2$ is always worse than cutoff $t_1^2$.

The inductive steps are similar. Consider a $k \geq 1$, and a strategy $(a_1, \ldots, a_N)$ such that $a_i \leq \bar{a}_i^k$ for all $i$ and $a_i > \bar{a}_i^{k+1}$ for some $i$. Because $\bar{a}_i^{k+1} = 1$ for $i > k + 1$, $a_i > \bar{a}_i^{k+1}$ implies $i \leq k + 1$. Let $i^*$ be the lowest index for which $a_{i^*} > \bar{a}_{i^*}^{k+1}$. Define a strategy $(b_1, \ldots, b_N)$ by letting $b_{i^*} = \bar{a}_{i^*}^{k+1}$, and $b_j = a_j$ for $j \neq i^*$. Then, $b_j \leq \bar{a}_j^k$ for all $j$, because $a_j \leq \bar{a}_j^k$ for all $j$ and $\bar{a}_j^{k+1} \leq \bar{a}_j^k$. The strategy $(b_1, \ldots, b_N)$ is monotonic.

\footnote{$t_i^*>0$ is indifferent between $a_i^*$ to $a_{i+1}^*$.}
Indeed, \( b_{i^*-1} \leq b_i \) (when \( i^* > 1 \)), because \( b_{i^*-1} = a_{i^*-1} \leq a_{i^*}^{i^*-1} \) and \( b_i = a_i^{i^*-1} \); \( b_i \leq b_{i^*+1} \), because \( b_i = a_i^{i^*-1} < a_{i^*+1} \), and \( b_j \leq b_{j+1} \) for all other \( j \), because \( a_j \leq a_{j+1} \). The comparison of the Receiver’s payoffs from playing strategies \((a_1, \ldots, a_N)\) and \((b_1, \ldots, b_N)\) reduces to the comparison of the payoffs of actions \( a_i \) and \( a_i^{i^*-1} \) against intervals \([t_{i*+1}, t_i] \) of the Sender’s strategies \((t_0, t_1, \ldots, t_N)\) such that \( t_{i+1} - t_i \) against \( t_i \). Therefore, it is strictly preferred against \((t_0, t_1, \ldots, t_N)\) when \( t_{i+1} - t_i \). It can happen that \( t_{i+1} - t_i \) is equal to \( t_i \) for all strategies \((t_0, t_1, \ldots, t_N)\) such that \( t_i \leq t_{i+1} \). In this case, \( a_i^{i^*-1} \) is strictly interim preferred to \( a_i \), but \( a_i^{i^*-1} \) is not strictly preferred to \( a_i \). This problem requires modifying our procedure. At the end of the proof, we will modify the procedure so that eliminated strategies are both weakly dominated and interim weakly dominated.

Consider now a strategy \((t_0, \ldots, t_N)\) such that \( t_i \leq t_i^k \) for all \( i \) and \( t_i \geq t_i^k+1 \) for some \( i \). Let \( i^* \) be the lowest index for which \( t_i^* > t_i^{k+1} \). Define a strategy \((s_0, \ldots, s_N)\) by letting \( s_i = t_i^k \) and \( s_j = t_j \) for \( j < i^* \). Then, \( s_{i^*} \leq t_{i^*} \) for all \( i \), because \( t_j \leq t_j^k \) for all \( j \) and \( t_i^k \). To show the strategy \((s_0, \ldots, s_N)\) is monotonic, observe that 0 = \( s_0 \leq s_1 \) and if \( i^* > 1 \), then \( s_{i*} \leq s_{i-1} \) because \( s_i = t_i^k \leq t_{i+1} = t_{i-1}^k \) and \( s_{i-1} = t_{i+1} \). Furthermore, \( s_i \geq s_i^{k+1} \), because \( s_i = t_i^{k+1} \leq t_i^{k+1} \). Finally \( s_i \leq s_i^{k+1} \) for all other \( ji \), because \( t_{i+1} > t_{i+1} \). The comparison of the Sender’s payoffs from playing strategies \((t_0, \ldots, t_N)\) and \((s_0, \ldots, s_N)\) reduces to the comparison of the payoffs from locating the \( i^* \)-th cutoff at \( t_i^* \) and \( s_i \) against the Receiver’s strategies \((a_1, \ldots, a_N)\) such that \( a_i \leq a_i^k \). Because \( a_i \leq a_i^k \), \( s_i = t_i^{k+1} \) yields a payoff higher than \( t_i^* \). This payoff is strictly higher for \((a_1, \ldots, a_N)\), because (i) \( t_i^{k+1} \) is a best response to \((a_1, \ldots, a_N)\) and (ii) \( a_i^k > 0 \), which implies that \( a_i^k \leq a_i^{k+1} \). To see \( a_i^k > 0 \), observe that \( 0 \leq t_i^{k+1} < t_i^* \leq t_i^{k} \), which implies that \( 0 < t_i^{k+1} = t_i^*-1 \), which in turn implies that \( a_i^{k+1} > 0 \).

We will now describe a modified procedure. Given \( l \geq 0 \), define \((t_0^l(\varepsilon), t_1^l(\varepsilon), \ldots, t_N^l(\varepsilon)) \) and \((a_1^l(\varepsilon), \ldots, a_N^l(\varepsilon)) \), for \( k = 0, 1, \ldots \), as in Section 4.4, except that \( t_i^k(\varepsilon) \neq 0 \) for \( i > 0 \); instead it will be placed within \( \varepsilon \) of 0. This is possible if \( \varepsilon \) is sufficiently small. We construct the sequences inductively.

Let

\[
0 = t_0^0(\varepsilon) < t_1^0(\varepsilon) = \cdots = t_N^0(\varepsilon) = 1 \quad \text{and} \quad a_1^0(\varepsilon) = \cdots = a_N^0(\varepsilon) = 1.
\]

Let \((a_1^1(\varepsilon), \ldots, a_N^1(\varepsilon)) \) be the interim best response of the Receiver to the strategy \((t_0^l(\varepsilon), t_1^l(\varepsilon), \ldots, t_N^l(\varepsilon)) \) of the Sender.

Define \( j^k \) to be \( N \) if \( u^s(\bar{a}_j^{k+1}(\varepsilon), 0) > u^s(a_j^k(\varepsilon), 0) \) for all \( j \) and

\[
\min \{ j : u^s(a_j^{k+1}(\varepsilon), 0) < u^s(a_j^k(\varepsilon), 0) \}
\]

otherwise. Define \( t_0^k(\varepsilon) = 1 \) if \( j^k = N \) and otherwise to be the type indifferent between \( a_j^{k+1}(\varepsilon) \) and \( a_j^k(\varepsilon) \). Hence \( t_0^k(\varepsilon) > 0 \). For \( i = j^k - 1, j^k - 2, \ldots, 1 \), let

\[
t_i^{k+1}(\varepsilon) \in (0, \min \{ \varepsilon, t_i^k(\varepsilon)/2, t_i^{k+1}(\varepsilon) \})
\]
This is possible because \( \varepsilon, \bar{t}_k^*(\varepsilon) / 2 \), and \( \bar{t}_k^{k+1}(\varepsilon) > 0 \). If type 0 prefers \( \bar{a}_{i+1}^k(\varepsilon) \) to \( \bar{a}_i^k(\varepsilon) \) for some \( k \), then this is so for all greater \( k \) when \( \varepsilon \) is small enough. Indeed, either \( \bar{a}_i^{k+1}(\varepsilon) = \bar{a}_i^k(\varepsilon) \) (for odd \( k \)) or \( \bar{a}_i^{k+1}(\varepsilon) < \bar{a}_i^k(\varepsilon) \) for all \( i \). Hence \( \bar{t}_k^* \) is non-decreasing. Consequently, \( \bar{t}_k^k(\varepsilon) \) are strictly positive, strictly increasing in \( i \), decreasing in \( k \), and \( \lim_{k \to \infty} \bar{t}_k^k(\varepsilon) = 0 \) if \( i = \lim_{k \to \infty} \bar{t}_k^k \).

By construction, this modified procedure satisfies condition (12) and avoids the problem described in the first paragraph of the inductive steps for the original procedure. The only argument (for the original procedure) that requires a change concerns why \( s_{i^*} = \bar{t}_i^{k+1}(\varepsilon) \) yields a payoff higher than \( t_i > \bar{t}_i^{k+1}(\varepsilon) \) against all \( (a_1, \ldots, a_N) \) such that \( a_{i^*} \leq \bar{a}_{i^*}^k(\varepsilon) \), and this payoff is strictly higher for some such strategies. If \( u^S(\bar{a}_i^k(\varepsilon), 0) > u^S(\bar{a}_i^{k+1}(\varepsilon), 0) \), the argument requires no change; and if \( u^S(\bar{a}_i^k(\varepsilon), 0) \leq u^S(\bar{a}_i^{k+1}(\varepsilon), 0) \), then \( u^S(\bar{a}_i^k(\varepsilon), \bar{t}_i^{k+1}(\varepsilon)) < u^S(\bar{a}_i^{k+1}(\varepsilon), \bar{t}_i^{k+1}(\varepsilon)) \), and so yields higher payoffs than any \( t_{i^*} > \bar{t}_i^{k+1}(\varepsilon) \).

\[ \Box \]

**Corollary 2.** Assume \( N \geq N^* \). If there exists a unique equilibrium type-action mapping that satisfies NITS, then there is a procedure of iterated deletion of weakly dominated and interim weakly dominated strategies that retains only this type-action mapping. Furthermore, the surviving strategy uses only the highest \( N^* \) messages with positive probability.

**Proof.** The first part of Corollary 2 follows from Theorem 2. The second part follows from Claim 2. \( \Box \)

We will now turn to showing the irrelevance of the order of elimination. We will restrict attention to allowable IDIWDs procedures.

**Theorem 3.** For any procedure of iterated deletion of allowable interim weakly dominated strategies, the sets \( \tilde{S}^* \) and \( \tilde{A}^* \) consist of the strategies that satisfy (5).

**Proof.** Consider an arbitrary IDIWDs procedure. The equilibrium strategies \( (\bar{t}_0^*, \bar{t}_1^*, \ldots, \bar{t}_N^*) \) and \( (\bar{a}_1^*, \ldots, \bar{a}_N^*) \) are never eliminated, because they uniquely interim best respond to each other; similarly the equilibrium strategies \( (\bar{t}_0^*, \bar{t}_1^*, \ldots, \bar{t}_{N+1}^*) \) and \( (\bar{a}_1^*, \ldots, \bar{a}_N^*) \) are never eliminated.

We will show that (12) and (13) hold true for any strategies \( (t_0, t_1, \ldots, t_N) \) and \( (a_1, \ldots, a_N) \) that belong to \( \tilde{S}^* \) and \( \tilde{A}^* \), respectively. Together with Lemma 1 this will complete the proof of Theorem 3. Denote by \( IBR_i^R(S) \) the set of \( i \)-th actions of the Receiver’s interim best responses to strategies in \( S \).

Let
\[
a_i^{\sup} = \sup IBR_i^R(\tilde{S}^*) \quad \text{and} \quad a_i^{\inf} = \inf IBR_i^R(\tilde{S}^*).
\]

Because \( (\bar{a}_1^*, \ldots, \bar{a}_N^*) \) is the interim best response to \( (\bar{t}_0^*, \bar{t}_1^*, \ldots, \bar{t}_N^*) \), which belongs to \( \tilde{S}^* \), action \( a_i^{\sup} \) is the supremum of a nonempty set, and \( \bar{a}_i^* \leq a_i^{\sup} \) for \( i = 1, \ldots, N \). It follows directly from the definition that \( a_1^{\sup} \leq \cdots \leq a_N^{\sup} \). We first establish the following result.

**Claim 3.** If \( (a_1, \ldots, a_N) \in \tilde{A}^* \), then \( a_i \leq a_i^{\sup} \) for \( i = 1, \ldots, N \).

**Proof.** Suppose that \( (a_1, \ldots, a_N) \) is such that \( a_{i^*} > a_{i^*}^{\sup} \) for an \( i^* = 1, \ldots, N \). We will define a strategy \( (b_1, \ldots, b_N) \in \tilde{A}^* \) that interim weakly dominates \( (a_1, \ldots, a_N) \), provided
that the Sender is restricted to playing only strategies from \( \tilde{S}^* \). This will imply that \((a_1, \ldots, a_N) \notin \mathcal{A}^*\). Let \( b_i \) be defined by

\[
    b_i = \begin{cases} 
        a_i^{\sup} & \text{if } a_i \geq a_i^{\sup} \\
        a_i & \text{if } a_i \in (a_i^{\inf}, a_i^{\sup}) \\
        a_i^{\inf} & \text{if } a_i \leq a_i^{\inf}
    \end{cases}
\]

We first show that strategy \((b_1, \ldots, b_N)\) is monotonic, that is, that \( b_i \leq b_{i+1} \) for \( i = 1, \ldots, N-1 \). Note that \( b_j \leq a_j^{\sup} \) for all \( j \). So, \( b_i \leq b_{i+1} \) when \( b_{i+1} = a_{i+1}^{\sup} \). This last equality holds if \( a_{i+1} \geq a_{i+1}^{\sup} \). If \( a_{i+1} \in (a_{i+1}^{\inf}, a_{i+1}^{\sup}) \), then \( b_{i+1} = a_{i+1} \geq \max\{a_i, a_i^{\inf}\} \). Thus, \( b_i \leq b_{i+1} \). Finally, if \( a_{i+1} < a_i^{\inf} \), then

\[
    a_i \leq a_{i+1} < a_{i+1}^{\inf} = b_{i+1}.
\] (14)

If \( a_i \geq b_i \), then \( b_i \leq b_{i+1} \) by (14). Otherwise, \( b_i = a_i^{\inf} \leq a_{i+1}^{\inf} \leq b_{i+1} \). This establishes monotonicity of \( b \).

We next show that \((b_1, \ldots, b_N)\) weakly interim dominates strategy \((a_1, \ldots, a_N)\), given the Sender is restricted to playing the strategies from \( \tilde{S}^* \). If \( a_i > a_i^{\sup} \), then \( b_i = a_i^{\sup} \) yields a payoff strictly higher than that from playing \( a_i \), because all interim best responses to the strategies from \( \tilde{S}^* \) have the \( i \)-th action smaller than \( a_i^{\sup} \). If \( a_i \in (a_i^{\inf}, a_i^{\sup}) \), then \( b_i = a_i \); if \( a_i \leq a_i^{\inf} \), \( b_i \) is the infimum of interim best responses to the strategies from \( \tilde{S}^* \), which are all greater than \( a_i \).

Finally, \((b_1, \ldots, b_N)\) belongs to \( \tilde{A}^* \). Indeed, for all \( i \) either \( b_i \) is the limit of the actions that best respond to the \( i \)-th interval of the Sender’s types for some strategy from \( \tilde{S}^* \), or there are actions smaller and greater than \( b_i \) that best respond to the \( i \)-th interval of the Sender’s types for some strategy from \( \tilde{S}^* \). Therefore, \((b_1, \ldots, b_N)\) cannot be interim weakly dominated.

Note, in addition, that \((a_1^{\sup}, \ldots, a_N^{\sup}) \in \tilde{A}^* \). Indeed, suppose that \((a_1, \ldots, a_N)\) weakly interim dominates \((a_1^{\sup}, \ldots, a_N^{\sup})\). If \( a_i \neq a_i^{\sup} \) for some \( i \), then \( a^{\sup} \) would be a strictly better action than \( a_i \) against any strategy from \( \tilde{S}^* \) to which the Receiver’s interim best response is sufficiently close to \( a_i^{\sup} \). Thus, \( a_i = a_i^{\sup} \) for all \( i \), but then \((a_1, \ldots, a_N)\) cannot dominate \((a_1^{\sup}, \ldots, a_N^{\sup})\).

Denote by \( \text{IBR}^S_i(\tilde{A}^*) \) the set of \( i \)-th cutoffs of the Sender’s allowable best responses to strategies in \( \tilde{A}^* \). Let

\[
    t_i^{\inf} = \inf \text{IBR}^S_i(\tilde{A}^*),
\]

and let the strategy with cutoffs \( 0 = \tilde{t}_0 \leq \cdots \leq \tilde{t}_N = 1 \) be the Sender’s unique allowable best response to \((a_1^{\sup}, \ldots, a_N^{\sup})\). Note that \( \tilde{t} \geq t^{\inf} \).

Claim 4. If \((t_0, \ldots, t_N) \in \tilde{S}^* \), then \( t_i \leq \tilde{t}_i \) for \( i = 0, \ldots, N \).

Proof. Suppose that \((t_0, \ldots, t_N) \in \tilde{S}^* \) and \( t_{i^*} > \tilde{t}_{i^*} \) for some \( i^* \). Let \((s_0, \ldots, s_N)\) be defined by

\footnote{Recall that interim best responses are unique.}
the types which are indifferent between the Receiver from

\[ s_i = \begin{cases} 
  \tilde{t}_i & \text{if } t_i \geq \tilde{t}_i \\
  t_i & \text{if } t_i \in (t_i^{\inf}, \tilde{t}_i) \\
  t_i^{\inf} & \text{if } t_i \leq t_i^{\inf} 
\end{cases} \]

It is apparent that \( s_0 = 0 \) and \( s_N = 1 \).

We first show that strategy \((s_0, \ldots, s_N)\) is monotonic, that is, that \( s_i \leq s_{i+1} \) for \( i = 1, \ldots, N - 1 \). Note that \( s_j \leq \tilde{t}_j \) for all \( j \). So, \( s_i \leq s_{i+1} \) when \( s_{i+1} = \tilde{t}_{i+1} \). This last equality holds if \( t_{i+1} \geq \tilde{t}_{i+1} \). If \( t_{i+1} \in (t_i^{\inf}, \tilde{t}_{i+1}) \), then \( s_{i+1} = t_{i+1} \geq \max\{t_i, t_i^{\inf}\} \). Thus, \( s_i \leq s_{i+1} \). Finally, suppose that \( t_{i+1} \leq t_i^{\inf} \). If \( s_i > t_i \), then \( s_i = t_i^{\inf} \leq t_i^{\inf} = s_{i+1} \). Otherwise \( s_i \leq t_i \leq t_{i+1} \leq t_i^{\inf} \leq s_{i+1} \). This establishes monotonicity of \( s \).

We next show that \((s_0, \ldots, s_N)\) weakly dominates strategy \((t_0, \ldots, t_N)\), given the Receiver is restricted to playing the strategies from \( \tilde{A}^* \). If \( t_i > t_i \), then replacing \( t_i \) with \( s_i = \tilde{t}_i \) yields higher payoffs for types between \( \tilde{t}_i \) and \( t_i \), and the payoffs are strictly higher when \( a_i^{\sup} < a_i^{\sup} \). If \( a_i^{\sup} = a_i^{\sup} \), then \( s_i = t_i \) still dominates \( t_i > \tilde{t}_i \) by the assumption that allowable dominance. If \( t_i \in (t_i^{\inf}, \tilde{t}_i) \), then \( s_i = t_i \); and if \( t_i \leq t_i^{\inf} \), then replacing \( t_i \) with \( s_i = t_i^{\inf} \) yields higher payoffs for types between \( t_i \) and \( t_i^{\inf} \).

It remains to show that \((s_0, \ldots, s_N) \in \tilde{S}^* \). Indeed, for all \( i \) either \( s_i \) is the limit of the types which are indifferent between the \( i \)-th and \((i + 1)\)-st action of a strategy of the Receiver from \( \tilde{A}^* \), or there are types smaller and greater than \( s_i \) that are indifferent between the \( i \)-th and \((i + 1)\)-st action of a strategy of the Receiver from \( \tilde{A}^* \). Therefore, \((s_0, \ldots, s_N)\) cannot be weakly dominated.

Note, in addition, that \((\tilde{t}_0, \ldots, \tilde{t}_N)\) belongs to \( \tilde{S}^* \) as the allowable best response to the strategy \((a_1^{\sup}, \ldots, a_N^{\sup})\) from \( \tilde{A}^* \). This and Claim 3 imply that \((a_1^{\sup}, \ldots, a_N^{\sup})\) is the interim best response to \((\tilde{t}_0, \ldots, \tilde{t}_N)\). So, the strategies \((a_1^{\sup}, \ldots, a_N^{\sup})\) and \((\tilde{t}_0, \ldots, \tilde{t}_N)\) are an equilibrium. This cannot happen if \( \tilde{t}_i > t_i^* \) or \( a_i^{\sup} > a_i^{\sup} \) for some \( i \). This completes the proof of (12). The proof of (13) is analogous. \(\square\)

**Appendix B**

We present here an example of the basic cheap-talk game which has two equilibria that satisfy NITS. Whenever there exist multiple equilibria satisfying NITS, type-action mappings in the best-response dynamics can converge to the type-action mapping induced by either of them. Indeed, the type-action mapping induced by an NITS equilibrium (with off-path messages inducing the best response to the lowest type) is a constant best-response sequence. An additional feature that our example illustrates is that the number of partition intervals in different NITS equilibria can be different.

Suppose the distribution of the Sender’s types is uniform, and the utilities of the Sender and the Receiver are \( u^S(a, t) = -(a - t - c(t))^2 \) and \( u^R(a, t) = -(a - t)^2 \), where \( c(t) > 0 \) for all \( t \). So, the only departure from the uniform-quadratic example is that the bias \( c(t) \) depends on \( t \). Assume that the bias satisfies the following properties:

\[ c(0) = 7/96; \quad c(8/96) = 2/96; \quad c(16/96) = 8/96; \quad c(20/96) = 14/96; \quad c(24/96) = 14/96. \]
When $c(\cdot)$ satisfies these conditions, $a^S(t) = t + c(t)$ restricted to $t \in \{0, 8/96, 16/96, 20/96, 24/96\}$ is strictly increasing. Hence it is possible to extend the definition of $c(t)$ so that $a^S(t) = t + c(t)$ is strictly increasing for all $t \in [0, 1]$.

This game has an equilibrium in which the types from $[0, 20/96)$ induce action $a^*_2 = 10/96$, and the types from $(20/96, 1]$ induce action $a^*_3 = 58/96$. If there are three messages $m_1 < m_2 < m_3$, then $m_1$ is an off-path message which induces action $a^*_1 = 0$, and $m_i$ induces action $a^*_i$ for $i = 2, 3$.

The game also has an equilibrium, in which: the types from $[0, 8/96)$ induce action $\overline{a}^*_1 = 4/96$, the types from $(8/96, 24/96)$ induce action $\overline{a}^*_2 = 16/96$, and the types from $(24/96, 1]$ induce action $\overline{a}^*_3 = 60/96$. Both equilibria satisfy NITS because both $a^*_2$ and $\overline{a}^*_1$ are closer to $a^S(0) = c(0)$ than 0.
References


