Effective Communication in Cheap Talk Games*

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March 6, 2021

Abstract

This paper presents arguments based on weak dominance and learning for selecting informative equilibria in a model of cheap-talk communication. The results also predict a monotonic relationship between messages and the actions they induce. We reformulate the communication game as one in which the strategy of the informed player is an interval partition of the type space instead of type-contingent messages. We show that in this game there is a largest (informally, most informative) equilibrium. The largest equilibrium in the reformulated game is the largest equilibrium that survives iterated deletion of weakly dominated strategies and that, under a standard regularity condition, this is the only equilibrium that survives deletion of weakly dominated strategies. We interpret these results in terms of learning. Our arguments establish that a class of adaptive dynamics converge to the largest equilibrium from a rich set of initial conditions.

Journal of Economic Literature Classification Numbers: C72, D83. Keywords: Communication, Learning, Equilibrium Selection.

*This paper combines results from three projects: Gordon [11], Kartik and Sobel [16], and Lo and Olszewski [18]. We are grateful to many seminar audiences and to Pierpaolo Battigalli, Andreas Blume, Richard Brady, Vincent Crawford, Sjaak Hurkens, Philip Neary, Alexsandr Levkun, Olivier Tercieux, and Yuehui Wang for useful comments.

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1 Introduction

Talk is a useful way to communicate private information in strategic situations. At least since the formal models of Crawford and Sobel [7] and Green and Stokey [12], the possibility that cheap talk can be influential in games has been understood. However, these early papers recognized that equilibrium analysis is generally indeterminate. Models of cheap-talk communication have multiple equilibria and, in particular, they typically have a babbling equilibrium in which players do not take advantage of opportunities to communicate. A central concern of this research has been finding conditions under which communication is effective, that is in which the predicted outcome involves non-trivial information transmission. We hope this paper advances the literature on effective communication.

We present related arguments that lead to strong equilibrium selection results in simple cheap-talk games. We establish that deletion of weakly dominated strategies[1] selects an outcome with effective communication when such an outcome exists. We also show that a family of adaptive dynamics lead to the same selection. These arguments are subtle because they require a reformulation of the strategic situation in order to work. The paper presents three different ways to think about the reformulation: one reformulation studies a game in with the players are restricted to monotonic strategies; another reformulation redefines the strategy set of one of the players, replacing signaling strategies with partitions of the state space; the third reformulation looks at learning processes. In the first two cases, our solution concept involves iterated deletion of weakly dominated strategies. In the third case, it is a form of dynamic stability.

Underlying our analysis is a basic model of strategic communication. An informed Sender sends a message to an uninformed Receiver. The Receiver responds to the message by making a decision that is payoff relevant to both players. Talk is cheap because the payoffs of the players do not depend directly on the Sender's message. Crawford and Sobel [7] (hereafter CS) characterize the set of equilibrium outcomes in a one-dimensional model of cheap talk with a particular kind of conflict of interest. CS demonstrate that there is a finite upper bound, N^*, to the number of distinct actions that the Receiver takes in equilibrium, and that for each $N = 1, \ldots, N^*$, there is an equilibrium in which the Receiver takes N actions. In addition, when a technical condition holds, CS demonstrate that for all $N = 1, \ldots, N^*$, there is a unique equilibrium outcome in which the Receiver takes N distinct actions, and the ex-ante expected payoff for both Sender and Receiver is strictly increasing in N. The equilibrium with N^* actions is often the outcome selected for analysis in applications.

The multiple-equilibria problem arises in three different ways in cheap-talk games. Typically, some messages are sent with zero probability in equilibrium. There will often be multiple ways to specify behavior off the path of play. This first kind of multiplicity, off-path indeterminacy, is familiar in games with incomplete information and need not be essential. Changing off-path behavior does not change what we observe. If cheap-talk games had a unique equilibrium outcome, then we would be able to confidently classify situations in which communication is effective even if many different equilibria

[1] Our results require *iterated* deletion of weakly dominated strategies, but we frequently omit the adjective "iterated."

supported that outcome. The second kind of multiplicity, message indeterminacy, is that the meaning of messages is arbitrary. Given any equilibrium, one can generate another equilibrium by changing the use and interpretation of messages. This kind of problem identifies a way in which language is arbitrary. The word used to describe the color of a white house in Paris is blanche and in Warsaw is biały. Predictions are still possible with this kind of indeterminacy when the different equilibria induce the same relationship between types and actions. What matters is that French speakers and Polish speakers classify the same set of houses as "white" (and their audiences understand that) rather than the particular word they use to describe the color. The third type of multiplicity, type-action indeterminacy, is fundamental. Cheap-talk games typically have an uninformative equilibrium (or, to be precise, many uninformative equilibria when one takes into account the first two kinds of multiplicity) and may have qualitatively different equilibria in which the Receiver takes at least two different actions with positive probability. It is this type of multiplicity that we wish to examine, but our approach shows how one can eliminate the problem of message indeterminacy can resolve type-action indeterminacy.

We begin the analysis by replacing the standard cheap-talk game with a related formulation, first introduced by Gordon [10]. Under the maintained assumptions in CS, it is well known that in any equilibrium, the Sender's strategy breaks the unit interval of types into a finite set of intervals. Rather than assuming the Sender's strategy is a signaling rule (a function from type into message), we instead assume the Sender's strategy is an increasing set of cutpoints. The interpretation is that adjacent cutpoints identify an interval and types send a common message if and only if they are in the same interval. This formulation removes messages from the description of the game. Hence it eliminates message indeterminacy. It is straightforward to see that an equilibrium of the CS game generates a type-action distribution if and only if it is an equilibrium of the cutpoint game. Hence, the reformulation alone does not eliminate type-action indeterminacy. We show, however, under assumptions that are common in the cheap-talk literature, that a family of adaptive dynamics converge to an equilibrium with N^* partition elements and that all such equilibria are associated with the same type-action distribution. More generally, the babbling outcome will not be a stable point of adaptive dynamics if non-trivial equilibria exist. We prove this result by demonstrating that deletion of weakly dominated strategies refines the set of equilibria in the cutpoint game. To establish this result, we first note that the strategies in our game are ordered. We form an increasing (decreasing) sequence of strategies by starting with the lowest (highest) strategy of the Receiver and iterating best responses. These sequences converge monotonically to equilibria and, under standard assumptions, converge to the same equilibrium. We also show that only strategies lower (higher) than the decreasing (increasing) sequence survive iterated deletion of weakly dominated strategies and that limit points of adaptive dynamics must lie between the lower and upper bounds.

The cutpoint game eliminates message indeterminacy. By doing so, one can focus on multiple-equilibrium problems not caused by the fact that messages lack pre-assigned meaning. Nevertheless, the cutpoint games change the analysis substantively and technically. The substantive difference is a clear limitation. In cutpoint games there are no messages, so the analysis does not explain how the Sender communicates to the Receiver. We put words back into the analysis in two different ways. First, we imagine that there is an exogenous order on messages and restrict players to strategies that are monotonic with respect to this order. We view this as a way to incorporate "exogenous meaning" into the communication game. Players enter a strategic setting with a shared ordering and it is common knowledge that they will behave in a way that is consistent with the order in the communication game. In the CS game when players are restricted to monotonic strategies, the Sender's strategy can be described by cutpoints. The monotonic cheap-talk game has the same set of equilibrium type-action distributions as the cutpoint game. Therefore, the result that deletion of weakly dominated strategies selects an equilibrium in the cutpoint game provides a selection result for monotone cheap-talk games. This analysis has a bonus. Imposing monotonicity actually selects the messages that are used in equilibrium. That is, the monotonicity condition (combined with the rest of the analysis) eliminates message indeterminacy. We find the selection intuitive. When the Sender has an upward bias (for each type, the ideal action of the Sender is strictly greater than the ideal action of the Receiver), the selected equilibrium uses only the highest messages. That is, upward bias leads to exaggeration in equilibrium.

The technical limitation of the cutpoint game is that the equivalence between monotone cheap-talk games and the cutpoint game is not perfect. Although it is true that monotone strategies can be described using cutpoints, it need not be the case that best responses to non-trivial mixtures of monotone strategies are intervals. This difference is not important for equilibrium analysis because players do not need to randomize in equilibrium.[2] The difference is important if we study learning processes in which players respond to mixtures of opponents' past behavior. We present an analysis of dynamics in cutpoint games and provide conditions under which the limits of learning processes converge. For a class of processes that include best-response dynamics, we find that play converges to a non-babbling equilibrium under mild restrictions on initial conditions. Specifically, a sufficient condition for convergence is a richness assumption guaranteeing that initially it is possible to induce a variety of actions. These results correspond to convergence results for cheap-talk games in the case of best-response dynamics, but in general are tied to the cutpoint game.

The paper proceeds as follows. Section 2 introduces the cutpoint game and the cheap-talk game and shows how they are related. Section 3 presents two examples that illustrate our results. Section 4 reviews preliminary results. Section 5 states the main result and provides intuition. Section 6 describes some convergence results. Section 7 discusses the implication of our approach for cheap-talk models under alternative restrictions on preferences. Section 8 interprets the main result and connects it to the literature. The proofs are in the appendices.

2 Model

This section describes two games and shows how they are related. Both games have two players, the Sender S and the Receiver R. The players have utility functions u^S and u^R

[2]Furthermore, randomization in equilibrium must lead to a probability distribution over types and actions that agrees with a distribution generated by a pure-strategy equilibrium.

defined on $\mathbb{R} \times [0,1]$. We assume that for $j = S$ and R, $u^j : \mathbb{R} \times [0,1] \to \mathbb{R}$ is twice continuously differentiable, strictly concave in its first argument, and has strictly positive mixed partial derivative. The first argument of $u^j(\cdot)$ is an action, a. The second argument is the type of the Sender, θ. We assume that θ has a strictly positive, continuous density f on $[0,1]$.

We extend u^j to mixed strategies by linearity; that is, if $\mu(\cdot)$ is a probability measure on $[0,1]$, then we extend $u^j(\cdot)$ by defining $u^j(\mu,\theta) = \int u^j(a,\theta)\,d\mu$. We assume that $\max_{a\in\mathbb{R}} u^j(a,\theta)$ exists and let $a^j(\theta) = \arg\max_{a\in\mathbb{R}} u^j(a,\theta)$ for $j = R$ and S (strict concavity guarantees that the maximizer of $u^j(\cdot,\theta)$ is unique). For $\theta' < \theta''$, let

$$\bar{a}(\theta',\theta'') = \arg\max_{a\in\mathbb{R}} \int_{\theta'}^{\theta''} u^R(a,\theta)\, f(\theta)\, d\theta.$$

For $\theta' = \theta''$, let $\bar{a}(\theta',\theta') = a^R(\theta')$. If $a^R(\theta) < (>) a^S(\theta)$ for all θ, we say that there is positive (negative) bias. Unless otherwise noted, we assume a positive bias. We discuss the extent to which our results generalize in Section 7. We normalize preferences so that $a^R(0) = 0$ and $a^R(1) = 1$. With this normalization, actions outside of the unit interval are dominated for R and would not be part of any equilibrium outcome. We therefore restrict the action set to be $[0,1]$.

2.1 The Cutpoint Game

Given a nonnegative integer L, an interval partition of order L is a list $t = (t(0),\ldots,t(L))$ such that

$$0 = t(0) \le t(1) \le \cdots \le t(L) = 1.$$

Each $t(n)$ is called a cutpoint. An interval partition of order L generates L intervals, of the form $I_n = [t(n-1), t(n)]$ for $n = 1,\ldots,L$.[3] The intervals are nondegenerate if and only if the cutpoints are distinct. Let T_L be the set of all interval partitions of order no greater than L and let T be the set of all interval partitions of any order.

For $L = 1, 2, \ldots$, an action list of order L is a list $a = (a(1),\ldots,a(L))$ of L elements such that

$$a(1) \le a(2) \le \cdots \le a(L).$$

A_L is the set of action lists of order L.

We assume the order of action lists and interval partitions is bounded by some nonnegative integer M, and for $L < M$ we identify T_L with T_M by mapping $t \in T_L$ to $\hat{t} \in T_M$ by inserting 0s at the beginning of the list. That is, we set

$$\hat{t}(n) = \begin{cases} 0 & \text{if } n < M - L \\ t(n-(M-L)) & \text{if } n \ge M - L \end{cases}.$$

[3]An interval partition is not really a partition because $I_n \cap I_{n+1} = t(n) \ne \emptyset$. It is the case that the union of partition elements is equal to $[0,1]$ and that different partition elements have disjoint interiors. If $t(n-1) = t(n)$, then $I_n = \{t_n\}$.

Similarly, we identify A_L with A_M by mapping the action lists from $a \in A_L$ to the action lists from $\hat{a} \in A_M$ by

$$\hat{a}(n) = \begin{cases} 0 & \text{if } n \leq M - L \\ a(n - (M - L)) & \text{if } n > M - L \end{cases}.$$

With the identification above, each f, u^S, and u^R determine the following **cutpoint game**. The pure strategy set is $A_M \times T_M$, where A_M is the set of action lists available to the Receiver and T_M is the set of strategies available to the Sender. The payoffs are given by

$$U^j(a, t) = \sum_{n=1}^{M} \int_{t(n-1)}^{t(n)} u^j(a(n), \theta) f(\theta)\, d\theta$$

for $j = S, R$. We extend $U^j(\cdot)$ to mixed strategies by linearity.

2.2 The Cheap-Talk Game

A **cheap-talk game** starts with the same basic information as the cutpoint game: S and R; $u^S(\cdot)$ and $u^R(\cdot)$; and $f(\cdot)$. It adds a finite set $\mathcal{M}$ of messages.[4] A pure strategy for S is a mapping $s : [0, 1] \to \mathcal{M}$ that associates with every type θ the message $s(\theta)$. A pure strategy for R is a mapping $b : \mathcal{M} \to \mathbb{R}$ that associates with every message m an action $b(m)$. Given (s, b) the payoff to player j is $\int_0^1 u^j(b(s(\theta)), \theta) f(\theta)\, d\theta$. We denote R's pure strategy set by $\mathcal{B}$ and S's strategy set by $\mathcal{S}$.

It is useful to consider a restriction on strategy spaces. Assume $\mathcal{M}$ is linearly ordered and denote the order by $\geq$. A pure strategy $s(\cdot)$ for S is monotonic if $\theta > \theta'$ implies that $s(\theta) \geq s(\theta')$, and a pure strategy $b(\cdot)$ for R is monotonic if $m > m'$ implies that $b(m) \geq b(m')$. A **monotonic cheap-talk game** is derived from the cheap-talk game by assuming that R's strategy set is $\mathcal{B}_0 = \{b \in \mathcal{B} : m > m' \implies b(m) \geq b(m')\}$ and S's strategy set is $\mathcal{S}_0 = \{s \in \mathcal{S} : \theta > \theta' \implies s(\theta) \geq s(\theta')\}$.

The restriction to monotonic strategies would follow if it were common knowledge that players have monotonic beliefs. Let $\beta^R(\cdot \mid m)$ denote the Receiver's posterior belief about the type after receiving the message m. We assume that if $m > m'$, then $\beta^R(\cdot \mid m)$ dominates $\beta^R(\cdot \mid m')$ in the sense of first-order stochastic dominance (so that higher types are expected to send higher messages). Similarly, let $\beta^S(\cdot \mid m)$ denote the Sender's beliefs about the action induced by each message. We assume that if $m > m'$, then $\beta^S(\cdot \mid m)$ dominates $\beta^S(\cdot \mid m')$ in the sense of first-order stochastic dominance. When beliefs are monotonic, there always exists a monotonic best response. If all strategies are available in the cheap-talk game, it is not necessary for beliefs to be monotonic, but the assumption is consistent with rationality provided that, as we have assumed, the u^j have strictly positive mixed partial derivatives.

The monotonicity assumption is restrictive in the trivial sense that non-monotonic strategies and beliefs exist. It is also restrictive in the stronger sense that non-monotonic

[4]The finiteness assumption is primarily for technical convenience, but results in Section 7 depend on the assumption.

strategies may be best responses even if the opponent is restricted to play monotonic strategies. Finally, it is restrictive because there are some non-monotonic strategies that have only non-monotonic best responses. The monotonicity assumption is not restrictive in two ways. First, any equilibrium type-action distribution[5] for the original game can be derived from monotonic strategies. To see this, note (see Section 4.1) that the equilibrium distribution involves a finite partition of the Sender's types into adjacent intervals. Construct a strategy for the Sender in which higher partitions elements send higher messages. Any best response to this strategy will be monotonic on the equilibrium path. One can define specific off-the-path actions to preserve monotonicity and support the equilibrium. Second, S's best response to a strictly monotonic strategy must be monotonic and S always has a monotonic best response to a monotonic strategy. Similarly, if S plays a monotonic strategy, then R has a monotonic best response, but R may also have a non-monotonic best response off-the-equilibrium path.

The restriction to a game in which players have monotonic beliefs is different from the restriction to a game in which all pure strategies are monotonic. If R is uncertain about which monotonic pure strategy S is playing, then R's best response may be non-monotonic. The next example illustrates the possibility.

Example 1. *Assume R believes that S is equally likely to play strategy s or s', with*

$$s(\theta) = \begin{cases} 0 & \text{if } \theta \in [0, .5) \\ 2 & \text{if } \theta \in [.5, 1] \end{cases}$$

and

$$s'(\theta) = \begin{cases} 1 & \text{if } \theta \in [0, .25) \\ 2 & \text{if } \theta \in [.25, 1] \end{cases}.$$

Then $\beta^R(\cdot \mid 0)$ will strictly dominate $\beta^R(\cdot \mid 1)$ and R would not have a monotonic best response.

2.3 Connections

We observe that (1) every cutpoint strategy has a corresponding monotonic cheap-talk strategy for S (and hence also a cheap-talk strategy in the cheap-talk game); and (2) every monotonic cheap-talk strategy for S has a cutpoint counterpart, but that non-monotonic cheap talk strategies cannot be represented using cutpoints.

Formally, if $\mathcal{M}$ has elements $m(1), \ldots, m(L)$, then any strategy profile $t \in T_L, a \in A_L$ corresponds to a strategy s for the Sender in the cheap-talk game formed by setting $s(\theta) = m(n)$ for $\theta \in (t(n-1), t(n))$ ($s(t(n))$ can be either $m(n)$ or $m(n+1)$) and a strategy b for the Receiver $b(m(n)) = a(n)$. If $\mathcal{M}$ has more than L elements, then one can let $b(m) = b(m(1))$ for all messages not in the image of $s(\cdot)$. With this association, it is straightforward to see that (a, t) is an equilibrium of the cutpoint game if and only if (b, s) is an equilibrium of the cheap-talk game.

[5] Given a (mixed) strategy profile, (α^*, σ^*), the associated type-action distribution is $\gamma(a \mid t) = \sum_m \alpha^*(a \mid m)\sigma^*(m \mid t)$.

The converse implication does not hold. There are strategies s for S in the cheap talk game that do not correspond to cutpoint strategies when $s^{-1}(m)$ is not an interval for some m. However, for any strategy of R in a cheap-talk game, S has a best response in which $s^{-1}(m)$ is an interval (or empty) for every m. To make the connection between cheap-talk and cutpoint games precise, we restrict attention to monotonic cheap-talk games.

The cutpoint game is related to the standard cheap-talk game, but is not equivalent to it. However, there is a correspondence between the monotonic cheap-talk game and the cutpoint game. Given a cutpoint strategy t for the Sender, we have explained how one can construct a monotonic signaling strategy for the Sender. Given a monotonic signaling strategy s such that $s(\theta) = m(j(\theta))$ for $j(\theta) \in \{1, \ldots, L\}$ and $\theta' > \theta$ implies $j(\theta') \geq j(\theta)$, one can construct cutpoints ($t(n) = \sup\{\theta : j(\theta) = n\}$).

We will show that equilibrium refinement arguments have power to select outcomes in monotonic cheap-talk games when they do not in general. We use the correspondence between the cutpoint game and monotonic cheap-talk games to illustrate how the lack of exogenous meaning is intimately connected to the fundamental indeterminacy in cheap-talk games. The cutpoint formulation eliminates explicit reference to messages. Hence in this formulation there is no message indeterminacy associated with the possibility that messages lack intrinsic meanings. The correspondence between monotonic cheap-talk games and the cutpoint game allows us to interpret the cutpoint game as a cheap-talk game in which there is a restriction on the use of messages determined outside of the game. We interpret our selection results as demonstrating that this exogenous restriction – a way to model a common understanding among players that higher messages are associated with higher actions – is enough to identify a unique (refined) equilibrium and to identify cases in which messages have the same interpretation in outcomes that satisfy our solution concept.

3 Examples

This section illustrates some of our results using two examples. Section 3.1 describes the multiple-equilibrium problem in a simple coordination game and demonstrates how weak dominance arguments combined with monotonicity restrictions lead to a selection. Section 3.2 presents examples using the standard uniform-quadratic model.

3.1 Coordination Game Example

Consider a signaling game in which there are two types (high and low), two actions (high and low), and two messages (high and low). The types are equally likely. The Sender and Receiver have common interests. The players receive a payoff of two if the action matches the type and a payoff of zero otherwise. Limiting attention to binary types and actions and identical preferences makes the analysis transparent, although our abstract analysis typically assumes that S and R have different preferences. The strategic form of the game is given in the following table.

A strategy for the Sender is a pair (i, j) where the Sender sends message i when her type is low and j when her type is high. Similarly, the first component in the Receiver's strategy is his response to a low message and the second is his response to a high messages. Hence (l, h) is the Sender strategy that reports a message that matches the state, (h, l) is the strategy that sends the high message when the state is low and the low message when the state is high. Similarly, (H, L) is the strategy of the Receiver that responds to the low message with the high action and the high message with the low action.

	(H,H)	(L,H)	(H,L)	(L,L)
(h,h)	$1, 1$	$1, 1$	$1, 1$	$1, 1$
(h,l)	$1, 1$	$0, 0$	$2, 2$	$1, 1$
(l,h)	$1, 1$	$2, 2$	$0, 0$	$1, 1$
(l,l)	$1, 1$	$1, 1$	$1, 1$	$1, 1$

The game has a babbling equilibrium in which S mixes equally between (h, l) and (l, h) and R mixes equally between (H, L) and (L, H). (There are also inefficient pure-strategy equilibria and other inefficient mixed equilibria.) There are also two efficient equilibria in which the Sender distinguishes between the states and the Receiver correctly interprets this information. The mixed-strategy equilibrium satisfies standard refinements (from perfection to strategic stability) although it is, intuitively, implausible. Our approach is to replace the original game by a game in which non-monotonic strategies are not available.

	(H,H)	(L,H)	(L,L)
(h,h)	$1, 1$	$1, 1$	$1, 1$
(l,h)	$1, 1$	$2, 2$	$1, 1$
(l,l)	$1, 1$	$1, 1$	$1, 1$

We obtain this game by deleting the non-monotonic strategies from the original game. Doing so eliminates some inefficient equilibria, but it does not eliminate any equilibrium payoffs. The monotonic game has multiple equilibria, but weak dominance selects the efficient one.

The general argument requires multiple rounds of deletion, but the example captures an essential feature of the construction. Restricting attention to monotonic strategies eliminates some coordination problems. Without the restriction, every non-babbling equilibrium type-action distribution can be supported in multiple ways by permuting the assignment of types to messages. Imposing an order on messages removes this indeterminancy.

Treating this as a cutpoint game, imagine that the type variable θ is uniformly distributed on $[0, 1]$, but all $\theta < .5$ are low and all $\theta > .5$ are high. With this interpretation it is natural to imagine strategies with only one cutpoint. That is, consider Sender strategies of the form $t = (0, t(1), 1)$ with $t(1) \in (0, 1)$. Note that the Receiver's best response to t is to set $a_1 = L$ and $a_2 = H$. Consequently, t is an equilibrium strategy if and only if $t(1) = .5$. $(a^*, t^*) = ((L, H), (0, .5, 1))$ is the efficient equilibrium in which low types use one message and high types use another message. Learning dynamics converge to this equilibrium, which is especially easy to see for the case of best-response dynamics. Consider a sequence $t_k(1)$ starting with an arbitrary $t_0(1) \in (0,1)$ and with $(0, t_k(1), 1)$ defined to be the Sender's best response to the Receiver's best response to $(0, t_{k-1}(1), 1)$. Because the Receiver's best response to $(0, t_0(1), 1)$ is $(a_1, a_2) = (L, H)$ and the Sender's best response this strategy is $(0, .5, 1)$, we have $t_k(1) = .5$. for $k > 0$.

We know that there is also a babbling equilibrium. Why is it not a limit? The answer is we have looked at Sender partitions that have two elements. The babbling outcome is the equilibrium when S restricted to the trivial t with order one ($t = (t(0), t(1)) = (0, 1)$).

3.2 Uniform-Quadratic Example

Assume that the Sender's type is distributed uniformly on interval $[0,1]$, and the players' utilities are: $u^S(a,\theta) = -(a-\theta-c)^2$ and $u^R(a,\theta) = -(a-\theta)^2$ for some $c > 0$. For this game, there is an $N^* \geq 1$ such that for every $N \leq N^*$ there exists a unique equilibrium type-action distribution with N partition intervals (i.e., with N equilibrium actions).

Take first $c = 0.1$. In this case, the game has only two kinds of equilibria: babbling and the equilibrium in which the types from $[0, 0.3]$ induce action 0.15, and the types from $[0.3, 1]$ induce action 0.65.

We study how a simple learning process, best-response dynamics, can generate a sequence of partitions and action lists that converge only to the informative equilibrium in the cutpoint game. Consider any initial two-interval strategy of the Sender, $0 = t_1(0) \leq t_1(1) \leq t_1(2) = 1$, and suppose that the strategy of the Sender in period $k-1$, for $k > 1$, of the best-response dynamics is $0 = t_{k-1}(0) \leq t_{k-1}(1) \leq t_{k-1}(2) = 1$. Then, the best response of the Receiver is $a_{k-1}(1) = (0 + t_{k-1}(1))/2$ and $a_{k-1}(2) = (t_{k-1}(1)+1)/2$, and the best response of the Sender to this strategy of the Receiver is $0 = t_k(0) \leq t_k(1) \leq t_k(2) = 1$, where

$$t_k(1) + 0.1 - \frac{0 + t_{k-1}(1)}{2} = \frac{t_{k-1}(1) + 1}{2} - t_k(1) - 0.1. \tag{1}$$

It follows from formula (1) that the sequence $(t_k(1))_{k=1}^{\infty}$ is monotonic (decreasing or increasing, depending on $t_1(1)$), and converges to 0.3. This implies that the strategies of the Sender converge to the Sender's strategy in the informative equilibrium of the cutpoint game. This in turn implies that the strategies of the Receiver converge to the Receiver's strategy in the informative equilibrium.

Next take the perspective of a monotonic cheap-talk game and examine how one can discard dominated strategies in a way that parallels the dynamics above. Consider two messages: $m_1 \leq m_2$. Suppose that a higher type of the Sender must send a weakly higher message, and the Receiver's response to the higher message must be weakly higher. Then, the undominated strategies of the Sender have the property that the types from $[y_0^*, 1]$ send message m_2 for $y_0^* = .9$. Given this restriction, it is weakly dominated for R to respond to m_1 with an action higher than .45. Repeating the process, we get a sequence of actions (x_k^*, y_k^*), where $x_0^* = 0$ and for $\theta \in [0, x_k^*]$, m_2 has been deleted and for $\theta \in [y_k^*, 1]$, m_1 has been deleted. Given (x_{k-1}^*, y_{k-1}^*), in order for m_2 to be undominated it must be that $\theta \geq x_k^*$, where

$$x_k^* + .1 - \frac{x_{k-1}^*}{2} = \frac{1 + x_{k-1}^*}{2} - x_k^* - .1, \tag{2}$$

and in order for m_1 to be undominated it must be that $\theta \leq y_k^*$, where

$$y_k^* + .1 - \frac{y_{k-1}^*}{2} = \frac{1 + y_{k-1}^*}{2} - y_k^* - .1. \tag{3}$$

To interpret expression (2), observe that if it holds, then x_k^* is indifferent between the lowest action that a rational Receiver would take given m_1 knowing that $\theta \leq x_{k-1}^*$ must send m_1 (this is $x_{k-1}^*/2$) and the lowest action that a rational Receiver would take given m_2 knowing that $\theta \leq x_{k-1}^*$ must send m_1 (this is $(1 + x_{k-1}^*)/2$). In a supplementary file, we provide a spreadsheet that computes bounds for several iterations and illustrates which strategies are deleting in each step of the process.

Equations (2) and (3) determine monotonic sequences: an increasing sequence $(x_k^*)_{k=1}^{\infty}$, and a decreasing sequence $(y_k^*)_{k=1}^{\infty}$, both converging to 0.3. Consequently, only the informative equilibrium survives iterative deletion of weakly dominated strategies (IDWDS).[6]

The analysis becomes more involved when the Sender induces more than two actions in equilibrium. Take now $c = 0.05$. In this case, the game has three equilibrium type-action distributions. In the outcome with the largest number of receiver actions, the types from $[0, 4/30]$ induce action $2/30$, the types from $[4/30, 14/30]$ induce action $9/30$, and the types from $[14/30, 1]$ induce action $22/30$.

Let $0 = t_k(0) \leq t_k(1) \leq t_k(2) \leq t_k(3) = 1$ denote the strategy of the Sender in period k, for $k \geq 1$, of the best-response dynamics. Then,

$$t_k(1) + 0.05 - \frac{t_{k-1}(1)}{2} = \frac{t_{k-1}(1) + t_{k-1}(2)}{2} - t_k(1) - 0.05,^7 \tag{4}$$

and

$$t_k(2) + 0.05 - \frac{t_{k-1}(1) + t_{k-1}(2)}{2} = \frac{1 + t_{k-1}(2)}{2} - t_k(2) - 0.05. \tag{5}$$

However, formulas (4) and (5) do not guarantee that sequences $(t_k(1))_{k=1}^{\infty}$ and $(t_k(2))_{k=1}^{\infty}$ are monotonic. It depends on the values of $t_1(1)$ and $t_1(2)$. For example, when $t_1(1) = 0.25$ and $t_1(2) = 0.75$, then $t_1(1) < t_2(1)$, but $t_k(1) > t_{k+1}(1)$ for $k > 1$, while $t_k(2) > t_{k-1}(2)$ for $k > 1$. It is true that $(t_k(1))_{k=1}^{\infty}$ converges to $4/30$ and $(t_k(2))_{k=1}^{\infty}$ converges to $14/30$, but it requires a more subtle argument.

To prove convergence, consider two specific initial conditions $t_1^{**}(1) = t_1^{**}(2) = 0$ (or $t_1^{**}(2) > t_1^{**}(1) > 0$ but both being very close to 0), and $t_1^*(1) = t_1^*(2) = 1$ (or $t_1^*(1) < t_1^*(2) < 1$ but both being very close to 1). In the former case, $t_2^{**}(1) > t_1^{**}(1)$ and $t_2^{**}(2) > t_1^{**}(2)$, and then, (4) and (5) imply that the sequence $(t_k^{**}(1))_{k=1}^{\infty}$ is increasing and converges to $4/30$. And in the latter case, $t_2^*(1) < t_1^*(1)$ and $t_2^*(2) < t_1^*(2)$, and then, (4) and (5) imply that the sequence $(t_k^*(2))_{k=1}^{\infty}$ is decreasing and converges to $14/30$. This implies that any $(t_k(1))_{k=1}^{\infty}$ converges to $4/30$, and any $(t_k(2))_{k=1}^{\infty}$ converges to $14/30$, because they are "sandwiched" between the sequences $(t_k^{**})_{k=1}^{\infty}$ and $(t_k^*)_{k=1}^{\infty}$.

[6] We describe procedures of deleting dominated strategies formally in Section 5.1.

[7] If this equation has no solution in $[0, 1]$, set $t_k(1) = 0$.

Consider now three messages: $m_1 \le m_2 \le m_3$, and suppose players are restricted to play monotonic strategies. The cutpoints $t_1(2) = t_1(1) = 0$ determine such a strategy of the Sender. The strategy that responds with actions $b_1(1) = b_1(2) = 0$ and $b_1(3) = 0.5$ is the Receiver's best response to this strategy of the Sender. And any strategy $b'(1) = b'(2) = 0$ and $b('3) < b_1(3)$ is dominated by $b_1(1) = b_1(2) = 0$ and $b_1(3) = 0.5$. Then, the strategy determined by $t_2(1) = 0$ and $t_2(2) = 0.2$ is the best response of the Sender to b_1, and any strategy determined by cutpoints $t'(1) = 0$ and $t'(2) < t_2(2)$ is dominated given that the Receiver does not play any strategy b' with $b'(1) = b'(2) = 0$ and $b('3) < b_1(3)$.

More generally, inductive reasoning shows that if $b_k(1)$, $b_k(2), b_k(3)$, is the Receiver's best response to the Sender's strategy determined by cutpoints $t_k(1)$ and $t_k(2)$, then any $b' = (b'(1), b'(2), b'(3))$ such that $b'(m_n) < b_k(m_n)$ for at least one n is dominated by the strategy $\tilde{b}_k = (\tilde{b}_k(1), \tilde{b}_k(2), \tilde{b}_k(3))$, such that $\tilde{b}_k(m_n) = \max\{b'(m_n), b_k(m_n)\}$, provided that the Sender is restricted to playing strategies with the property that if $\theta \ge t_k(n)$, then $s(\theta) \ge m_{t_k(n)+1}$ for all n. Furthermore, after eliminating dominated strategies of the Receiver (that is, leaving, strategies b' such that $b_k(m_n) \le b'(m_n)$ for all n, any strategy s of the Sender such that $s(t_k(n)) < m_{t_{k+1}(n)}$ for at least one n is dominated by the strategy $\tilde{s}$ where if $\theta \in (t_{k+1}(n-1), t_{k+1}(n))$, then $\tilde{s}(\theta) = \max\{s(\theta), m_{t_{k+1}(n)}\}$.

Because $(t_k(1))_{k=1}^{\infty}$ converges to $4/30$, and $(t_k(2))_{k=1}^{\infty}$ converges to $14/30$, all strategies with $s(\theta) = m_2$ for $\theta < 4/30$ or $s(\theta) = m_3$ for $\theta < 14/30$ do not survive IDWDS. By starting with $s_1(\theta) \equiv m_1$ and $t_1(n) \equiv 1$, and applying analogous arguments, we obtain that all strategies s of the Sender such that $s(\theta) = m_1$ for $\theta > 4/30$ or $s(\theta) = m_2$ for $\theta > 14/30$ do not survive IDWDS. Thus, we conclude that only the equilibrium outcome inducing three actions survives IDWDS.

4 Basic Results

To make the paper self-contained, this section reviews relevant results from the literature.

4.1 Structure of Equilibria in the Cheap-Talk Game

Under the positive-bias assumption, CS demonstrate that there exists a positive integer N^* such that for every integer N with $1 \le N \le N^*$, there exists at least one equilibrium in which there are N induced actions, and moreover, there is no equilibrium that induces strictly more than N^* actions. An equilibrium can be characterized by a partition of the set of types, $t_N = (t_N(0), \dots, t_N(N))$ with $0 = t_N(0) < t_N(1) < \cdots < t_N(N) = 1$, and actions $b(n)$, $n = 1, \dots, N$, such that for all $n = 1, \dots, N-1$

$$u^S(b(n+1), t_N(n)) - u^S(b(n), t_N(n)) = 0, \tag{6}$$

and for all $n = 1, \dots, N$,

$$b(n) = \overline{a}(t_N(n-1), t_N(n)). \tag{7}$$

Furthermore, all equilibrium outcomes can be described in this way.[8] In an equilibrium, adjacent types pool and send a common message. Condition (6) states that the Sender

[8]One caveat is in order. There can be an equilibrium where type 0 reveals herself and is indifferent between doing this and sending a signal that she is in the adjacent interval. We ignore this equilibrium,

types on the boundary of a partition element are indifferent between pooling with types immediately below or immediately above. Condition (7) states that R best responds to the information in S's message.

CS make another assumption that permits them to strengthen this result. For $t(n-1) \leq t(n) \leq t(n+1)$, let

$$V(t(n-1), t(n), t(n+1)) \equiv u^S(\bar{a}(t(n), t(n+1)), t(n)) - u^S(\bar{a}(t(n-1), t(n)), t(n)).$$

A (forward) solution to (6) of length L is a sequence $\{t(0), \ldots, t(L)\}$ such that $V(t(n-1), t(n), t(n+1)) = 0$ for $0 < n < L$ and $t(0) < t(1)$.

Definition 1. *The cheap-talk game satisfies the Regularity Condition (RC) if for any two solutions to (6) of length L, $\hat{t}$ and $\tilde{t}$ with $\hat{t}(0) = \tilde{t}(0)$ and $\hat{t}(1) > \tilde{t}(1)$, then $\hat{t}(n) > \tilde{t}(n)$ for all $n \geq 2$.*

(RC) is satisfied by the leading "uniform-quadratic" example in CS, which has been the focus of many applications. CS prove that if (RC) holds, then there is exactly one equilibrium partition for each $N = 1, \ldots, N^*$, and the ex-ante equilibrium expected utility for both S and R is increasing in N. These results provide an argument for the salience of the N^* equilibrium.

Another argument in support of this equilibrium outcome requires a definition.

Definition 2. An equilibrium (b^*, s^*) satisfies the *No Incentive to Separate* (NITS) Condition if $u^S(b^*(m(1)), 0) \geq u^S(a^R(0), 0)$.

NITS states that the lowest type of Sender prefers her equilibrium payoff to the payoff she would receive if the Receiver knew her type (and responded optimally). The definition extends naturally to cutpoint games: An equilibrium (a^*, t^*) of the cutpoint game satisfies NITS if $u^S(a^*(1), 0) \geq u^S(a^R(0), 0)$.

Chen, Kartik, and Sobel [6] show that the babbling equilibrium satisfies NITS if and only if $N^* = 1$ and only the (essentially unique[9]) equilibrium type-action distribution with N^* actions induced satisfies NITS when (RC) holds.

4.2 A Composed Best-Response Mapping

In this section we describe a best-response mapping studied by Gordon [10] and [11]. The mapping is essentially the composition of best-response functions of the Sender and the Receiver. The domain of the mapping is the set of cutpoints (the Sender's strategy space in the cutpoint game). To each strategy t in the cutpoint game we associate the Receiver's best response, which is the action list $\bar{a} = (\bar{a}(t(0), t(1)), \bar{a}(t(1), t(2)), \ldots, \bar{a}(t(L-1), t(L)))$. Using this action list we compute another cutpoint strategy, which is the Sender's best response to $\bar{a}$. We need to take care to define the mapping unambiguously when the Sender has multiple best responses.

since the set of actions it induces is identical to those in another equilibrium where type 0 instead pools with the adjacent interval. If we do so, then $t(1) > t(0)$ and so equilibria can be characterized by a strictly increasing sequence that solves (6) and the boundary conditions.

[9] Uniqueness fails because type $t_N(n)$ is indifferent between messages n and $n+1$ for $n = 1, \ldots, N-1$.

We define this mapping without assuming positive bias. The facts that we state in this section hold for positive bias. Let $t \in T_L$. Let $n_1 = 0$ if, for all $n = 1, \ldots, L-1$,

$$u^S(\bar{a}(t(n-1), t(n)), 0) \geq u^S(\bar{a}(t(n), t(n+1)), 0), \tag{8}$$

and otherwise let

$$n_1 = \max\{n : u^S(\bar{a}(t(n-1), t(n)), 0) < u^S(\bar{a}(t(n), t(n+1)), 0)\}.$$

If (8) holds, then the type $\theta = 0$ Sender prefers $\bar{a}(t(0), t(1))$ to $\bar{a}(t(n), t(n+1))$ for any $n = 1, \ldots, L-1$. If (8) does not hold, then the type 0 Sender's preferred action from $\{\bar{a}(t(n'), t(n'+1))\}$ for $n' = 1, \ldots, L-1$ must be $\bar{a}(t(n), t(n+1))$ for $n \geq n_1$.

Similarly, let $n_2 = L$ if for all $n = 1, \ldots, L-1$,

$$u^S(\bar{a}(t(n-1), t(n)), 1) \leq u^S(\bar{a}(t(n), t(n+1)), 1), \tag{9}$$

and otherwise let

$$n_2 = \min\{n : u^S(\bar{a}(t(n-1), t(n)), 1) > u^S(\bar{a}(t(n), t(n+1)), 1)\}.$$

If (9) holds, then the type $\theta = 1$ Sender prefers $\bar{a}(t(L-1), t(L))$ to $\bar{a}(t(n), t(n+1))$ for any $n = 0, \ldots, L-2$. When we assume positive bias, $n_2 = L$.

It must be the case that $n_2 > n_1$. If $n_2 - n_1 > 1$, then for each $n = 1, \ldots, n_2 - n_1 - 1$, if $\bar{a}(t(n_1+n-1), t(n_1+n)) \neq \bar{a}(t(n_1+n), t(n_1+n+1))$, then let $t'(n)$ be the unique solution to

$$u^S(\bar{a}(t(n_1+n-1), t(n_1+n)), t'(n)) = u^S(\bar{a}(t(n_1+n), t(n_1+n+1)), t'(n)); \tag{10}$$

if $\bar{a}(t(n_1+n-1), t(n_1+n)) = \bar{a}(t(n_1+n), t(n_1+n+1)) = a$, then let

$$t'(n) = \begin{cases} 0 & \text{if } a \leq a^S(0) \\ (a^S)^{-1}(a) & \text{if } a \in (a^S(0), a^S(1)) \\ 1 & \text{if } a \geq a^S(1) \end{cases}, \tag{11}$$

and let

$$D(t) = \begin{cases} (0, t'(1), \ldots, t'(n_2 - n_1 - 1), 1) & \text{if } n_2 - n_1 > 1 \\ (0, 1) & \text{if } n_2 - n_1 = 1. \end{cases}$$

Equation (10) states $t'(n)$ is indifferent between inducing $\bar{a}(t(n_1+n-1), t(n_1+n))$ and $\bar{a}(t(n_1+n), t(n_1+n+1))$. Equation (11) selects $t'(n)$ in degenerate cases to guarantee that it serves as a cutpoint.

It follows that $D : T \to T$ is well defined.

D associates with each strategy of S another strategy for S. The association can be viewed as a composition of best responses: S's strategy t determines actions $\bar{a}(t(n-1), t(n))$ (a best response of R to t); $D(t)$ is then the Sender's best response to R's strategy. With this interpretation, the next result is intuitive and straightforward to confirm:

Fact 1. *If $t^* \in T_L$ and $D(t^*) = t^*$, then t^* is an equilibrium partition. Furthermore, if $a^S(\theta) > a^R(\theta)$ for all θ, then L is bounded and $t^*(L-1) < t^*(L)$. If $a^S(\theta) < a^R(\theta)$ for all θ, then L is bounded and $t^*(0) < t^*(1)$. If $t^* \in T_L$ is an equilibrium partition and $0 = t^*(0) < t^*(1) < \cdots < t^*(L-1) < t^*(L) = 1$, then $D(t^*) = t^*$.*

Fact 1 essentially states that the fixed points of D are the equilibria of the game. There is a technicality that becomes important in the analysis. If $t(n-1) = t(n)$, then the n^{th} interval determined by t is degenerate. Any value of $a(n)$ can be part of the best response to t. Our specification, $a(n) = a^R(t(n))$, makes a natural selection from the set of best responses, but multiple best responses are possible when t induces degenerate intervals. Similarly, if $a(n-1) = a(n)$, then S will typically have multiple best responses to a. If t is nondegenerate ($t(n-1) < t(n)$ for all n), then R's best response will satisfy $a(1) < \cdots < a(L)$. In this case $D(t)$ is uniquely determined as S's best response to R's best response to t.

For $t, t' \in T_L$, say $t \leq t'$ if $t(n) \leq t'(n)$ for all n; $t < t'$ if $t \leq t'$ and $t \neq t'$; $t \ll t'$ if $t(n) < t'(n)$ for all $n = 1, \ldots, L-1$.

$G : T_L \to T_L$ is nondecreasing if $t < t'$ implies $G(t) \leq G(t')$ and strictly increasing if $t < t'$ implies $G(t) < G(t')$.

A sequence $\{t_k\}$ with $t_k \in T_L$ for $k = 1, 2, \ldots,$ is nondecreasing (resp. increasing, strongly increasing) if $t_k \leq t_{k+1}$ ($t_k <, \ll t_{k+1}$).

Fact 2. *The mapping $D(\cdot)$ is increasing on $\{t \in T_L : D(t) \in T_L\}$.*

Note that for $t \in T_L$, it is possible that $D(t) \in T_{L'}$ for $L' < L$. It is for this reason that we restrict D's domain. If we identify $t' \in T_{L'}$ with $t \in T_L$ with $t(n) = 0$ for $n < L - L'$, then D is nondecreasing on T_L.

Fact 3. *There is a maximal equilibrium (a^{Max}, t^{Max}) such that if (a, t) is an equilibrium, then $t \leq t^{Max}$. The order of (a^{Max}, t^{Max}) is N^*.*

Gordon [11] proves Fact 3. Gordon observes that T_L is a complete lattice and that all equilibria of the cutpoint game are fixed points of D when $L \geq N^*$. Hence, by Tarski's Fixed-Point Theorem, the set of fixed points of $D(\cdot)$ (and hence the set of equilibria) has a largest element. We refer to the equilibrium $(a^{\text{Max}}, t^{\text{Max}})$ as the largest equilibrium. The largest equilibrium will satisfy NITS. When (RC) holds, the largest equilibrium induces the only equilibrium type-action distribution with N^* non-empty steps. Our dominance arguments delete strategies greater than the largest equilibrium. Our convergence results identify processes with the property that if one starts with a Sender strategy greater than the largest equilibrium strategy, subsequent strategies converge to the largest equilibrium strategy.

Our selection results provide reasons why the largest equilibrium is distinguished. It has two related properties. First, we will show that t^{Max} is the largest strategy that survives iterated deletion of weakly dominated strategies. Second, we describe a family of adaptive processes defined on cutpoints with the property that if the initial condition is greater that t^{Max}, then the process converges monotonically to t^{Max}.

5 Characterization of Undominated Strategies

We provide a characterization of the set of strategies of the cutpoint game that survive deletion of weakly dominated strategies.

5.1 Deletion of Weakly Dominated Strategies

In this subsection we define iterated deletion of weakly dominated strategies (IDWDS) in the cutpoint game.

For a set $X \subset \mathbb{R}^l$, let $co(X)$ denote the convex hull of X. We identify the convex hull of a set of pure strategies with the associated set of mixed strategies.

The strategy $x_i' \in co(X_i)$ weakly dominates $x_i \in X_i$ if for all $x_j \in X_j$, $U^i(x_i', x_j) \geq U^i(x_i, x_j)$ and for some $x_j' \in X_j$, $U^i(x_i', x_j') > U^i(x_i, x_j')$. Here, $i, j = S$ or R, $i \neq j$. If $i = S$, then $X_i \subset T_M$; if $i = R$, then $X_i \subset A_M$.

A general procedure of deleting strategies produces a sequence of sets $\tilde{\mathcal{R}}_k$ and $\tilde{\mathcal{T}}_k$ such that:

1. $\tilde{\mathcal{R}}_0 = A_M$, $\tilde{\mathcal{T}}_0 = T_M$;
2. $\tilde{\mathcal{R}}_k$ is a subset of $\tilde{\mathcal{R}}_{k-1}$ obtained by deleting a subset of R's weakly dominated strategies in the game with strategy sets $(\tilde{\mathcal{R}}_{k-1}, \tilde{\mathcal{T}}_{k-1})$;
3. $\tilde{\mathcal{T}}_k$ is a subset of $\tilde{\mathcal{T}}_{k-1}$ obtained by deleting a subset of S's weakly dominated strategies in the game with strategy sets $(\tilde{\mathcal{R}}_{k-1}, \tilde{\mathcal{T}}_{k-1})$;
4. The sets $\tilde{\mathcal{R}}^* = \cap_{k \geq 0} \tilde{\mathcal{R}}_k$ and $\tilde{\mathcal{T}}^* = \cap_{k \geq 0} \tilde{\mathcal{T}}_k$ are well defined and non-empty.
5. There are no weakly dominated strategies in either $\tilde{\mathcal{R}}^*$ or $\tilde{\mathcal{T}}^*$.

The second and third conditions permit the deletion of only some weakly dominated strategies (and for deletions to be made simultaneously). The fourth condition guarantees that the limit of the process exists. This condition is satisfied if $\tilde{\mathcal{R}}_k$ and $\tilde{\mathcal{T}}_k$ are closed for all k. The fifth condition guarantees that the process continues as long as weakly dominated strategies remain.

In general, $(\tilde{\mathcal{R}}, \tilde{\mathcal{T}})$ depend on the order of deletion. In Appendix B we show that our main result does not depend on the order of deletion.[10]

Deletion of weakly dominated strategies is a powerful and general technique for refining predictions in games. Brandenburger, Friedenberg, and Keisler [4] analyze the epistemic foundations of a related procedure. Hillas and Samet [13] is a recent reconsideration of the foundations of weak dominance. Our main result provides a way to select a unique equilibrium outcome (eliminating both message indeterminacy and type-action indeterminacy) for simple cheap-talk games that satisfy the CS regularity condition.

[10] The result uses a slightly different definition of dominance, which we introduce in Appendix B.

5.2 Main Result

Theorem 1 is the main result of the paper.

Theorem 1. *The largest equilibrium (a^{Max}, t^{Max}) of the cutpoint game survives deletion of weakly dominated strategies. In any equilibrium that survives IDWDS, NITS holds.*

Chen, Kartik, and Sobel [6] show that a babbling equilibrium (only one action induced) survives NITS only if no informative equilibrium exists. The next result therefore follows from Theorem 1.

Corollary 1. *If $N^* > 1$, then the babbling equilibrium outcome does not survive IDWDS.*

Appendix A contains a proof of Theorem 1. Here we provide intuition and an outline of the argument.

Let $\theta^* = \min_\theta\{a^S(\theta) \geq a^R(1)\}$. It follows that $\theta^* < 1$ and for $\theta \in (\theta^*, 1]$, a type-θ Sender will always prefer the highest action that she can induce because R will never take an action greater than $a^R(1)$. Consequently, cutpoints for which $t(L-1) > \theta^*$ are weakly dominated. Hence IDWDS must remove some strategies of the Sender. In particular, IDWDS eventually restricts Sender to strategies in which $t(L-1) \leq \theta^*$. Once higher cutpoints have been removed, it will be weakly dominated for the Receiver to use a strategy in which $a(L) \geq \bar{a}(\theta^*, 1)$. Consequently, the set of responses to the highest element of a partition will be strictly less than $a^R(1)$ and more Sender types will prefer to be in the highest element of the partition. These observations demonstrate that IDWDS will discard strategies and that iterating the process will delete more strategies. So there is some hope that IDWDS can refine the set of predictions. Note that these heuristic arguments make use of weak (rather than strong) dominance and iteration.

The proof of the theorem constructs two sequences of strategy profiles that bound the set of strategies that survive IDWDS. The sequence of upper bounds, $\{a_k^*, t_k^*\}$, $k = 0, \ldots,$ starts with the largest strategy profile and then generates a sequence by taking best responses. The key insight is that any strategy that is not less than or equal to the Sender's (largest) best response to the Receiver's largest remaining strategy is weakly dominated and, likewise, any strategy that is not less than or equal to the Receiver's (largest) best response to the Sender's largest remaining strategy is weakly dominated. The sequence of best responses will be a monotonically decreasing sequence. Standard arguments guarantee that the limit exists and is an equilibrium. The monotonicity of the best-response mapping guarantees (when the initial strategy profile is greater than the largest equilibrium) that the limit is, in fact, the largest equilibrium. The sequence of lower bounds, $\{a_k^{**}, t_k^{**}\}$, $k = 0, \ldots,$ is constructed analogously. Hence strategies that survive IDWDS are sandwiched between the limit of the upper bounds ((a^*, t^*)) and the limit of the lower bounds ((a^{**}, t^{**})). An important observation is that the limit of this sequence must satisfy the NITS condition. Consequently, both limit equilibria satisfy NITS. It follows that if there is only one type-action distribution that satisfies NITS, the limit of the upper bounds is equal to the limit of the lower bounds and satisfies NITS.

To see that the limits of the sequences satisfy NITS, we show that the limit of the sequence of upper bounds is the largest equilibrium. Loosely speaking, this result follows because the sequence of upper bounds starts above the largest equilibrium and cannot jump below it. This guarantees that the limit of the upper bounds satisfies NITS.

The conclusion that the limit of the lower bounds satisfies NITS follows from three claims: (1) the Sender's limit cutpoint strategy has no more than N^* partition elements with nonempty interior; (2) only the lowest partition elements have empty interior; (3) responses to the lowest partition elements are equal to $a^R(0)$. The first claim follows because best responses to partition elements with nonempty interiors must be distinct and the Receiver can take no more that N^* actions in any equilibrium. We prove the second claim by showing by induction that if $t^{**}_{k'}(n) > t^{**}_{k'}(n-1)$ for some n, then $t^{**}_k(n') > t^{**}_k(n'-1)$ for all $n' > n$ and that $\lim_{k\to\infty} t^{**}_k(n) - t^{**}_k(n-1) > 0$. Provided that $M > N^*$, Claims (1) and (2) guarantee that $t^{**}(n) = 0$ for $n > 0$, which implies Claim (3). Claim (3) guarantees that $u^S(a^{**}(1), 0) \geq u^S(a^R(0), 0)$, consequently NITS holds.

Specifying these sequences is easy when best responses are unique, but require making choices when there are multiple best responses. There may be multiple cutpoint strategies that respond optimally to a strategy a_k if $a_k(n-1) = a_k(n)$ for some n. For the sequence of upper bounds, we select the largest best response. The Receiver may have multiple best responses to messages sent with probability zero (n such that $t_k(n-1) = t_k(n)$). For the decreasing sequence, we select the largest best responses whenever $t_k(n) > 0$ to preserve monotonicity and we require that off-path actions cannot decrease. The appendix describes the definition in detail. It is intuitive that if we wish to use the sequence $\{a^*_k, t^*_k\}$ as an upper bound to undominated strategies then it is sensible to look at largest best responses.

The formal proof of the argument sketched above relies on deleting weakly dominated strategies in a particular order. To show that the result does not depend on the order, we argue that the procedure will always reach a point at which if a strategy t_k not greater than or equal to t^*_k has not been deleted, then there exists a strategy that has not been deleted that dominates t_k and that a dominating strategy will remain as long t_k has not been deleted. Combined with a similar property bounding the remaining strategies of the Receiver, we conclude that deletion of weakly dominated strategies always removes strategies that are not greater than or equal to t^*.

Due to the correspondence between monotonic cheap-talk games and cutpoint games, we can interpret Theorem 1 as a selection result for monotonic cheap-talk games. It demonstrates that iterative deletion of weakly dominated strategies leads to an equilibrium selection in the monotonic cheap-talk game. The largest equilibrium outcome must be an outcome that induces the maximal number of actions; this is the unique equilibrium type-action distribution for games that satisfy (RC). In contrast to the characterization result in Crawford and Sobel [7] (in which one cannot draw any inference about how R interprets messages in equilibrium), the proof of Theorem 1 demonstrates that the use of messages is not arbitrary. Provided that there are enough messages ($M > N^*$), the Sender does not use the lowest messages.

When there is a unique outcome that satisfies NITS (for example, when (RC) holds), the process of iterative deletion of weakly dominated strategies selects the outcome without the need to invoke equilibrium.

Corollary 2. *If there exists a unique equilibrium that satisfies NITS, then there is a* *unique outcome that survives deletion of weakly dominated strategies of the monotonic cheap-talk game. This outcome is the NITS equilibrium outcome.*

6 Convergence Results

This section shows that, subject to mild initial conditions, the limits of a class of dynamic processes must lie in the set of strategies that survive IDWDS. It follows that when there is a unique type-action distribution that satisfies NITS the processes converge to an equilibrium that satisfies NITS. The class of dynamic processes includes best-response dynamics.

Subsection 6.1 describes the results using the cutpoint mapping. Subsection 6.2 introduces the class of dynamic processes and some useful notation. Subsection 6.3 states the result and describes the logic behind the proof. Appendix C contains the proof.

6.1 Stability of the Composed Best-Response Mapping

We begin this section by describing some convergence results in terms of the mapping $D(\cdot)$. We present a discussion in terms of learning later in the section.

The mapping $D(\cdot)$ suggests an adaptive dynamic process. Starting from an initial t, follow the path of iterates of $D(\cdot)$: $t, D(t), D^2(t), \ldots$. The process tracks best-response dynamics for the cutpoint game. A limit of this sequence must be an equilibrium of the cutpoint game because the fixed points of $D(\cdot)$ are the equilibria of the cutpoint game. We claim that there is a sense in which the maximal equilibrium is the natural limit of the process. Suppose that the initial condition is of the form $t^U = (0, 1, \ldots, 1)$. This initial condition is highly degenerate (in that all but one of the intervals it describes are equal to $\{1\}$). It is clear that t^U is the largest strategy for the Sender. The monotonicity of $D(\cdot)$ guarantees that the sequence $\{t^U, D(t^U), \ldots, D^k(t^U), \ldots\}$ is monotonically decreasing and bounded below by the largest equilibrium cutoffs. It therefore converges and the limit must be greater than or equal to t^{Max}. The limit point is a fixed point of $D(\cdot)$ and hence it must be equal to the largest equilibrium. This leads to Proposition 1.

Proposition 1. *The sequence $\{t^U, D(t^U), \ldots, D^k(t^U), \ldots\}$ converges down to t^{Max}.*

One can also consider the sequences that start from a low initial condition: $t^B = (0, 0, 0, \ldots, 1)$. It is possible to show that starting from this initial condition, iterates of D converge to an equilibrium and that equilibrium also satisfies NITS. Because D is monotonic, this means that any time the iterates converge, they will converge to something between the limit starting from t^B and the limit starting from t^U. This leads to the following two results.

Proposition 2. *If $t \in T_M$ for $M \geq N^*$, then if $\{t, D(t), \ldots, D^k(t), \ldots\}$ converges, it converges to an equilibrium that satisfies NITS.*

Corollary 3. *If the game has a unique equilibrium that satisfies NITS and $t \in T_M$ for $M \geq N^*$, then $\{t, D(t), \ldots, D^k(t), \ldots\}$ converges to the largest equilibrium.*

6.2 Adaptive Dynamics

Section 6.1 describes limits of the best-response mapping. We wish to discuss stability in the cutpoint game under a larger set of adaptive dynamics.

We study adaptive processes in which players in the cutpoint game select best responses to weighted averages of past actions. These processes include best-response dynamics (best responding to the most recent strategy of the opponent), but also apply to the case in which players respond to longer histories. Permitting players to respond to non-trivial mixtures creates a problem. The Receiver's best response to a mixture of cutpoint strategies need not be monotonic. Assume that the Receiver selects a best response to a mixture of strategies of the form t_k, for $k = 1, \ldots, K$, placing weight q_k on t_k. It is possible that the best response to a mixture of intervals $(t_k(n-1), t_k(n))$ for $k = 1, \ldots, K$ is greater than the best response to a mixture of intervals $(t_k(n), t_k(n+1))$ for $k = 1, \ldots, K$. Example 1 in Section 2.2 illustrates how this problem arises.

Our interpretation of the cutpoint strategy t is "types in $(t(n-1), t(n))$ prefer the n^{th} smallest action." That is, the Sender best responds to an ordered list. Hence we deal with the problem that the Receiver's best responses need not be monotonic by reordering them. The possibility that partition elements determined by a cutpoint strategy may have empty interiors does not create problems for the analysis. The possibility that the Receiver's best response to interval n is less than his best response to interval $n+1$ means that it is possible that the Receiver's response to interval n is equal to his response to interval $n+1$. This creates the possibility that the Sender is indifferent between inducing the n^{th} smallest action and the $(n+1)^{\text{th}}$ smallest action. When this happens, we need to specify how the Sender adjusts her strategy. We propose a specific rule below.

For any list of actions $(a(1), \ldots, a(L))$ define the ordered list $(\breve{a}(1), \ldots, \breve{a}(L))$. The ordered action list $\breve{a}$ permutes the actions in the set $\{a(1), \ldots, a(L)\}$ so that they are listed in non-decreasing order.

Definition 3. *A sequence $(a_k, t_k)_{k=1}^{\infty} \in A^M \times T_M$ and $a_0 \in [0,1]^M$ is an adaptive sequence with initial condition a_0 if*

1. $a_0(1) < a_0(2) < \cdots < a_0(n) < \cdots < a_0(M)$;

2. *t_k is a best response to a mixture of strategies $\breve{a}_j$ from periods $j < k$ and there exists C_k such that $\lim_{k\to\infty} C_k = \infty$ and the total probability placed on strategies from periods $j \leq C_k$ is 0;*

3. *a_k is a best response to a mixture of strategies t_j from periods $j \leq k$ and there exists C'_k such that $\lim_{k\to\infty} C'_k = \infty$ and the total probability placed on strategies from periods $j \leq C'_k$ is 0.*

In the definition, $a_k \in A^M$, which means that it is a list of actions; $\breve{a}_k \in A_M$, which is an ordered list of actions.

Adaptive sequences $(a_k, t_k)_{k=1}^{\infty}$ have the property that a_k is a best response to a mixture of t_j for $j \leq k$ and t_k is a best response to a mixture of $\breve{a}_j$ for $j < k$. We discuss the role of the bounds C_k and C'_k in a moment. In the adjustment process, the Receiver responds to past cutoff strategies of the Sender and the Sender responds to the past actions of the Receiver, with the modification that these actions are in increasing order. That is, we interpret a cutoff t as signaling "types in $(t(n-1), t(n))$ induce the n^{th} highest action." It is straightforward to show (and details are in Appendix C) that the Sender's best response can always be taken to be an interval partition (if type θ prefers to induce the n^{th} highest action to lower actions, then so does $\theta' > \theta$). Consequently it is possible to find an adaptive sequence given any initial condition. It could be the case that $\breve{a}_k(n) = \breve{a}_k(n+1)$. In this case, we need to add a condition to the definition of adaptive sequence. We make a selection from the set of best responses. Given $(a_k, t_k)_{k=1}^{\infty}$, we say that messages n and n' are equivalent at stage k if $\breve{a}_j(n) = \breve{a}_j(n')$ for all $j \geq C_k$ and we let $E_k = \{n :$ there exists $n' \neq n$ such that n and n' are equivalent at $k\}$. For $n \notin E_k$, $t_k(n)$ is uniquely defined by Condition 2. For $n \in E_k$, let $n_k^+ = \min\{t_k(n') : n' \notin E_k, n' > n\}$ ($n_k^+ = M$ if $n' \in E_k$ for all $n' > n$) and $n_k^- = \max\{t_k(n') : n' \notin E_k, n' < n\}$ ($n_k^- = 0$ if $n' \in E_k$ for all $n' < n$). It follows that $n_k^+ \geq n_k^-$. We say that an adaptive sequence is **regular**, if

$$t_k(n) = \begin{cases} n_k^- & \text{if } t_{k-1}(n) < n_k^- \\ n_k^+ & \text{if } t_{k-1}(n) > n_k^+ \\ t_{k-1}(n) & \text{if } t_{k-1}(n) \in [n_k^-, n_k^+] \end{cases} . \tag{12}$$

The regularity condition (12) states that when the Sender is indifferent, she only changes her cutpoint to preserve monotonicity.

The second and third conditions in Definition 3 simply state that strategies in an adaptive sequence respond optimally to a mixture of past strategies in the sequence. The requirement that $\lim_{k\to\infty} C_k(C'_k) = \infty$ means that we restrict attention to mixtures that eventually place zero weight on strategies used in initial periods. In best-response dynamics, players respond to the strategy that their opponent played in the previous period, which is a degenerate mixture that places zero weight on strategies from earlier periods. Hence best-response dynamics satisfies Conditions (2) and (3) of Definition 3. Fictitious play, in which agents respond to a uniform mixture of past strategies, does not satisfy the definition. We cannot extend our results to fictitious play for the following reason. We cannot rule out the possibility that some intervals $(t_k(n-1), t_k(n))$ converge to a point as k approaches infinity. If this happens, then for fictitious play, the Sender's behavior in early periods will determine the Receiver's response and limiting behavior will depend on initial conditions. We avoid this case by requiring that weights on early strategies must eventually be set equal to zero.

6.3 Convergence Result

Recall that t^* are the cutpoints associated with the largest equilibrium and t^{**} are the cutpoints associated with the smallest equilibrium.

Theorem 2. *Let $a_0 = (a_0(1), \ldots, a_0(M))$ for $M \geq N^*$ and $a_0(n) \neq a_0(n')$ for $n \neq n'$. Let (a_k, t_k) be a regular adaptive sequence with initial condition a_0. For any $\varepsilon > 0$, there exists K such that if $k > K$, then $t_k(n) \in (t_k^{**}(n-1) - \varepsilon, t_k^*(n) + \varepsilon)$. If the game has a unique equilibrium type-action distribution that satisfies NITS, then (a_k, t_k) converges to an equilibrium that satisfies NITS.*

The theorem makes two assumptions on initial conditions: a_0 must contain at least N^* actions and they must be distinct. If there is an equilibrium that does not satisfy NITS, then one can always set the initial actions equal to actions from an equilibrium that does not satisfy NITS. In this case, the adaptive sequence will remain at the initial condition rather than converge to an outcome that satisfies NITS.

Appendix D presents an example which demonstrates that if a game has more than one equilibrium that satisfy NITS, the play may not converge to the equilibrium with the highest number of induced actions even though the number of initially induced actions exceeds this number.

If there is a unique type-action distribution that satisfies NITS, then $t^{**} = t^*$. In this case the theorem guarantees convergence to an equilibrium and the limit equilibrium will satisfy NITS. If $t^{**} < t^*$, the theorem only states that all limit points of the process are contained in the set of cutpoints that survive IDWDS. Convergence to a particular equilibrium is not guaranteed. Olszewski [24] proves that best-response dynamics for cheap-talk games must converge to an equilibrium for cheap-talk games with a positive bias.

The assumption that the initial actions $a_0(m)$ are distinct guarantees that t_1 will be an interval partition, which combined with the regularity assumption, guarantees that the sequence is well defined.

We prove Theorem 2 by induction, taking advantage of bounds on strategies constructed in the proof of Theorem 1. In that proof, we construct sequences that bound the set of best responses and demonstrate that these bounds converge. We derive the bounds from shrinking strategy spaces by removing dominated strategies. In fact, the bounds are, when restricted to on-path behavior, respectively the highest and lowest best responses relative to surviving strategies. Using this fact, we show that any adaptive sequence must eventually stay within the bounds. The conclusion of Theorem 2 follows because we know that the upper and lower bounds are equal when there is a unique equilibrium outcome that satisfies NITS.

7 Other Biases

Thus far we have maintained the assumption that $a^S(\theta) > a^R(\theta)$ for all θ. Our analysis contains the elements needed to handle other cases. The case in which $a^S(\theta) < a^R(\theta)$ for all θ is completely symmetric. In all other cases, there will be θ such that $a^R(\theta) = a^S(\theta)$. Provided that $a^S(1) \geq a^R(1)$, D will be non decreasing on T_L (that is, Fact 2 does not require positive bias) and there will be a maximal equilibrium when D is restricted to partitions of size L (that is, the first line of Fact 3 does not require positive bias).

When $a^S(1) > a^R(1)$ we have seen that if t survives IDWDS, then $t(M-1) \leq \theta$ if $a^S(\theta) > a^R(1)$. That is, IDWDS determines the strategy of a set of positive measure of Sender types. Similarly, when $a^S(0) < a^R(0)$, $t(1) > 0$ for all undominated strategies. The next result makes this intuition more precise. In the statement, "largest" and "smallest" are defined over the set of interval partitions with exactly L elements.

Theorem 3. *Assume that $a^S(0) < a^R(0)$ and $a^S(1) > a^R(1)$. For each $L > 0$, the cutpoint game in which S's strategy set is T_L has a largest equilibrium $\bar{t}L$ such that $0 = \bar{t}^L(0) < \bar{t}^L(1) < \cdots < \bar{t}^L(L) = 1$ and a smallest equilibrium $\underline{t}^S$ such that $0 = \underline{t}^L(0) < \underline{t}^L(1) < \cdots < \underline{t}^L(L) = 1$. The largest and smallest equilibria survive iterative deletion of weakly dominated strategies and if t is another equilibrium strategy that survives iterative deletion of weakly dominated strategies, then $t(n) \in [\underline{t}^L(n), \bar{t}^L(n)]$ for all n.*

When the Sender has more extreme preferences than the Receiver, both low and high types of the Sender use extreme messages to signal their information. The process of deletion of weakly dominated strategies deletes all outcomes that do not use all messages. The outcomes must be informative when $L > 1$. Unlike the positive bias case of CS, in this case there exists an equilibrium in which the Sender induces infinitely many actions (see Gordon [10]). Hence, the restriction to a finite message space is a binding constraint. When (RC) holds, for each L there is exactly one set of L actions that can be induced in an equilibrium. If there are L messages, then the only strategy profile that survives deletion of weakly dominated strategies supports this equilibrium.

A qualitatively distinct case arises the bias is inward. When $a^S(0) > a^R(0)$ and $a^S(1) < a^R(1)$, extreme types may not use the most extreme messages. We can assert that whenever these messages are not used, the extreme type has no incentive to separate. (This is the standard NITS condition when $t = 0$ and the condition that $u^S(b^*(s^*(1)), 1) \geq u^S(a^R(1), 1)$ at $t = 1$.) Let $\underline{m}$ denote the lowest message and $\bar{m}$ denote the highest message.

Theorem 4. *Assume that $a^S(0) \geq a^R(0)$ and $a^S(1) \leq a^R(1)$. For each message space, there exists an equilibrium of the monotonic cheap-talk game that survived iterated deletion of weakly dominated strategies. All equilibria of the monotonic cheap-talk game are equilibria of the original game. Let (b^*, s^*) be an equilibrium of the monotonic cheap-talk game that survives iterated deletion of weakly dominated strategies. If $s^*(0) > \underline{m}$, then $u^S(b^*(s^*(0)), 0) \geq u^S(a^R(0), 0)$. If $s^*(1) < \bar{m}$, then $u^S(b^*(s^*(1)), 1) \geq u^S(a^R(1), 1)$.*

This result is limited because it does not guarantee that NITS holds at one end or the other. It does not even rule out pooling. Gordon [10] describes some properties of the equilibria to this game (in particular, there may or may not be a finite upper bound to the number of actions induced). Gordon [11] provides sharper predictions, but they have a similar qualitative flavor (NITS at one end).

8 Discussion

The literature contains different theoretical arguments that suggest why, under positive bias, the equilibrium with N^* actions is salient. Under their regularity condition, CS demonstrate that there is an essentially unique equilibrium outcome with N^* actions and that, under some conditions, this equilibrium is ex ante preferred to all other equilibria by both the Sender and the Receiver.

Mensch [21] notes that monotonicity restrictions in cheap-talk games can lead to the kind of selection that we describe. Mensch imposes a monotonicity condition on off-path beliefs and argues that this leads to a selection of equilibria that satisfy NITS.

Milgrom and Roberts [22] and Vives [28] study the class of supermodular games introduced by Topkis [26]. In a supermodular game, each player's strategy set is partially ordered and there are strategic complementarities that cause a player's best response to be increasing in opponents' strategies. Milgrom and Roberts [22] demonstrate that supermodular games have a largest and smallest equilibrium and that these extreme equilibria can be obtained by iterating the best-response correspondence. Our argument uses similar techniques. There are two differences. Our game is not a supermodular game. In particular, it does not satisfy the increasing difference condition of Milgrom and Roberts. In addition, Milgrom and Roberts study the implications of deletion of strictly dominated strategies. Our analysis uses weak dominance. Sobel [25] shows how Milgrom and Roberts's general arguments extend to a broader class of games and a more restrictive solution concept. He points out that the monotonic cheap-talk game satisfies a weak form of supermodularity that makes it possible to bound the set of strategies that survive deletion of weakly dominated strategies using arguments similar to ours. Sobel does not provide conditions under which the process leads to a unique prediction.

Words have commonly accepted meanings. When there are no conflicts of interest, it is natural to assume that agents will use words in conventional ways. In strategic situations, however, sophisticated agents will not take words at face value. Standard models of cheap talk abstract from the conventional meaning of words in order to focus on strategic problems. A limitation of this approach is that meaning is determined completely endogenously. An equilibrium type-action distribution determines the minimum number of distinct messages that the Sender must use, but does not specify which message is associated with which action. If there is to be a connection between the equilibrium use of messages and exogenous meaning, then we must impose additional assumptions. The literature has approached this issue in several ways.

Farrell [9] introduced the first attempt to refine the equilibrium set in cheap-talk games. Farrell's notion of neologism-proof equilibrium models the idea that messages have commonly accepted meanings and that players are able to use these statements provided that they were consistent with strategy constraints. This general idea does refine the set of equilibria in cheap-talk games, but lacks general existence properties.[11]

Chen [5] and Kartik [15] assume that the message space is equal to the type space, which suggests a natural correspondence between types and messages. They make this connection operational by modifying the game. Chen assumes that with positive probability the Sender sends a message equal to her type (and with positive probability the Receiver interprets the message literally). Kartik assumes that the Sender has a cost of "lying." These perturbations create an exogenous meaning for messages. In these models, the limits of equilibria in monotonic strategies as the perturbations vanish converge to an equilibrium that satisfies NITS.[12] Furthermore, the limit equilibrium involves the use of "inflated" messages. Hence these arguments are alternative ways to make the same selection that we make. Our result imposes the monotonicity condition directly on the game and makes a selection without perturbations.

Dilmé [8] also provides an argument that selects equilibrium outcomes with commu-

[11]In particular, typically no equilibrium is neologism-proof in the uniform-quadratic special case of the CS model.

[12]Chen, Kartik, and Sobel [6] introduce the NITS criterion, which we described in Section 4.

nication. Dilmé studies cheap-talk games in which payoffs are perturbed. He then looks for equilibria of the underlying game that are robust, where a robust equilibrium is close to some equilibrium in every nearby game. He shows that in games with a uniform bias satisfying the standard regularity condition, only the equilibrium with the maximal number of actions induced is robust. He extends this result to more general cheap-talk games. Dilmé selection generally coincides with the outcomes we select.[13] His approach has a superficial similarity to Chen, Kartik, and Sobel [6, Section 4.4], in that both operate by perturbing payoffs. Chen, Kartik, and Sobel study a particular kind of signaling cost introduced in Kartik [15] and impose an equilibrium refinement (restriction to monotone strategies), while Dilmé uses the freedom to specify signaling costs to attain a selection result. Dilmé approach is also related to solution concepts like strategic stability (Kohlberg and Mertens [17]) or truly perfect equilibria (Van Damme [27]) that require robustness with respect to a large family of perturbations. In addition to reaching similar conclusions, the source DIlmé's results is similar to ours. Both approaches exploit the fact that there are a limited number of specifications of off-path behavior that are consistent with equilibrium. For example, in a cheap-talk model in which the Sender is upward biased, equilibrium requires that off-path actions either agree with on-path actions or are strictly lower than the lowest on-path action. Furthermore, when a regularity condition holds, only the equilibrium with the maximal number of actions induced can be supported using low off-path responses. Dilmé's argument, like ours, operates by showing that some messages must lead to low off-path actions.

Antić and Persico [1] study a game in which the players make a costly investment can alter ideal points prior to playing a cheap-talk game. They study equilibria of the two-stage game that satisfy a forward-induction refinement. A fixed cheap-talk game can be viewed as a two-stage game in which players face infinite costs associated with changing their biases. Antić and Persico identify conditions on the underlying cheap-talk game and the investment-cost function that imply that only an outcome that satisfies NITS is the limit of refined equilibria of the two-stage game as the investment costs grow to infinity. This argument selects the same type-action distribution as Chen [5] and Kartik [15] by examining limits of equilibria, but the logic of the arguments appears to be different. Chen and Kartik perturb payoffs, while Antić and Persico perturb strategy spaces. Furthermore, Chen and Kartik's selection, like ours, results the message-indeterminacy problem while Antić and Persico do not.

Blume [2] and [3] propose refinements for finite cheap-talk games based on Kalai and Samet's [14] concept of persistent equilibrium. Blume [2] introduces perturbations to Sender strategies. The perturbations guarantee that there are no off-path messages. Unused messages cannot be interpreted arbitrarily. Rather, they take on exogenous meanings (messages are associated with subsets of types) independent of the strategic context. Blume [3] demonstrates that these perturbations to the Sender's messages determine the relationship between types and messages in the equilibria selected by his refinement. These perturbations, like the initial conditions in our dynamic arguments, solve the message-indeterminancy problem.

Olszewski [23] investigates the stability of equilibria in cheap-talk game with respect

[13] Dilmé does not resolve message indeterminacy.

to the introduction of new messages and shows through examples that this idea destabilizes "implausible" equilibria. The initial conditions of our adaptive processes act like new messages do in Olszewski's paper. Hence the approaches share the feature of investigating conditions under which the introduction of novel interpretations of messages (either through the addition of new message that the Receiver interprets randomly as in Olszewski or a rich initial condition that the Receiver responds to optimally as in our paper) and adaptive dynamics can select equilibria.

Lo [19] imposes restrictions on the set of strategies available to agents in a discrete cheap-talk game and then studies the outcomes that survive deletion of weakly dominated strategies.[14] Like Lo, we impose restrictions on strategies and study the implications of IDWDS. Our results differ from hers because we impose only the restrictions that messages are linearly ordered, that higher sender types send weakly higher signals and that the receiver takes weakly higher actions for higher signals. These restrictions do not eliminate any equilibrium outcomes of the original game. Lo makes further restrictions on the strategy space and shows that these can actually lead to outcomes that are not equilibria of the original game.

There are several criticisms of IDWDS. It is well known that, unlike deletion of strongly dominated strategies, the order of deletion may matter. In some games with large strategy spaces, equilibrium may fail to exist in weakly undominated strategies.[15] The process of eliminating weakly dominated strategies may introduce new equilibria. It also leads to strong predictions that are not behaviorally accurate in common settings (like the centipede game).

We show that the technical problems with IDWDS do not hold in our setting. We establish that the main result is independent of the order of deleting dominated strategies. Undominated best responses exist in our setting even though the game is not finite. The process introduces no new equilibrium outcomes.

[14]Lo [20] applies similar arguments to study cheap-talk extensions of games with complete information.

[15]A simple example is a first-price auction in which two surplus-maximizing agents bid for an item with known, common value. The only equilibrium of the game involves both players bidding the common value, but this strategy is weakly dominated by bidding less.

Appendix A

This appendix contains a proof of Theorem 1.

In the main text, we concentrated on the case in which $a^S(\theta) > a^R(\theta)$ for all θ. However, we stated results for other biases in Section 7. In order to establish these results, we do not make any assumption about the relationship between a^S and a^R in this appendix unless we state it explicitly.

We introduce a specific order for deleting dominated strategies. In Appendix B we show that the result does not depend on the choice of order.

$\mathcal{T}_0 = \{t = (t(0), t(1), \ldots, t(M)) : 0 = t(0) \le t(1) \cdots \le t(M) = 1\}$ and $\mathcal{R}_0 = \{a = (a(1), a(2), \ldots, a(M)) : 0 \le a(1) \le \cdots \le a(M) \le 1\}$.

For all $k \ge 1$, form $\mathcal{T}_k$ and $\mathcal{R}_k$ inductively as follows:

$$\begin{aligned}\mathcal{R}_k = \{a_k \in \mathcal{R}_{k-1} \quad : \quad & \nexists \alpha_k \in co(\mathcal{R}_{k-1}) \text{ s.t.}\\ & U^R(\alpha_k, t_{k-1}) \ge U^R(a_k, t_{k-1}) \text{ for every } t_{k-1} \in \mathcal{T}_{k-1}\\ & U^R(\alpha_k, t_{k-1}) > U^R(a_k, t_{k-1}) \text{ for some } t_{k-1} \in \mathcal{T}_{k-1}\}\end{aligned}$$

$$\begin{aligned}\mathcal{T}_k = \{t_k \in \mathcal{T}_{k-1} \quad : \quad & \nexists \tau_k \in co(\mathcal{T}_{k-1}) \text{ s.t.}\\ & \tau_k(t) = 0 \text{ if } t(k+1) < 1\\ & U^S(a_k, \tau_k) \ge U^S(a_k, t_k) \text{ for every } a_k \in \mathcal{R}_k\\ & U^S(a_k, \tau_k) > U^S(a_k, t_k) \text{ for some } a_k \in \mathcal{R}_k\}\end{aligned}$$

Thus, the order of deletion is that in any round $k \ge 1$, we first delete all weakly dominated strategies for the Receiver (using Sender strategies that remain from the previous round), then we delete some strategies that are weakly dominated strategies for the Sender relative to the remaining strategies of the Receiver, and then continue. In the first M rounds we do not necessarily delete all of the weakly dominated strategies of the Sender because we require that the dominating strategy of the Sender place positive probability only on strategies satisfying $t(k+1) = 1$. This additional restriction is convenient for our construction. The constraint that the Sender set high cutpoints equal to one is not binding when $k \ge M - 1$. We emphasize that the main result does not depend on this choice of order.

Let $a_0^* = (1, 1, \ldots, 1)$ and $t_0^* = (0, 1, 1, \ldots, 1)$. Given a_0^* and t_0^*, define a_k^* and t_k^* inductively. Let

$$a_k^*(M) = \begin{cases} a^R(1) & \text{if } t_{k-1}^*(M-1) = 1\\ \bar{a}(t_{k-1}^*(M-1), 1) & \text{if } t_{k-1}^*(M-1) < 1\end{cases}$$

and, after defining $a_k^*(n')$ for $n' = n+1, \ldots, M$,

$$a_k^*(n) = \begin{cases} \min\{a_{k-1}^*(n), a_k^*(n+1)\} & \text{if } t_{k-1}^*(n-1) = t_{k-1}^*(n)\\ \bar{a}(t_{k-1}^*(n-1), t_{k-1}^*(n)) & \text{if } t_{k-1}^*(n-1) < t_{k-1}^*(n).\end{cases}$$

In particular $a_1^* = (\bar{a}(0,1), a^R(1), \ldots, a^R(1))$. The fact that a_k^* is a best response to t_{k-1}^* is clear for those n such that $t_{k-1}^*(n-1) < t_{k-1}^*(n)$. If $t_{k-1}^*(n-1) = t_{k-1}^*(n)$, best responding places no restrictions on $a_k^*(n)$. The specification guarantees that $a_k^*(n) \leq a_k^*(n+1)$.

Let $t_k^*(M) = 1$ and $t_k^*(0) = 0$. For $n = 1, \ldots, M-1$, after defining $t_k^*(n')$ for $n' = n+1, \ldots, M$, let $t_k^*(n) = t_k^*(n+1)$ if $a_k^*(n') = a_k(n'+1)$ for $n' = n+1, \ldots M$ and otherwise let

$$t_k^*(n) = \begin{cases} t_k^*(n+1) & \text{if } u^S(a_k^*(n), 1) \geq u^S(a_k^*(n+1), 1) \\ 0 & \text{if } u^S(a_k^*(n), 0) \leq u^S(a_k^*(n+1), 0) \\ \min\{\theta : u^S(a_k^*(n), \theta) = u^S(a_k^*(n+1), \theta) & \text{otherwise.} \end{cases}$$

Given a_0^*, t_0^* satisfies this rule for $k = 0$. Hence it is not necessary to define t_0^* independently.

The specification guarantees that t_k^* is a best response to a_k^*. If $u^S(a_k^*(n), 1) \geq u^S(a_k^*(n+1), 1)$[16] and $a_k^*(n) < a_k^*(n+1)$ it must be that $u^S(a_k^*(n'), 1) \geq u^S(a_k^*(n'+1), 1)$ for $n' > n$ and so $t_k^*(n') = 1$ for $n' \geq n$ and this is a best response. If $u^S(a_k^*(n), 0) \leq u^S(a_k^*(n+1), 0)$ and $a_k^*(n) < a_k^*(n+1)$, then no type would want to induce $a_k^*(n')$ for $n' \leq n$. If $u^S(a_k^*(n), \theta) = u^S(a_k^*(n+1), \theta)$ and $a_k^*(n) < a_k^*(n+1)$, then types greater than θ will strictly prefer to induce an action $a_k^*(n')$ to $a_k^*(n)$ for some $n' > n$ and types less than θ will strictly prefer $a_k^*(n)$ to any $a_k^*(n')$ for $n' > n$. If $a_k^*(n) = a_k^*(n+1)$, then there are multiple possible best responses. Our specification of t^* breaks indifference in different ways. It specifies higher cutoffs (so that the Sender uses a lower message when indifferent) if all higher messages lead to the same action. Otherwise, it selects lower cutoffs. The tie-breaking rules guarantee that if $t_k^*(n) = 0$, then $t_k^*(n') = 0$ for all $n' < n$, and if $t_k^*(n) = 1$, then $t_k^*(n') = 1$ for all $n' > n$.

The specification of t_k^* also guarantees that $t_k^*(k+1) = 1$.

We construct the sequence (a_k^*, t_k^*), by starting with an initial a_0^* and then letting t_k^* be the best response to a_k^* and a_{k+1}^* be the best response to t_k^*. The definition is complicated because in some situations, best responses are not unique. The Sender will have more than one best response to a_k if there exists n such that $a_k(n) = a_k(n+1)$. In our specification, we assume that t_k^* and a_k^* are effectively the highest best responses. The tie breaking rules serve to make a_k^* and t_k^* essentially upper bounds to strategies remaining at stage k of the deletion process. Claim 1 below makes this statement precise.

We can now describe the important properties of the sequence $\{a_k^*, t_k^*\}$.

For each k let $\underline{i}_k = min\{n : t_k^*(n) < t_k^*(n+1)\}$ and $\bar{i}_k = max\{n > 0 : t_k^*(n-1) < t_k^*(n)\}$. It follows that

$$\underline{i}_k = \max\{n : t_k^*(n) = 0\} \text{ and } \bar{i}_k = \min\{n : t_k^*(n) = 1\}. \tag{13}$$

Claim 1. *For any $k \geq 0$,*

1. $t_k^*(n) < t_k^*(n+1)$ *for* $n \in [\underline{i}_k, \bar{i}_k)$; $a_{k+1}^*(n+1) > a_{k+1}^*(n)$ *for* $n \in (\underline{i}_k, \bar{i}_k]$;
2. $\underline{i}_k \leq \underline{i}_{k+1}$; $\bar{i}_k \leq \bar{i}_{k+1} \leq \bar{i}_k + 1$;
3. $a_{k+1}^*(n) \leq a_k^*(n)$;

[16] This condition can hold only if $a^S(1) \leq a^R(1)$.

4. $a^*_{k+1}(n) = a^R(1)$ *for* $n > \bar{i}_k$; $a^*_{k+1}(n) \le a^S(0)$ *for* $n \le \underline{i}_k$;

5. $t^*_{k+1}(n) \le t^*_k(n)$;

6. *For all* $a_k \in \mathcal{R}_k$, $a^*_k(n) \ge a_k(n)$ *for* $n \ge \min\{\underline{i}_k, \underline{i}_{k-1}+1\}$;

7. $a^*_k \in \mathcal{R}_k$;

8. *For all* $t_k \in \mathcal{T}_k$, $t_k \le t^*_k$.

9. $t^*_k \in \mathcal{T}_k$.

A careful examination of the proof reveals that the first five parts of the claim, which establish properties of a^*_k and t^*_k do not require the final four parts, which establish dominance properties.

Proof. We prove the result by induction. Assume that $k = 0$. The first part of Part (1) follows because $\underline{i}_0 = 0, \bar{i}_0 = 1$, and $t^*_0(0) = 0 < 1 = t^*_0(1)$ and the second part follows because $a^*_1(1) = a^R(0,1) < a^R(1) = a^*_1(2)$. Part (2) follows by the definitions of $\underline{i}_k$ and $\bar{i}_k$ because $\underline{i}_0 = 0$ and $\bar{i}_0 = 1$ are minimal values for $\underline{i}_k$ and $\bar{i}_k$. The definition of t^*_k implies that $t^*_1(2) = 1$; therefore $\bar{i}_1 \le 2$. Next we prove Part (4). If $n > \bar{i}_0$, then $t^*_0(n-1) = t^*_0(n) = 1$. Consequently, if $n = M$, then $a^*_1(n) = a^R(1)$ by definition. Otherwise, $a^*_1(n) = \min\{a^*_0(n), a^*_1(n+1)\}$, which implies the result by induction (decreasing n from M) because $a^*_0(n) = 1$. This establishes the first sentence of Part (4). Because $\underline{i}_0 = 0$ the second part of Part (4) is vacuous. Parts (3)-(9) hold because a^*_0 is the maximum strategy in $\mathcal{R}_0$ and t^*_0 is the maximum strategy in $\mathcal{T}_0$.

Now suppose that for $k \ge 1$, the claim is true for $j = 0, \dots, k-1$.

Part (1). We prove the first clause first.

If $\underline{i}_k + 1 = M$, then $t^*_k(\underline{i}_k) = 0 < 1 = t^*_k(\bar{i}_k)$. Consequently, the result follows.

Otherwise, $\underline{i}_k + 1 < M$. If $\underline{i}_k = 0$, then $t^*_k(0) < t^*_k(1)$ by definition. We therefore need only prove the first part of Part (1) for $n \ge \max\{1, \underline{i}_k\}$. Assume that $n \ge \max\{1, \underline{i}_k\}$. Part (2) implies that $\underline{i}_k \ge \underline{i}_{k-1}$. Therefore, by Part (1) (for $k-1$), $a^*_k(\underline{i}_k+2) > a^*_k(\underline{i}_k+1)$ (note that $\underline{i}_k+1 < M$ implies that $a^*_k(\underline{i}_k+2)$ is well defined). We claim that

$$a^*_k(\underline{i}_k + 2) > a^S(0). \tag{14}$$

To see this, assume that $a^*_k(\underline{i}_k + 2) \le a^S(0)$ and argue to a contradiction. Because $a^*_k(\underline{i}_k+2) > a^*_k(\underline{i}_k+1)$, all types would strictly prefer $a^*_k(\underline{i}_k+2)$ to $a^*_k(\underline{i}_k+1)$, which would imply that $t^*_k(\underline{i}_k+1) = t^*_k(\underline{i}_k) = 0$, contrary to the definition of $\underline{i}_k$. Hence (14) holds.

To see that $a^*_k(\bar{i}_k - 1) < a^S(1)$, assume $a^*_k(\bar{i}_k - 1) \ge a^S(1)$ and argue to a contradiction. If $a^*_k(\bar{i}_k - 1) \ge a^S(1)$ and $a^*_k(\bar{i}_k) > a^*_k(\bar{i}_k - 1)$, then all types strictly prefer $a^*_k(\bar{i}_k - 1)$ to $a^*_k(\bar{i}_k)$, which would imply that $t^*_k(\bar{i}_k - 1) = t^*_k(\bar{i}_k)$, contrary to the definition of $\bar{i}_k$. If $a^*_k(\bar{i}_k - 1) \ge a^S(1)$ and $a^*_k(\bar{i}_k) = a^*_k(\bar{i}_k - 1)$, then $a^*_k(n) = a^R(1)$ for $n \ge \bar{i}_k - 1$, so $t^*_k(\bar{i}_k - 1) = t^*_k(\bar{i}_k)$, which contradicts the definition of $\bar{i}_k$. It follows that for each $n \in (\underline{i}_k + 1, \bar{i}_k)$, $a^*_k(n) \in (a^S(0), a^S(1))$. For $n \in (\underline{i}_k + 1, \bar{i}_k)$ there is a unique t_n such that $a_k^*(n) = a^S(t_n)$ and $a_k^*(n') \neq a^S(t_n)$ for $n' \neq n$. Consequently $t_n \in (t_k^*(n-1), t_k^*(n))$, which implies $0 < t_k^*(\underline{i}_k+1) < t_k^*(\underline{i}_k+2) < \cdots < t_k^*(\bar{i}_k-1) < 1$. Because $t_k^*(\underline{i}_k) = 0$ and $t_k^*(\bar{i}_k) = 1$, the first statement in Part (1) is true.

It follows from the first sentence of Part (1) that $\bar{a}(t_k^*(n), t_k^*(n+1))$ is strictly increasing in n for $n \in [\underline{i}_k, \bar{i}_k)$ and the definition of a_{k+1}^* therefore implies that $a_{k+1}^*(n)$ is strictly increasing for $n \in (\underline{i}_k, \bar{i}_k]$. Furthermore, $a_{k+1}^*(\bar{i}_k - 1) = \bar{a}(t_k^*(\bar{i}_k - 1), t_k^*(\bar{i}_k)) < a^R(1)$, while $a_{k+1}^*(n) \geq a^R(1)$ for $n \geq \bar{i}_k$, which establishes the second clause in Part (1).

Part (2). (13) and Part (5) (for $k-1$) establish that $\underline{i}_k \leq \underline{i}_{k+1}$ and $\bar{i}_k \leq \bar{i}_{k+1}$. Finally, to show $\bar{i}_{k+1} \leq \bar{i}_k + 1$, note that $a_{k+1}^*(n) = a^R(1)$ for $n > \bar{i}_k$ and $t_{k+1}^*(\bar{i}_k + 1) = 1$ by the definition of t_k^*.

Part (3). Part (3) follows directly from the definition of a_k^* when $n \leq \underline{i}_k$. It follows from Parts (1) and (5) (for $k-1$) when $n \in (\underline{i}_k, \bar{i}_k]$. It follows from Part (5) when $n > \bar{i}_k$.

Part (4). If $n > \bar{i}_k$, then $t_k^*(n-1) = t_k^*(n) = 1$. Consequently, if $n = M$, then $a_{k+1}^*(n) = a^R(1)$ by definition. Otherwise, $a_{k+1}^*(n) = \min\{a_k^*(n), a_k^*(n+1)\}$, which implies the result by induction (decreasing n from M) because $a_k^*(n) = a^R(1)$ for $n > \bar{i}_{k-1}$ by Part (4) $(k-1)$ and $n > \bar{i}_k$ implies $n > \bar{i}_{k-1}$ by Part (2). This establishes the first clause of Part (4). To establish the second clause, we may assume $\underline{i}_k > 0$ as $a_k^*(n)$ is not defined for $n = 0$. If $\underline{i}_{k-1} > 0$, then

$$a_k^*(\underline{i}_k) \leq a_k^*(\underline{i}_{k-1}) \leq a^S(0),$$

where the first inequality follows from Part (2) $(k-1)$ and $\underline{i}_k \leq \underline{i}_{k-1}$ and the second inequality follows from the induction hypothesis. If $\underline{i}_{k-1} = 0$, then $\underline{i}_{k-1} < \underline{i}_k < \bar{i}_k \leq \bar{i}_{k-1} + 1$ and thus Part (1) implies that $a_k^*(\underline{i}_k) < a_k^*(\underline{i}_k + 1)$. If $a_k^*(\underline{i}_k) > a^S(0)$, then type 0 would strictly prefer $a_k^*(\underline{i}_k)$ to $a_k^*(\underline{i}_k + 1)$, which means that $t_k^*(\underline{i}_k) > 0$, in contradiction to the definition of $\underline{i}_k$. Consequently $a_k^*(\underline{i}_k) \leq a^S(0)$.

Part (5). t_k^* is a best response to a_k^*. It follows that

$$u^S(a_k^*(n), t_k^*(n)) = u^S(a_k^*(n+1), t_k^*(n)). \tag{15}$$

If $a_k^*(n) < a_k^*(n+1)$, then (15) and Part (3) imply that

$$u^S(a_{k+1}^*(n), t_k^*(n)) \leq u^S(a_{k+1}^*(n+1), t_k^*(n)).$$

Hence when $a_k^*(n) < a_k^*(n+1)$ and $a_{k+1}^*(n) < a_{k+1}^*(n+1)$, $t_{k+1}^*(n) \leq t_k^*(n)$ follows from the definition of t_k^*. Consequently, by Part (1) (for both $k-1$ and k) and Part (2) the result holds for $\underline{i}_k < n \leq \bar{i}_{k-1}$.

If $n \leq \underline{i}_k$, then $n \leq \underline{i}_{k+1}$ by Part (2). Consequently, by the definition of $\underline{i}_k$, $t_{k+1}^*(n) = 0 \leq t_k^*(n)$.

If $n > \bar{i}_{k-1}$, then $n > \bar{i}_k - 1$ by Part (2). Consequently, $n \geq \bar{i}_k$, and, by the definition of $\bar{i}_k$, $t_k^*(n) = 1 \geq t_{k+1}^*(n)$, so the result follows.

Part (6). Let $\hat{a}_k \in \mathcal{R}_k$ and assume that there exists n' such that $\hat{a}_k(n') > a_k^*(n')$. We will show that this leads to a contradiction. Observe that $\bar{i}_{k-1} \geq n'$ because if $n' > \bar{i}_{k-1}$, then $a_k^*(n') = a^R(1) \geq a_k(n)$ for all a_k.

When $n = \underline{i}_k = \underline{i}_{k-1}$,

$$a_k^*(\underline{i}_{k-1}) = \min\{a_{k-1}^*(\underline{i}_{k-1}), a_k^*(\underline{i}_{k-1} + 1)\}.$$

We know that $a_{k-1}(\underline{i}_{k-1}) \leq a_{k-1}^*(\underline{i}_{k-1})$ for all $a \in \mathcal{R}_{k-1}$ by the induction hypothesis, which means that in this case $a_k(\underline{i}_{k-1} + 1) \leq a_k^*(\underline{i}_{k-1} + 1)$. Hence the result follows because $a_k(\underline{i}_{k-1}) \leq a_k(\underline{i}_{k-1} + 1)$.

To complete the proof, we deal with $n > \underline{i}_{k-1}$.

Consider the strategy $\tilde{a}_k$ defined by

$$\tilde{a}_k(n) = \begin{cases} \hat{a}_k(n) & \text{if } n > n', \\ \min\{\hat{a}_k(n), a_k^*(n')\} & \text{if } n' \geq n. \end{cases} \tag{16}$$

We claim that $\tilde{a}_k$ weakly dominates $\hat{a}_k$. By definition, $\tilde{a}_k(n) = \hat{a}_k(n)$ when $n > n'$.

When $n \in (\underline{i}_{k-1}, \bar{i}_{k-1}]$, $a_k^*(n) = \bar{a}(t_{k-1}^*(n-1), t_{k-1}^*(n))$. For $n = n'$, it follows from Part (8) $(k-1)$ and $\hat{a}_k(n') > a_k^*(n')$ that $\tilde{a}_k(n') = a_k^*(n')$ does at least as well as $\hat{a}_k(n')$ against any $t_{k-1} \in \mathcal{T}_{k-1}$ and it does strictly better against t_{k-1}^*. If $n < n'$, either $\tilde{a}_k(n) = \hat{a}_k(n)$ so that the strategies have the same performance against n or

$$a_k^*(n') = \tilde{a}_k(n) < \hat{a}_k(n) \leq \hat{a}_k(n'). \tag{17}$$

By definition, $a_k^*(n')$ is an optimal response to $\theta \in [t_{k-1}^*(n'-1), t_{k-1}^*(n')]$. Part (8) $(k-1)$ implies that $t_{k-1} \leq t_{k-1}^*$ for all $t_{k-1} \in \mathcal{T}_{k-1}$.

Therefore, $\tilde{a}_k$ outperforms $\hat{a}_k$ when (17) holds. Hence when $\tilde{a}_k(n) < \hat{a}_k(n)$, $\tilde{a}_k(n)$ is a strictly better response than $\hat{a}_k(n)$ to all $t_{k-1} \in \mathcal{T}_{k-1}$.

When $\underline{i}_{k-1} + 1 < \underline{i}_k$, we have $a_k^*(n) = \min\{a_k^*(n+1), a_{k-1}^*(n)\}$. Consequently $a_k^*(n) \geq a_{k-1}(n)$ by Part (6) $(k-1)$. This establishes Part (6).

Part (7). a_k^* is a best response to t_{k-1}^*. If there were a strategy $\hat{a}_k \neq a_k^*$ that dominates a_k^*, then $\hat{a}_k$ would also be a best response to t_{k-1}^*. If $\hat{a}_k$ is a best response to t_{k-1}^*, then $\hat{a}_k(n) = a_k^*(n)$ whenever $t_{k-1}^*(n-1) < t_{k-1}^*(n)$. Hence, by Part (1), $\hat{a}_k$ could only dominate a_k^* if either (a) there exists $n' > \bar{i}_{k-1}$ such that $\hat{a}_k(n') < a^R(1)$; or (b) there exists $n' \leq \underline{i}_{k-1}$ such that $\hat{a}_k(n') \neq a_k^*(n')$.

Consider Case (a). Because $n' > \bar{i}_{k-1}$, $t_{k-1}^*(n') = 1$ and, by Part (4), $a_k^*(n') = a^R(1)$. If $t_{k-1}(n'-1) = 1$ for all $t_{k-1} \in \mathcal{T}_{k-1}$, then $\hat{a}_k$ cannot dominate a_k^* on the basis of behavior at n'. If there is $\hat{t}_{k-1} \in \mathcal{T}_{k-1}$ such that $\hat{t}_{k-1}(n'-1) < 1$, then let $\hat{t}_{k-1}$ be a best response to $\hat{a}_{k-1} \in \mathcal{R}_{k-1}$. It follows that $u^S(\hat{a}_{k-1}(n'-1), \theta) \leq u^S(\hat{a}_{k-1}(n'), \theta)$ for $\theta > \hat{t}_{k-1}(n'-1)$. Furthermore, because $t^*_{k-1}(n'-1) = 1$, $u^S(a^*_{k-1}(n'-1), 1) \geq u^S(a^*_{k-1}(n'), 1)$ and therefore $u^S(a^*_{k-1}(n'-1), \theta) \geq u^S(a^*_{k-1}(n'), \theta)$ for all θ. Consequently, given any $\theta \in (\hat{t}_{k-1}(n'-1), 1)$ there is a mixture of $\hat{a}_{k-1}$ and a^*_{k-1}, say $\tilde{\alpha}_{k-1} = \delta\ \hat{a}_{k-1} + (1-\delta)a^*_{k-1}$ such that that $u^S(\tilde{\alpha}_{k-1}(n'-1), \theta) = u^S(\tilde{\alpha}_{k-1}(n'), \theta)$. Therefore the best response to $\tilde{\alpha}_{k-1}$, $\tilde{t}_k \in \mathcal{T}_{k-1}$, will have the property that $\tilde{t}_{k-1}(n'-1)$ is arbitrarily close to one. Hence, it will be the case that $\hat{a}_k(n')$ is inferior to $a^*_k(n')$ as a response to $\tilde{t}_{k-1}$. Hence $\hat{a}_k$ cannot dominate a^*_k.

Consider Case (b). $t_{k-1} \in \mathcal{T}_{k-1}$ implies that $t_{k-1}(\underline{i}_{k-1}) = 0$ by Part (8). Hence $\hat{a}_k$ cannot dominate a^*_k on the basis of behavior at n'.

Part (8). When $n \geq \bar{i}_k$, $t^*_k(n) = 1 \geq t_k(t)$ for $t_k \in \mathcal{T}_k$.

When $n \in [\underline{i}_{k-1}+1, \bar{i}_k - 1]$, we have

$$u^S(a^*_k(n), t^*_k(n)) \leq u^S(a^*_k(n+1), t^*_k(n)). \tag{18}$$

Part (1) $(k-1)$ implies $\bar{i}_k \leq \bar{i}_{k-1}+1$ and so $n \in [\underline{i}_{k-1}+1, \bar{i}_{k-1}]$. Therefore $a^*_k(n) < a^*_k(n+1)$ by Part (2) $(k-1)$. Part (6) and Inequality (18) imply that

$$u^S(a_k(n), \theta) \leq u^S(a_k(n+1), \theta) \text{ for all } \theta > t^*_k(n) \text{ and all } a_k \in \mathcal{R}_k$$

with strict inequality for $a_k = a^*_k$. By Part (7), $a^*_k \in \mathcal{R}_k$. Therefore $t_k(n) \leq t^*_k(n)$ for all $t_k \in \mathcal{T}_k$.

When $n \leq \underline{i}_{k-1}$, $t^*_{k-1}(n) = 0$. By Part (8) $(k-1)$, $t(n) \leq t^*_{k-1}(n)$ for all $t \in \mathcal{T}_{k-1}$. Because $\mathcal{T}_k \subset \mathcal{T}_{k-1}$, it follows that $t_k(n) = 0$.

Part (9). t^*_k is a best response to a^*_k. Further, it follows from the second clause of Part (1) $(k-1)$ that $t^*_k(n)$ is the unique best response to a^*_k for $n \in (\underline{i}_{k-1}, \bar{i}_{k-1}]$.

Next consider the case where $n \leq \underline{i}_{k-1}$. If, in addition, $n < \underline{i}_k$, then $a^*_k(n) \leq a^S(0)$ by Part (4), so all types weakly prefer $a^*_k(\underline{i}_k)$ to $a^*_k(n)$ and hence, by Part (6) they also prefer $a_k(\underline{i}_k)$ to $a_k(n)$ for all $a_k \in \mathcal{R}_k$. Hence it is not possible to dominate t^*_k for these n. Otherwise, it must be that $n = \underline{i}_k$. By the definition of t^*_k if $a^*_k(n) = a^*_k(n+1)$, then $t^*_k(n) = t^*_k(n+1)$. But, by definition of $\underline{i}_k$, $0 = t^*_k(\underline{i}_k) < t^*_k(\underline{i}_k+1)$. We conclude that $n = \underline{i}_k$ implies that $a^*_k(n) < a^*_k(n+1)$; so types θ close to zero strictly prefer $a^*_k(n)$ to $a^*_k(n+1)$ and hence t^*_k cannot be weakly dominated at n.

Hence if t_k weakly dominates t^*_k it must be that, with positive probability, $t_k(n) < 1$ for $n \geq \bar{i}_k$, which implies that $t_k(\bar{i}_k) < 1$. If $\bar{i}_k \geq k+1$, then $k < M$ and $t_k(\bar{i}_k) = 1$ for all $t_k \in \mathcal{T}_k$ by the definition of $\mathcal{T}_k$. In this case, $t_k(\bar{i}_k) < 1$ is impossible. If $\bar{i}_k < k+1$, then, by the definition of t^*_k, $u^S(a^*_k(\bar{i}_k), 1) \geq u^S(a^R(1), 1)$. Also, $a^*_k(\bar{i}_k) \leq a^R(1)$. Consequently, $u^S(a^*_k(\bar{i}_k), \theta) \geq u^S(a^R(1), \theta)$ for all θ and so t_k with $t_k(\bar{i}_k) < 1$ cannot weakly dominate t^*_k.

□

The claim constructs decreasing sequences of Sender and Receiver strategies that provide bounds for strategies remaining during the process of deletion. It is straightforward to confirm that these sequences converge to an equilibrium.

Claim 2. *The sequence $\{(a_k^*, t_k^*)\}$ converges. The limit, $\{(a^*, t^*)\}$, is a Nash equilibrium for the game.*

Proof. Consider the sequence $\{a_k^*\}$. It follows from Part (3) of Claim 1 that the sequence $\{a_k^*(m)\}$ is monotonically decreasing and bounded. Hence it converges. Similarly, from Part (5), $\{t_k^*\}$ converges. Let $(a^*, t^*) = \lim_{k\to\infty}(a_k^*, t_k^*)$. By the definition of the sequences $\{a_k^*\}$ and $\{t_k^*\}$, (a^*, t^*) is a Nash equilibrium. □

The next claim guarantees that the partition intervals $[t^*(n-1), t^*(n)]$ are non-degenerate or equal to one of the endpoints.

Claim 3. *If $n < M$ and $1 > t^*(n) > t^*(n-1) \geq 0$, then $t^*(n+1) > t^*(n)$.*

Proof. In order to reach a contradiction, assume that there exists $n' > n$ such that

$$t^*(n'+1) > t^*(n') = t^*(n) > t^*(n-1).$$

By definition, (a^*, t^*) is the limit of (a_k^*, t_k^*). Therefore, $a^*(n) = \bar{a}(t^*(n-1), t^*(n))$, $a^*(m) = a^R(t^*(n))$ for $m = n+1, \ldots, n'$, and $a^*(n'+1) = \bar{a}(t^*(n'), t^*(n'+1))$. In particular, $a^*(n'+1) > a^*(n') > a^*(n)$. However, because (a^*, t^*) is an equilibrium, $u^S(a^*(n), t^*(n)) = u^S(a^*(n'), t^*(n)) = u^S(a^*(n'+1), t^*(n))$. Because $u^S(\cdot, t^*(n))$ cannot take on the same value at three district points, we have a contradiction, which establishes the claim. □

We add to these claims an observation that depends on boundary conditions:

Claim 4. *If $a^S(1) > a^R(1)$, then there exists $\delta > 0$ such that $t^*(M-1) \leq 1 - \delta$.*

Proof. By assumption, there exists $\delta > 0$ such that a type $\theta \leq 1 - \delta$ of Sender prefers $a^R(1)$ to any smaller action. Because $a_k^*(M) = \max_n a_k^*(n)$, if $\bar{i}_k < M$, then $t_{k+1}^*(\bar{i}_k) < 1$ and, because $t_k^*(\bar{i}_k) = 1$ by Part (2), $\bar{i}_k < \bar{i}_{k+1}$. It follows that there exists $\bar{k}$ such that $\bar{i}_{\bar{k}} = M$ for $k \geq \bar{k}$. Hence $t^*(M-1) < 1$ by Part (5). □

Theorem 1 now follows from the following claim.

Claim 5. *If $a^S(\theta) > a^R(\theta)$ for all θ, $\{(a^*, t^*)\}$ satisfies NITS and the Sender uses only the highest N^* messages with positive probability.*

Proof. From Claim 4, $t^*(M-1) < 1$. It follows from Claim 3 that there is an $\underline{n}^*$ such that $t^*(n) = 0$ for $n \leq \underline{n}^*$ and $t^*(n+1) > t^*(n)$, for $n \in [\underline{n}^*, M)$. $\{a^*(\underline{n}^*+1), \ldots, a^*(M)\}$ are therefore $M - \underline{n}^*$ actions induced by the equilibrium (a^*, t^*). It follows that if $M > N^*$, then $0 < \underline{n}^*$. If there is such an n such that $t_k^*(n) = 0$, then Parts (4) and (3) Claim 1 imply that $a^*(n) \leq a^S(0)$. Otherwise $t_k^*(n)$ decreases to zero and $a^*(n) = a^R(0) < a^S(0)$ for $n \leq \underline{n}^*$. Consequently NITS is satisfied. □

It is possible to mimic the construction of Claim 1 starting with the lowest strategy for S ($t = (0, 0, \ldots, 0, 1)$) and the lowest strategy for R ($a(n) \equiv 0$). We present the argument in Appendix E, which is available as a supplement. The procedure generates an increasing sequence of S strategies t_k^{**} and an increasing sequence of R strategies a_k^{**} that provide lower bounds on undominated strategies. These sequences will converge to an equilibrium strategy profile (a^{**}, t^{**}) that satisfies NITS. If there is only one NITS outcome, then $t^* = t^{**}$ and $a^* = a^{**}$ on the equilibrium path. Consequently all strategy profiles that survive deletion of weakly dominated strategies must induce the same equilibrium outcome and Theorem 1 holds.

Corollary 2 follows: When only one equilibrium type-action distribution satisfies NITS, then $(a^*, t^*) = (a^{**}, t^{**})$.

The construction that determines (a^{**}, t^{**}) allows us to prove an analog to Claim 4:

Claim 6. *If $a^S(0) < a^R(0)$, then there exists $\delta > 0$ such that $t^{**}(1) \geq \delta$.*

Theorem 3 follows from Claims 4 and 6. Theorem 4 follows from Claim 1.

We have described an iterative process that associates with any pair (a, t) another pair $F(a, t)$. Starting from a high initial condition, the process generates a decreasing sequence that converges to an equilibrium. Starting from a low initial condition, the process generates an increasing sequence that converges to an equilibrium. Any equilibrium outcome of the process is a fixed-point of the process and the process satisfies $F(a, t) \geq F(a', t')$ if $(a, t) \geq (a', t')$. Consequently the argument guarantees that the decreasing sequence converges to the biggest NITS equilibrium and the increasing sequence converges to the smallest NITS equilibrium. That is, if NITS is not unique, then the set of strategies that survive deletion of weakly dominated strategies contains at least two equilibrium outcomes. The example in Appendix D illustrates this property.

Appendix B

This Appendix states and proves Claim 7, which establishes that the results do not depend on the order in which we delete weakly dominated strategies. The core of the argument appears in the proof of Claim 1. The statement of the result requires a new definition and more notation.

In the following definition, $(\int_{t(n-1)}^{t(n)} u^R(a(n), \theta) f(\theta)\, d\theta)/(\int_{t(n-1)}^{t(n)} f(\theta)\, d\theta) = u^R(a(n), t(n))$ when $t(n-1) = t(n)$.

Given strategy sets $(\mathcal{T}, \mathcal{R})$, the strategy $a' \in \mathcal{R}$ interim dominates $a \in \mathcal{R}$ if for all n and $t \in \mathcal{T}$,

$$\frac{\int_{t(n-1)}^{t(n)} u^R(a'(n), \theta) f(\theta)\, d\theta}{\int_{t(n-1)}^{t(n)} f(\theta)\, d\theta} \geq \frac{\int_{t(n-1)}^{t(n)} u^R(a(n), \theta) f(\theta)\, d\theta}{\int_{t(n-1)}^{t(n)} f(\theta)\, d\theta},$$

and for some n and $t' \in \mathcal{T}$ and for ,

$$\frac{\int_{t'(n-1)}^{t'(n)} u^R(a'(n), \theta) f(\theta)\, d\theta}{\int_{t'(n-1)}^{t'(n)} f(\theta)\, d\theta} > \frac{\int_{t'(n-1)}^{t'(n)} u^R(a(n), \theta) f(\theta)\, d\theta}{\int_{t'(n-1)}^{t'(n)} f(\theta)\, d\theta}.$$

We modify IDWDS by replacing weak dominance by interim dominance. That is we study Iterated Deletion of Interim Weakly Dominated Strategy (IDIWDS), which is defined as a procedure that satisfies the following properties.

A general procedure of deleting strategies produces a sequence of sets $\tilde{\mathcal{R}}_k$ and $\tilde{\mathcal{T}}_k$ such that:

1. $\tilde{\mathcal{R}}_0 = A_M$, $\tilde{\mathcal{T}}_0 = T_M$;

2. $\tilde{\mathcal{R}}_k$ is a subset of $\tilde{\mathcal{R}}_{k-1}$ obtained by deleting a subset of R's interim dominated strategies given the strategy sets $(\tilde{\mathcal{R}}_{k-1}, \tilde{\mathcal{T}}_{k-1})$;

3. $\tilde{\mathcal{T}}_k$ is a subset of $\tilde{\mathcal{T}}_{k-1}$ obtained by deleting a subset of S's weakly dominated strategies given the strategy sets $(\tilde{\mathcal{R}}_{k-1}, \tilde{\mathcal{T}}_{k-1})$;

4. The sets $\tilde{\mathcal{R}}^* = \cap_{k\geq 0}\tilde{\mathcal{R}}_k$ and $\tilde{\mathcal{T}}^* = \cap_{k\geq 0}\tilde{\mathcal{T}}_k$ are well defined and non-empty.

5. Given the strategy sets $(\tilde{\mathcal{R}}^*, \tilde{\mathcal{T}}^*)$ there are no weakly dominated strategies in $\tilde{\mathcal{T}}^*$ and no interim dominated strategies in $\tilde{\mathcal{R}}^*$.

Interim dominance implies weak dominance.[17] Relative to weak dominance, interim dominance places an additional restriction for the Receiver's strategy a' to dominate a. The dominating strategy must not only lead to expected utility at least as great as the dominated strategy for every strategy of the Sender, but it must do so for every message. The additional restriction only matters for Sender strategies t for which $t(n-1) = t(n)$. In this case, for a' is not interim dominated, then $a'(t(n)) = a^R(t(n))$. Weak dominance does not impose this restriction because when $t(n-1) = t(n)$, the Receiver's response to n does not influence his expected utility. Our main result (Theorem 1) holds for both IDWDS and interim IDWDS. Our proof that the conclusion of Theorem 1) holds independent of the order of deletion of weakly dominated strategies requires interim IDWDS.

The sequences $\bar{i}_k$ and $\underline{i}_k$ are also bounded and monotone, so they converge. We denote the limits by $\bar{i}^*$ and $\underline{i}^*$.

Claim 7. *If $t \in \tilde{\mathcal{T}}^*$, then $t \leq t^*$. If $a \in \tilde{\mathcal{R}}^*$, then $a(n) \leq a^*(n)$.*

Proof. It suffices to show for all k:

$$\text{if } t \in \tilde{\mathcal{T}}^*, \quad \text{then } t \leq t^*_k \tag{19}$$

and

$$\text{if } a \in \tilde{\mathcal{R}}^*, \quad \text{then } a(n) \leq a^*_k(n). \tag{20}$$

We prove that (19) and (20) hold by induction. They plainly hold for $k = 0$. Assume that they are true for $k \leq k' - 1$. We will show that they are true for $k = k'$.

Suppose that there exists $\hat{t} \in \tilde{\mathcal{T}}^*$ that is not less than or equal to $\hat{t}^*_{k'}$. That is, assume that there exists $\varepsilon > 0$ and n such

[17] The only difference between the definition of a general IDWDS procedure and the definition of a IDIWDS procedure is replacing "weakly" by "interim" in the second condition.

$$\hat{t}(n) \geq t^*_{k'}(n) + \varepsilon. \tag{21}$$

By the induction hypothesis,

$$t^*_{k'-1}(n) > \hat{t}(n). \tag{22}$$

Further suppose that n be the smallest partition element in which the claim fails. That is, suppose that

$$\text{for all } n' < n,\ t^*_{k'}(n') \geq \hat{t}(n'). \tag{23}$$

Note that (21) implies $t^*_{k'}(n) < 1$ and (22) implies that $t^*_{k'-1}(n) > 0$, it follows that $n < \bar{i}_{k'}$ and $n > \underline{i}_{k'-1}$.

It follows from Claim 1 (Part (2)) that $n \in (\underline{i}_{k'-1}, \bar{i}_{k'-1}]$. Consequently, Claim 1 (Part (1)) implies that

$$a^*_{k'}(n+1) > a^*_{k'}(n). \tag{24}$$

Because $u^S(a^*_{k'}(n+1), t^*_{k'}(n)) = u^S(a^*_{k'}(n), t^*_{k'}(n))$, inequality (24) implies that

$$u^S(a^*_{k'}(n+1), \theta) > u^S(a^*_{k'}(n), \theta) \text{ for } \theta > t^*_{k'}(n). \tag{25}$$

The induction hypothesis implies that for any $\delta > 0$, we can pick $\underline{j}$ sufficiently large that if $a \in \tilde{\mathcal{R}}_j$ and $j > \underline{j}$ and $n \geq \underline{i}_{k'}$, then $a(n) \leq a^*_{k'}(n) + \delta$. Inequality (25) implies that $a^*_{k'}(n) < a^R(t^*_{k'}(n))$. Consequently we can take δ sufficiently small that $a(n) < a^R(t^*_{k'}(n))$. Inequality (25) implies that $u^S(a(n+1), \theta) \geq u^S(a(n), \theta)$ for all $a \in \tilde{\mathcal{R}}_j$ and $\theta > t_{k'}(n)$. From (24), we know that $t^*_{k'}$ dominates $\hat{t}$. The strategy $\hat{t}$ is also dominated by any strategy that dominates $t^*_{k'}$. Furthermore, the domination must persist as long as there exists a strategy $a \in \tilde{\mathcal{R}}_j$ with $a(n) < a(n+1)$. We claim that as long as $\hat{t} \in \tilde{\mathcal{T}}_j$, there exists $a \in \tilde{\mathcal{R}}_j$ such that $a(n+1) > a(n)$. To prove this claim, note that if there exists $t \in \tilde{\mathcal{T}}_j$ such that $t(n-1) = \hat{t}(n-1)$, $t(n) = \hat{t}(n)$, and $t(n+1) > \hat{t}(n)$, then any best response to t must satisfy

$$a(n+1) = \bar{a}(t(n), t(n+1)) > a^R(t(n)) > \bar{a}(t(n-1), t(n)) = a(n),$$

where the second inequality follows because (21) and (23) imply that $\hat{t}_n > \hat{t}_{n-1}$. Also, if every $t \in \tilde{\mathcal{T}}_j$ such that $t(n-1) = \hat{t}(n-1)$ and $t(n) = \hat{t}(n)$ satisfies $t(n+1) = \hat{t}(n)$, then a strategy that best responds to $\hat{t}$ and satisfies $a(n+1) = a^R(\hat{t}(n)) > \bar{a}(t(n-1), t(n)) = a(n)$ cannot be deleted by interim dominance.

Next, let $\hat{a} \in \tilde{\mathcal{R}}^*$ and assume that there exists $\varepsilon > 0$ and $n > \underline{i}_{k'}$ such that $\hat{a}(n) \geq a^*_{k'}(n) + \varepsilon$. We take n to be the smallest such n' so that there exists $\underline{j}$ such that if $j > \underline{j}$, then $\hat{a}(n') < a^*_{k'}(n') + \varepsilon$ for $n' < n$. This means that $\hat{a}(n) \geq \hat{a}(n-1) + \eta$ for some $\eta > 0$. Because $a^*_{k'}(n') = a^R(1) \geq \hat{a}(n')$ for $n' > \bar{i}_{k'}$ by Claim 1 (Parts (1) and (4)), we know that $n \leq \bar{i}_{k'}$. It follows from Claim 1 (Part 1), that $t^*_{k'}(n) > t^*_{k'}(n-1)$. Consider the strategy $\tilde{a}$ defined by

$$\tilde{a}(n') = \begin{cases} \hat{a}(n') & \text{if } n' > n, \\ \min\{\hat{a}(n'), a^*_{k'}(n)\} & \text{if } n \geq n'. \end{cases} \tag{26}$$

We will show that $\tilde{a}$ weakly dominates $\hat{a}$. To see this, note that $\tilde{a}$ and $\hat{a}$ are identical against messages n when $n' > n$. If $n' \leq n$, either $\tilde{a}(n') = \hat{a}(n')$ so that the strategies have the same performance against n' or $a^*_k(n') = \tilde{a}(n') < \hat{a}(n')$. To show that $\tilde{a}$ outperforms $\hat{a}$ in the second case, note that by the induction hypothesis, for any $\delta > 0$, we can pick $\underline{j}$ sufficiently large so that if $t \in \tilde{\mathcal{T}}_{\underline{j}}$, then $t(n') \leq t^*_{k'-1}(n') + \delta$. It follows from the induction hypothesis and $\hat{a}(n') > a^*_{k'}(n')$ that we can pick $\underline{j}$ large enough so that $\tilde{a}(n') = a^*_{k'}(n')$ does at least as well as $\hat{a}(n')$ against any $t \in \tilde{\mathcal{T}}_{\underline{j}}$ and it does strictly better against t with $t(n') > t(n'-1)$. If $t(n') = t(n'-1)$ for all $t \in \tilde{\mathcal{T}}_{\underline{j}}$, then $\tilde{a}(n')$ interim dominates $\hat{a}(n')$. Because $\hat{a}(n')$ and $\tilde{a}(n')$ perform equally well when $t(n') = 0$, we have shown that $\tilde{a}$ interim dominates $\hat{a}$: it performs at least as well for all n' and when $n' = n$ it either performs strictly better for one t (if $t(n) > t(n-1)$ for some t) or does strictly better conditional on n (if $t(n) = t(n-1)$ for all $t \in \tilde{\mathcal{T}}_{\underline{j}}$).

It remains to examine the behavior of $\hat{a}$ for $n < \underline{i}_{k'}$. In this case, interim dominance and $t(n) = 0$ implies that any a with $a(n) > a^R(0)$ will eventually be deleted by interim dominance. □

It follows that if $t \in \tilde{\mathcal{T}}^*$, then $t \leq t^*$; if $a \in \tilde{\mathcal{R}}^*$, then $a(n) \leq a^*(n)$ for all $n > \underline{i}^*$ and $a(n) \leq a^S(0)$ for all $n \leq \underline{i}^*$. Consequently the only strategies that survive deletion of weakly dominated strategies use only the highest messages and permit the type $\theta = 0$ Sender to obtain utility equal to at least $u^S(a^R(0), 0)$. Consequently, NITS holds when $a^S(\theta) > a^R(\theta)$ for all θ. A symmetric argument demonstrates that main result holds independent of the order of deletion of dominated strategies when $a^S(\theta) < a^R(\theta)$ for all θ.

Appendix C

The Appendix contains the proof of Theorem 2.

Claim 8. *If there exists $\breve{J}_k$ such that for all n, $\breve{a}_j(n) \le a_k^*(n)$ for all $j > \breve{J}_k$, then there exists K_k such that for all n, $t_j(n) \le t_k^*(n)$ for $j > K_k$.*

Claim 9. *If there exists K_k such that $t \le t_j^*$ for all $j > K_k$, then there exists J_{k+1} such that $a_j(n) \le a_{k+1}^*(n)$ for all $j > J_{k+1}$ and all n.*

Claim 10. *If $a_j(n') \le a_k^*(n)$ for all $n' \le n$, then $\breve{a}_j(n) \le a_k^*(n)$ for all $n' \le n$.*

Suppose that the claims hold. We know that $\breve{a}_j \le a_0^*$ for all j because a_0^* is the largest strategy. It follows from induction that for any k, there exists K_k such that $t_j(n) \le t_k^*(n)$ for all $j > K_k$ and there is $\breve{J}_k$ such that $\breve{a}_j(n) \le a_k^*(n)$ for all $j > \breve{J}_k$. Claim 10 guarantees that we can take $\breve{J}_k = J_k$.

This argument (and the convergence of t^*) guarantees that the upper bounds in the statement of Theorem 2 hold. A symmetric argument holds for the lower bounds. When there is a unique type-action distribution that satisfies NITS, $t^* = t^{**}$ and consequently the sequence $\{t_k\}_{k=0}^{\infty}$ converges to t^* and $\{a_k(n)\}_{k=0}^{\infty}$ converges to $a_k^*(n)$ provided that $t^*(n) > 0$.

It remains to prove Claims 8, 9, and 10.

Consider first Claim 10. To prove the claim, note that

$$\breve{a}_j(n) \le \max\{a_j(1), \ldots, a_j(n)\} \tag{27}$$

because at most $n-1$ components of a_j can be strictly lower than $\breve{a}_j(n)$ (recall that $\breve{a}_j(n)$ is the n^{th} smallest of $a_j(n')$, $n' = 0, \ldots, M$). Furthermore, $a_k^*(n-1) \le a_k^*(n)$ implies that

$$\max\{a_j(1), \ldots, a_j(n)\} \le a_k^*(n) \tag{28}$$

whenever $a_j(n) \le a_k^*(n)$ The claim follows from inequalities (27) and (28).

To prove Claim 9, we use the following result.

Lemma 1. *If $t_j(n-1) \le t_k^*(n-1)$, $t_j(n) \le t_k^*(n)$, and $t_j(n-1) < t_j(n)$, then $\int_{t_j(n-1)}^{t_j(n)} \left(u^R(a_{k+1}^*(n), \theta) - u^R(a, \theta)\right) f(\theta)\, d\theta \ge 0$ for all $a \ge a_{k+1}^*(n)$.*

Proof. If $t_k^*(n-1) = t_k^*(n) = 1$, then $a_{k+1}^*(n) = a^R(1)$ and the result follows. Otherwise, $t_k^*(n-1) < t_k^*(n)$ and $a_{k+1}^*(n)$ maximizes $\int_{t_k^*(n-1)}^{t_k^*(n)} u^R(a, \theta)\, f(\theta)\, d\theta$. It follows that

$$\int_{t_k^*(n-1)}^{t_k^*(n)} \left(u^R(a_{k+1}^*(n), \theta) - u^R(a, \theta)\right) f(\theta)\, d\theta \ge 0 \tag{29}$$

and when $a > a_{k+1}^*(n)$ the assumption that the mixed partial of $u^R(\cdot)$ is positive implies that $u^R(a_{k+1}^*(n+1), \theta)) - u^R(a, \theta)$ is strictly decreasing in θ and is equal to zero at some $\theta^* \in (t_k^*(n-1), t_k^*(n))$. Consequently, lowering the lower limit of integration from $t_k^*(n-1)$ increases the integral in (29); lowering the upper limit of integration from $t_k^*(n)$ to $t \ge \theta^*$

increases the integral in (29); and $\int_\tau^t \left(u^R(a^*_{k+1}(n),\theta) - u^R(a,\theta)\right) f(\theta)\, f(\theta)\, d\theta > 0$ for all $0 \le \tau < t \le \theta^*$. Hence $t_j(l) \le t^*_k(l)$ for $l = n-1, n$ implies that there exists $a < a^*_{k+1}$ such that

$$\int_{t_j(n-1)}^{t_j(n)} \left(u^R(a^*_{k+1}(n),\theta) - u^R(a,\theta)\right)\, f(\theta)\, d\theta \le 0$$

and, when $a > a^*_{k+1}(n)$,

$$\int_{t_j(n-1)}^{t_j(n)} \left(u^R(a^*_{k+1}(n),\theta) - u^R(a,\theta)\right)\, f(\theta)\, d\theta \ge 0,$$

with strict inequality if either $t_j(n-1) < t^*_k(n-1)$ or $t_j(n) < t^*_k(n)$.

□

Suppose that there exists K_k such that $t \le t^*_k$ for all $j > K_k$. It follows that for $j > K_k + C$, $a_j(n)$ is a best response to a mixture of strategies in which $\theta \in [t_l(n-1), t_l(n)]$ for $t_l(n-1) \le t^*_k(n-1)$ and $t_l(n-1) \le t^*_k(n-1)$. If $t_l(n-1) = t_l(n)$ for all $l = j-C+1, \ldots, j$, then $t_l(n-1) \le t_k(n-1)$ implies

$$a_j(n) = a^*(t_j(n-1)) \le \bar{a}(t_k(n-1), t_k(n)) = a^*_{k+1}(n).$$

Otherwise, Lemma 1 implies that $a_{k+1}(n)$ does at least as well $a > a^*_{k+1}(n)$ to a mixture of S's previous C strategies. Hence, $a_j(n) \le a_{k+1}(n)$. This establishes Claim 9 (using $J_{k+1} = K_k + C$).

To prove Claim 8, we first note the following.

Lemma 2. *Assume $t^*_k(n) < 1$. If $\breve{a}_j(n) \le a^*_k(n)$ and $\breve{a}_j(n+1) \le a^*_k(n+1)$, then $u^S(\breve{a}_j(n+1), t^*_k(n)) \ge u^S(\breve{a}_j(n), t^*_k(n))$, with strict inequality if $\breve{a}_j(n+1) > \breve{a}_j(n)$ and $t^*_k(n) > 0$.*

Proof. If $t^*_k(n) = 0$, then $u^S(a^*_k(n+1), 0) \ge u^S(a^*_k(n), 0)$, it follows from Claim 1 that $a^S(0) \ge a^*_k(n)$. Either $a^S(0) \ge a^*_k(n+1) \ge a^*_k(n)$ or $a^*_k(n+1) > a^S(0) > a^*_k(n)$. Hence either

$$a^S(0) \ge a^*_k(n+1) \ge \breve{a}_j(n)$$

(which could hold in either case) or

$$a^*_k(n+1) \ge \breve{a}_j(n+1) > a^S(0) > \breve{a}_j(n)$$

(which could hold only in the second case), implying that

$$u^S(\breve{a}_j(n+1), 0) \ge u^S(a^*_k(n+1), 0) \ge u^S(a^*_k(n), 0) \ge u^S(\breve{a}_j(n), 0).$$

In any event, $u^S(\breve{a}_j(n+1), t^*_k(n)) \ge u^S(\breve{a}_j(n+1), t^*_k(n))$.

Assume now that $t^*_k(n)$ is uniquely defined by

$$u^S(a^*_k(n+1), t^*_k(n)) = u^S(a^*_k(n), t^*_k(n)). \tag{30}$$

It follows from (30) that $a^*_k(n+1) > a^S(t_k(n)) > a^*_k(n)$. We now claim that

$$u^S(\breve{a}_j(n+1), t^*_k(n)) \ge u^S(\breve{a}_j(n), t^*_k(n)). \tag{31}$$

To verify inequality (31), note that either

$$a^S(t_k^*(n)) > \breve{a}_j(n+1) \geq \breve{a}_j(n)$$

or

$$a_k^*(n+1) \geq \breve{a}_j(n+1) \geq a^S(t_k^*(n)) > a_k^*(n) \geq \breve{a}_j(n).$$

In the first case, $u^S(a, t_k^*(n))$ is strictly increasing for $a \in (a_k^*(n), a_k^*(n+1))$. In the second case,

$$u^S(\breve{a}_j(n+1), t_k^*(n)) \geq u^S(a_k^*(n+1), t_k^*(n)) = u^S(a_k^*(n), t_k^*(n)) \geq u^S(\breve{a}_j(n), t_k^*(n)) \quad (32)$$

and $u^S(\breve{a}_j(n+1), t_k^*(n)) > u^S(\breve{a}_j(n), t_k^*(n))$ when $\breve{a}_j(n+1) > \breve{a}_j(n)$. □

Now we can prove Claim 8. There is nothing to show if $t_k^*(n) = 1$. Assume that $t_k^*(n) < 1$, that $\breve{J}_k$ satisfies the conditions of Claim 8, and that $j > \breve{J}_k$. Assume $\breve{a}_l(n+1) \neq \breve{a}_l(n)$ for some $l = j-C+1, \ldots, j$ that receives positive weight when S responds in stage j. If there exists $\breve{J}_k$ such that for all n, $\breve{a}_j(n) \leq a_k^*(n)$ for all $j > \breve{J}_k$, then for $K_k \geq \breve{J}_k + C$, $\breve{a}_l(n) \leq a_l^*(n)$ for $l = j-C+1, \ldots, j$. It follows from Lemma 2 that each term in the sum

$$\sum_{l=j-C+1}^{j} q_l \left(u^S(\breve{a}_l(n+1), t_k^*(n)) - u^S(\breve{a}_l(n), t_k^*(n))\right) \quad (33)$$

is nonnegative and the sum is strictly positive. Because the mixed partial of $u^S(\cdot)$ is strictly positive, this guarantees that $t_j(n) \leq t_k^*(n)$. Hence Claim 8 holds if we set $K_k > \breve{J}_k + 1$.

To complete the proof, consider the case where $\breve{a}_l(n+1) = \breve{a}_l(n)$ for all $l = j-C+1, \ldots, j$. In this case, S is indifferent between inducing the n^{th} and $(n+1)^{\text{th}}$ highest action in period j. The argument above demonstrates that eventually $t_j(n) \leq t_k^*(n)$ unless

$$\breve{a}_j(n+1) = \breve{a}_j(n) \text{ for all } j > \breve{J}_k. \quad (34)$$

Assume (34) holds. It follows that $t_j(n)$ is determined by the tie-breaking rule (12). We will how that Claim 8 holds by induction on n. The claim holds when $n = 0$. Assume that it holds for all $n' < n$. We will show that it also holds for n. If $t_j(n)$ is equal to $t_j(n-1)$ for any $j > \breve{J}_k$, then we are done because for all sufficiently large j, $t_j(n-1) \leq t_k^*(n-1) \leq t_k^*(n)$, where the first inequality follows from the induction hypothesis. If $t_j(n)$ is equal to $t_j(n+1)$ for any $j > \breve{J}_k$, then they will remain equal by (12). This guarantees that for sufficiently large j, $\breve{a}_j(n+1) = a_j(n+1)$ and $\breve{a}_j(n) = a_j(n)$. Consequently,

$$\breve{a}_j(n+1) = a_j(n+1) = a^R(t_j(n+1)) > \bar{a}(t_{k+1}^*(n-1), t_j(n)) \geq a_j(n) = \breve{a}_j(n), \quad (35)$$

which contradicts (34). Otherwise, $t_j(n)$ does not change with j for j sufficiently large. This guarantees that for sufficiently large j, $\breve{a}_j(n+1) = a_j(n+1)$ and $\breve{a}_j(n) = a_j(n)$. Consequently,

$$\breve{a}_j(n+1) = a_j(n+1) \geq a^R(t_j(n+1)) > \bar{a}(t_{k+1}^*(n-1), t_j(n)) \geq a_j(n) = \breve{a}_j(n), \quad (36)$$

which contradicts (34).

Appendix D

This section presents an example which demonstrates that if a game has more than one equilibrium that satisfy NITS, the play may not converge to the equilibrium with the highest number of induced actions, even though the number of initially induced actions exceeds this number.

Suppose the distribution of the Sender's types is uniform, and the utilities of the Sender and the Receiver are $u^S(b,\theta) = -(b-\theta-c(\theta))^2$ and $u^R(b,\theta) = -(b-\theta)^2$ for some $c(\theta) > 0$. So, the only departure from the uniform-quadratic example is that the bias $c(\theta)$ depends on θ. Assume that the bias has the following properties:
(a) $c(0) = 7/96$; **(b)** $c(8/96) = 2/96$; **(c)** $c(16/96) = 8/96$; **(d)** $c(20/96) = 14/96$; **(e)** $c(24/96) = 14/96$. It is clearly possible to specify $c(\cdot)$ that satisfies these properties with the additional property that $a^S(\theta) = \theta + c(\theta)$ is strictly increasing.

This game has an equilibrium in which the types from $[0, 20/96]$ induce action 10/96, and the types from $[20/96, 1]$ induce action 58/96. The game also has an equilibrium, in which the types from $[0, 8/96]$ induce action 4/96, the types from $[8/96, 24/96]$ induce action 16/96, and the types from $[24/96, 1]$ induce action 60/96. Both equilibria satisfy NITS.

Suppose now that initially there are three messages: m_1 induces action $b(m_1) = 4/96$, m_2 induces action $b(m_2) = 8/96$, and message m_3 induces action $b(m_3) = 40/96$. We will show that under the best-response dynamics, the play converges to the two-action equilibrium. Indeed, in period 1, all types prefer m_2 to m_1, and type 16/96 is indifferent between sending m_2 and m_3. This means that action 0 is the response to m_1 in period 2, action 8/96 is the response to m_2, and action 56/96 is the response to m_3. The type that is indifferent between these two responses lies in the interval $(16/96, 20/96)$. Thus, the responses to m_2 and m_3 increase in period 3, compared to those in period 2. So, they gradually increase in the subsequent periods, and converge to those in the two-action equilibrium.

Notice that in our example $b(1) \geq b^*(1)$, $b(2) \geq b^*(2)$ and $b(3) < b^*(3)$, where $b^*(1)$, $b^*(2)$, and $b^*(3)$ are the actions induced in the three-action equilibrium. One may wonder if the play would have to converge to the three-action equilibrium if $b(n) \geq b^*(n)$ for $n = 1, 2, 3$. The proof of our main result demonstrates that the answer to this question is positive. The play must converge to the equilibrium with highest number of actions if initially some message induces a weakly higher action than the highest action in the equilibrium with highest number of actions, two messages induce weakly higher actions than the second highest action in the equilibrium with highest number of actions, etc.

When there are several equilibria that satisfy NITS, our arguments guarantee the existence of a sequence that converges down to the largest equilibrium and another sequence that converges up to the smallest equilibrium that satisfies NITS.

Appendix E

This Appendix provides the analog to the arguments in Appendix A for sequences of lower bounds.

It is possible to mimic the construction of Claim 1 starting with the lowest strategy for S ($t = (0, 0, \ldots, 0, 1)$) and the lowest strategy for R ($a(m) \equiv 0$). The procedure generates an increasing sequence of S strategies t_k^{**} and an increasing sequence of R strategies a_k^{**} that provide lower bounds on undominated strategies. These sequences will converge to an equilibrium strategy profile (a^{**}, t^{**}) that satisfies NITS. If there is only one NITS outcome, then $t^* = t^{**}$ and $a^* = a^{**}$ on the equilibrium path. Consequently all strategy profiles that survive deletion of weakly dominated strategies must induce the same equilibrium outcome and the main theorem holds.

We introduce a specific order for deleting dominated strategies. A variation of the argument in Appendix A (Claim (7)) demonstrates that the results do not depend on the order.

$\mathcal{T}_0 = \{t = (t(0), t(1), \ldots, t(M)) : 0 = t(0) \leq t(1) \leq \cdots \leq t(M) = 1\}$ and $\mathcal{R}_0 = \{a = (a(1), a(2), \ldots, a(M)) : 0 \leq a(1) \leq \cdots \leq a(M) \leq 1\}$.

For all $k \geq 1$, form $\mathcal{T}_k$ and $\mathcal{R}_k$ inductively as follows:

$$\begin{aligned} \mathcal{R}_k = \{a_k \in \mathcal{R}_{k-1} \quad : \quad & \nexists \alpha_k \in co(\mathcal{R}_{k-1}) \text{ s.t.} \\ & U^R(\alpha_k, t_{k-1}) \geq U^R(a_k, t_{k-1}) \text{ for every } t_{k-1} \in \mathcal{T}_{k-1} \\ & U^R(\alpha_k, t_{k-1}) > U^R(a_k, t_{k-1}) \text{ for some } t_{k-1} \in \mathcal{T}_{k-1}\} \end{aligned}$$

$$\begin{aligned} \mathcal{T}_k = \{t_k \in \mathcal{T}_{k-1} \quad : \quad & \nexists \tau_k \in co(\mathcal{T}_{k-1}) \text{ s.t.} \\ & \tau_k(t) = 0 \text{ if } t(M-1-k) > 0 \\ & U^S(a_k, \tau_k) \geq U^S(a_k, t_k) \text{ for every } a_k \in \mathcal{R}_k \\ & U^S(a_k, \tau_k) > U^S(a_k, t_k) \text{ for some } a_k \in \mathcal{R}_k\} \end{aligned}$$

Thus, the order of deletion is that in any round $k \geq 1$, we first delete all weakly dominated strategies for the Receiver (using Sender strategies that remain from the previous round), then we delete some strategies that are weakly dominated strategies for the Sender relative to the remaining strategies of the Receiver, and then continue. In the first M rounds we do not necessarily delete all of the weakly dominated strategies of the Sender because we require that the dominating strategy of the Sender satisfy $t(M-1-k) = 0$. This additional restriction is convenient for our construction. The constraint that the Sender uses only the highest messages is not binding when $k \geq M-1$. The main result does not depend on this choice of order.

Let $a_0^{**} = (0, 0, \ldots, 0)$ and $t_0^{**} = (0, 0, \ldots, 0, 1)$. Given a_0^{**} and t_0^{**}, define a_k^{**} and t_k^{**} inductively.

Let

$$a_k^{**}(1) = \begin{cases} a^R(0) & \text{if } t_{k-1}^{**}(1) = 0 \\ \bar{a}(0, t_{k-1}^{**}(1)) & \text{if } t_{k-1}^{**}(1) > 0 \end{cases}$$

and, after defining $a_k^{**}(n')$ for $n' = 1, \ldots, n-1$,

$$a_k^{**}(n) = \begin{cases} \max\{a_{k-1}^{**}(n), a_k^{**}(n-1)\} & \text{if } t_{k-1}^{**}(n-1) = t_{k-1}^{**}(n) \\ \bar{a}(t_{k-1}^{**}(n-1), t_{k-1}^{**}(n)) & \text{if } t_{k-1}^{**}(n-1) < t_{k-1}^{**}(n). \end{cases}$$

In particular $a_1^{**} = (a^R(0), \ldots, a^R(0), \bar{a}(0,1))$. The fact that a_k^{**} is a best response to t_{k-1}^{**} is clear for those n such that $t_{k-1}^{**}(n-1) < t_{k-1}^{**}(n)$. If $t_{k-1}^{**}(n-1) = t_{k-1}^{**}(n)$, best responding places no restrictions on $a_k^{**}(n)$. The specification guarantees that $a_k^{**}(n) \leq a_k^{**}(n+1)$.

Let $t_k^{**}(0) = 0$ and $t_k^{**}(M) = 1$. For $n = 1, \ldots M-1$, after defining $t_k^{**}(n')$ for $n' = 0, \ldots, n-1$, let $t_k^{**}(n) = t_k^{**}(n-1)$ if $a_k^{**}(n') = a_k^{**}(n'+1)$ for $n' = 0, \ldots, n-1$, and, otherwise let

$$t_k^{**}(n) = \begin{cases} 1 & \text{if } u^S(a_k^{**}(n), 1) \geq u^S(a_k^{**}(n+1), 1) \\ t_k^{**}(n-1) & \text{if } u^S(a_k^{**}(n), 0) \leq u^S(a_k^{**}(n+1), 0) \\ \min\{\theta : u^S(a_k^{**}(n), \theta) = u^S(a_k^{**}(n+1), \theta)\} & \text{otherwise.} \end{cases}$$

The specification guarantees that t_k^{**} is a best response to a_k^{**}. If $u^S(a_k^{**}(n), 1) \geq u^S(a_k^{**}(n+1), 1)$[18] and $a_k^{**}(n) < a_k^{**}(n+1)$ it must be that $u^S(a_k^{**}(n'), 1) \geq u^S(a_k^{**}(n'+1), 1)$ for $n' > n$ and so $t_k^{**}(n') = 1$ for $n' \geq n$ and this is a best response. If $u^S(a_k^{**}(n), 0) \leq u^S(a_k^{**}(n+1), 0)$ and $a_k^{**}(n) < a_k^{**}(n+1)$, then no type would want to induce $a_k^{**}(n')$ for $n' \leq n$. If $u^S(a_k^{**}(n), \theta) = u^S(a_k^{**}(n+1), \theta)$ and $a_k^{**}(n) < a_k^{**}(n+1)$, then types greater than θ will strictly prefer to induce an action $a_k^{**}(n')$ to $a_k^{**}(n)$ for some $n' > n$ and types less that θ will strictly prefer to induce $a_k^{**}(n)$ to $a_k^{**}(n')$ for $n' > n$. Our specification of t^{**} breaks indifference in different ways. It specifies lower cutoffs (so that the Sender uses a higher message when indifferent) if all lower messages lead to the same action. Otherwise, it selects higher cutoffs. The tie-breaking rules guarantee that if $t_k^{**}(n) = 0$, then $t_k^{**}(n') = 0$ for all $n' > n$, and if $t_k^{**}(n) = 1$, then $t_k^{**}(n') = 1$ for all $n' < n$.

The specification of t_k^{**} also guarantees that $t_k^{**}(M-k-1) = 0$ for $k \leq M-1$.

We construct the sequence (a_k^{**}, t_k^{**}), by starting with an initial a_0^{**} and then letting t_k^{**} be the best response to a_k^{**} and a_{k+1}^{**} be the best response to t_k^{**}. The definition is complicated because in some situations, best responses are not unique. The Sender will have more than one best response to a_k if there exists n such that $a_k(n) = a_k(n+1)$. In our specification, we assume that t_k^{**} and a_k^{**} are effectively the lowest best responses. The tie breaking rules serve to make a_k^{**} and t_k^{**} lower bounds to strategies remaining at stage k of the deletion process.

Claim 11 below makes this statement precise.

We can now describe the important properties of the sequence $\{a_k^{**}, t_k^{**}\}$.

For each k let $\underline{i}_k = min\{n : t_k^{**}(n) < t_k^{**}(n+1)\}$ and $\bar{i}_k = max\{n > 0 : t_k^{**}(n-1) < t_k^{*}(n)\}$. It follows that

$$\underline{i}_k = \max\{n : t_k^{**}(n) = 0\} \text{ and } \bar{i}_k = \min\{n : t_k^{**}(n) = 1\}. \tag{37}$$

Claim 11. *For any $k \geq 0$,*

[18]This condition can hold only if $a^S(1) \leq a^R(1)$.

1. $t_k^{**}(n) < t_k^{**}(n+1)$ *for* $n \in [\underline{i}_k, \bar{i}_k)$; $a_{k+1}^{**}(n+1) > a_{k+1}^{**}(n)$ *for* $n \in [\underline{i}_k, \bar{i}_k)$;

2. $\underline{i}_{k+1} + 1 \geq \underline{i}_k \geq \underline{i}_{k+1}$; $\bar{i}_k \geq \bar{i}_{k+1}$;

3. $a_{k+1}^{**}(n) \geq a_k^{**}(n)$;

4. $a_{k+1}^{**}(n) \geq a^S(1)$ *for* $n \geq \bar{i}_k$; $a_{k+1}^{**}(n) = a^R(0)$ *for* $n < \underline{i}_k$;

5. $t_{k+1}^{**}(n) \geq t_k^{**}(n)$;

6. *For all* $a_k \in \mathcal{R}_k$, $a_k^{**}(n) \leq a_k(n)$ *for* $n \leq \max\{\bar{i}_k, \bar{i}_{k-1} + 1\}$;

7. $a_k^{**} \in \mathcal{R}_k$;

8. *For all* $t_k \in \mathcal{T}_k$, $t_k \geq t_k^{**}$;

9. $t_k^{**} \in \mathcal{T}_k$.

Proof. We prove the result by induction. Assume that $k = 0$. The first part of Part (1) follows because $\underline{i}_0 = M - 1, \bar{i}_0 = M$, and $t_0^{**}(M-1) = 0 < 1 = t_0^{**}(M)$ and the second part follows because $a_1^{**}(M) = \bar{a}(0,1) > a^R(0) = a_1^{**}(M-1)$. Part (2) follows from the definitions of $\underline{i}_k$ and $\bar{i}_k$ because $\underline{i}_0 = M - 1$ and $\bar{i}_0 = M$ are maximal values for $\underline{i}_k$ and $\bar{i}_k$. The definition of t^{**} guarantees that $\underline{i}_1 \geq M - 2$. The definition of t_k^{**} implies that $t_1^{**}(M-2) = 0$; therefore $\bar{i}_1 \geq M - 1$. Because $\bar{i}_0 = M$ the first clause of Part (4) is vacuous. If $n \leq \underline{i}_0$, then $t_0^{**}(n-1) = t_0^{**}(n) = 0$. Consequently, if $n = 1$, then $a_1^*(n) = a^R(0)$ by definition. Otherwise, $a_1^*(n) = \max\{a_0^{**}(n), a_1^{**}(n-1)\}$, which implies the result by induction (increasing n from 1) because $a_0^{**}(n) = 0$. This establishes the second clause of Part (4).

Parts (3)-(9) hold because a_0^{**} is the minimum strategy in $\mathcal{R}_0$ and t_0^{**} is the minimum strategy in $\mathcal{T}_0$.

Now suppose that for $k \geq 1$, the claim is true for $j = 0, \ldots, k-1$.

Part (1). We prove the first clause first. If $\bar{i}_k = 1$, then $t_k^{**}(\underline{i}_k) = 0 < 1 = t_k^{**}(\bar{i}_k)$. Consequently, the result follows.

Otherwise, $\bar{i}_k > 1$. If $\bar{i}_k = M$, then $t_k^{**}(M-1) < t_k^{**}(M)$ by definition. We therefore need only prove the first part of Part (1) for $n \leq \min\{M-1, \bar{i}_k\}$. Assume that $n \leq \min\{M-1, \bar{i}_k\}$. Part (2) implies that $\bar{i}_k \leq \bar{i}_{k-1}$. Therefore, by Part (1) (for $k-1$), $a_k^{**}(\bar{i}_k) > a_k^{**}(\bar{i}_k - 1)$ (note $\bar{i}_k > 1$ implies $a_k^{**}(\bar{i}_k - 1)$ is well defined). We claim that

$$a_k^{**}(\bar{i}_k - 1) < a^S(1). \tag{38}$$

To see this, assume that $a_k^{**}(\bar{i}_k - 1) \geq a^S(1)$ and argue to a contradiction. Because $a_k^{**}(\bar{i}_k) > a_k^{**}(\underline{i}_k - 1)$, all types would strictly prefer $a_k^{**}(\underline{i}_k - 1)$ to $a_k^{**}(\underline{i}_k)$, which would imply that $t_k^{**}(\bar{i}_k - 1) = t_k^{**}(\bar{i}_k) = 0$, contrary to the definition of $\bar{i}_k$. Hence (38) holds.

Similarly, $a_k^{**}(\underline{i}_k + 2) > a^S(0)$. Otherwise, either $a_k^{**}(\underline{i}_k + 2) > a_k^{**}(\underline{i}_k + 1)$ and all types weakly prefer $a_k^{**}(\underline{i}_k + 2)$ to $a_k^{**}(\underline{i}_k + 1)$ or $a_k^{**}(\underline{i}_k + 2) = a_k^{**}(n)$ for $n < \underline{i}_k + 2$. Either imply that $t_k^{**}(\underline{i}_k + 1) = 0$, contrary to the definition of $\underline{i}_k$. It follows that for each $n \in (\underline{i}_k + 1, \bar{i}_k)$, $a_k^{**}(n) \in (a^S(0), a^S(1))$. For $n \in (\underline{i}_k + 1, \bar{i}_k)$ there is a unique t_n such that $a_k^{**}(n) = a^S(t_n)$ and $a_k^{**}(n') \neq a^S(t_n)$ for $n' \neq n$. Consequently $t_n \in (t_k^{**}(n-1), t_k^{**}(n))$, which implies $0 < t_k^{**}(\underline{i}_k+1) < t_k^{**}(\underline{i}_k+2) < \cdots < t_k^{**}(\bar{i}_k-1) < 1$. Because $t_k^{**}(\underline{i}_k) = 0$ and $t_k^{**}(\bar{i}_k) = 1$, the first statement in Part (1) is true.

It follows from the first sentence of Part (1) that $\bar{a}(t_k^{**}(n), t_k^{**}(n+1))$ is strictly increasing in n for $n \in [\underline{i}_k, \bar{i}_k)$ and the definition of a_{k+1}^{**} therefore implies that $a_{k+1}^{**}(n)$ is strictly increasing for $n \in (\underline{i}_k, \bar{i}_k]$. Furthermore, $a_{k+1}^{**}(\underline{i}_k + 1) = \bar{a}(t_k^{**}(\underline{i}_k), t_k^{**}(\underline{i}_k + 1)) > a^R(0)$, while $a_{k+1}^{**}(n) \leq a^R(0)$ for $n \leq \underline{i}_k$, which establishes the second clause in Part (1).

Part (2). (37) and Part (5) (for $k-1$) establish that $\underline{i}_k \geq \underline{i}_{k+1}$ and $\underline{i}_k \geq \underline{i}_{k+1}$. Finally, to show $\underline{i}_k \leq \underline{i}_{k+1} + 1$, note that $a_{k+1}^{**}(n) = a^R(0)$ for $n \leq \underline{i}_k$ and $t_{k+1}^{**}(\underline{i}_k) = 0$ by the definition of t_k^{**}.

Part (3). Part (3) follows directly from the definition of a_k^{**} when $n \geq \bar{i}_k$. It follows from Parts (1) and (5) (for $k-1$) when $n \in [\underline{i}_k, \bar{i}_k)$. It follows from Part (5) when $n < \underline{i}_k$.

Part (4). Assume that $a_{k+1}^{**}(\bar{i}_k) < a^S(1)$. It follows from Part (3) that $a_k^{**}(\bar{i}_k) < a^S(1)$, which means that $t_k^{**}(\bar{i}_k) < 1$, in contradiction to the definition of $\bar{i}_k$. This establishes the first clause of Part (4). If $n < \underline{i}_k$, then $t_k^{**}(n+1) = t_k^{**}(n) = 0$. Consequently, if $n = 1$, then $a_{k+1}^{**}(n) = a^R(0)$ by definition. Otherwise, $a_{k+1}^{**}(n) = \max\{a_k^{**}(n), a_k^{**}(n-1)\}$, which implies the result by induction (increasing n from 1) because $a_k^{**}(n) = a^R(0)$ for $n < \underline{i}_{k-1}$ by Part (4) $(k-1)$ and $n < \underline{i}_k$ implies $n < \underline{i}_{k-1}$ by Part (2).

Part (5). t_k^{**} is a best response to a_k^{**}. It follows that

$$u^S(a_k^{**}(n), t_k^{**}(n)) = u^S(a_k^{**}(n+1), t_k^{**}(n)). \tag{39}$$

If $a_k^{**}(n) < a_k^{**}(n+1)$, then (39) and Part (3) imply that

$$u^S(a_{k+1}^{**}(n), t_k^{**}(n)) \geq u^S(a_{k+1}^{**}(n+1), t_k^{**}(n)).$$

Hence when $a_k^{**}(n) < a_k^{**}(n+1)$ and $a_{k+1}^{**}(n) < a_{k+1}^{**}(n+1)$, $t_{k+1}^{**}(n) \geq t_k^{**}(n)$ follows from the definition of t_k^*. Consequently, by the second clause of Part (1) (for $k-1$ and k) and Part (2) the result holds for $\underline{i}_k \leq n < \bar{i}_{k-1}$.

If $n \geq \bar{i}_k$, then $n \geq \bar{i}_{k+1}$ by Part (2). Consequently, by the definition of $\bar{i}_k$, $t_{k+1}^{**}(n) = 1 \geq t_k^{**}(n)$.

If $n < \underline{i}_{k-1}$, then $n < \underline{i}_k + 1$ by Part (2). Consequently, $n \leq \bar{i}_k$, and, by the definition of $\underline{i}_k$, $t_k^{**}(n) = 0 \leq t_{k+1}^{**}(n)$, so the result follows.

Part (6). Let $\hat{a}_k \in \mathcal{R}_k$ and assume that there exists $n' \le \bar{i}_k$ such that $\hat{a}_k(n') < a_k^{**}(n')$. We will show that this leads to a contradiction. Observe that $\underline{i}_{k-1} < n'$ because if $n' \le \underline{i}_{k-1}$, then $a_k^{**}(n') = a^R(0) \le a_k(n)$ for all a_k.

Consider the strategy $\tilde{a}_k$ defined by

$$\tilde{a}_k(n) = \begin{cases} \hat{a}_k(n) & \text{if } n < n', \\ \max\{\hat{a}_k(n), a_k^{**}(n')\} & \text{if } n' \le n. \end{cases} \tag{40}$$

We claim that $\tilde{a}_k$ weakly dominates $\hat{a}_k$. By definition, $\tilde{a}_k(n) = \hat{a}_k(n)$ when $n < n'$. When $n \in (\underline{i}_{k-1}, \bar{i}_{k-1}]$, $a_k^{**}(n) = \bar{a}(t_{k-1}^{**}(n-1), t_{k-1}^{**}(n))$. For $n = n'$, it follows from Part (8) $(k-1)$ and $\hat{a}_k(n') < a_k^{**}(n')$ that $\tilde{a}_k(n') = a_k^{**}(n')$ does at least as well as $\hat{a}_k(n')$ against any $t_{k-1} \in \mathcal{T}_{k-1}$ and it does strictly better against t_{k-1}^{**}.

If $n \ge n'$, either $\tilde{a}_k(n) = \hat{a}_k(n)$ so that the strategies have the same performance against n or

$$a_k^{**}(n') = \tilde{a}_k(n) > \hat{a}_k(n) \ge \hat{a}_k(n'). \tag{41}$$

By definition, $a_k^{**}(n')$ is an optimal response to $\theta \in [t_{k-1}^{**}(n'-1), t_{k-1}^{**}(n')]$. From Part (8), $t_k \ge t_k^{**}$ for all $t_k \in \mathcal{T}_k$, $\tilde{a}_k$ outperforms $\hat{a}_k$ when (41) holds.

When $\bar{i}_{k-1} + 1 > \bar{i}_k$, we have $a_k^{**}(n) = \max\{a_k^{**}(n-1), a_{k-1}^{**}(n)\}$. Consequently $a_k^{**}(n) \le a_{k-1}(n)$ by Part (6) $(k-1)$. This establishes Part (6).

Part (7). a_k^{**} is a best response to t_{k-1}^{**}. If there were a strategy $\hat{a}_k \ne a_k^{**}$ that dominates a_k^{**}, then $\hat{a}_k$ would also be a best response to t_{k-1}^{**}. If $\hat{a}_k$ is a best response to t_{k-1}^{**}, then $\hat{a}_k(n) = a_k^{**}(n)$ whenever $t_{k-1}^{**}(n-1) < t_{k-1}^{**}(n)$. Hence, by Part (1), $\hat{a}_k$ could only dominate a_k^{**} if either (a) there exists $n' \le \underline{i}_{k-1}$ such that $\hat{a}_k(n') > a^R(0)$; or (b) there exists $n' > \bar{i}_{k-1}$ such that $\hat{a}_k(n') \ne a_k^{**}(n')$.

Consider Case (a). Because $n' \le \underline{i}_{k-1}$, $t_{k-1}^{**}(n') = 0$ and, by Part (4), $a_k^{**}(n') = a^R(0)$. If $t_{k-1}(n') = 0$ for all $t_{k-1} \in \mathcal{T}_{k-1}$, then $\hat{a}_k$ cannot dominate a_k^{**} on the basis of behavior at n'. If there is $\hat{t}_{k-1} \in \mathcal{T}_{k-1}$ such that $\hat{t}_{k-1}(n') > 0$, then let $\hat{t}_{k-1}$ be a best response to $\hat{a}_{k-1} \in \mathcal{R}_{k-1}$. It follows that $u^S(\hat{a}_{k-1}(n'), \theta) \ge u^S(\hat{a}_{k-1}(n'+1), \theta)$ for $\theta < \hat{t}_{k-1}(n')$. Furthermore, because $t_{k-1}^{**}(n') = 0$, $u^S(a_{k-1}^{**}(n'-1), 0) \le u^S(a_{k-1}^{**}(n'), 0)$ and therefore $u^S(a_{k-1}^{**}(n'-1), \theta) \le u^S(a_{k-1}^{**}(n'), \theta)$ for all θ. Consequently, given any $\theta \in (0, \hat{t}_{k-1}(n'))$ there is a mixture of $\hat{a}_{k-1}$ and a_{k-1}^{**}, say $\tilde{\alpha}_{k-1} = \delta\, \hat{a}_{k-1} + (1-\delta) a_{k-1}^{**}$ such that that $u^S(\tilde{\alpha}_{k-1}(n'), \theta) = u^S(\tilde{\alpha}_{k-1}(n'+1), \theta)$. Therefore the best response to $\tilde{a}_{k-1}$, $\tilde{t}_k \in \mathcal{T}_{k-1}$, will have the property that $\tilde{t}_{k-1}(n')$ is arbitrarily close to zero. Hence, it will be the case that $\hat{a}_k(n')$ is inferior to $a_k^{**}(n')$ as a response to $\tilde{t}_{k-1}$. Hence $\hat{a}_k$ cannot dominate a_k^{**}.

In Case (b), $t_{k-1} \in \mathcal{T}_{k-1}$ implies that $t_{k-1}(\bar{i}_{k-1}) = 1$ by Part (8). Hence $\hat{a}_k$ cannot dominate a_k^{**} on the basis of behavior at n'.

Part (8). When $n \le \bar{i}_k$, $t_k^{**}(n) \; = 0 \le t_k(t)$ for $t_k \in \mathcal{T}_k$.

When $n \in [\underline{i}_k + 1, \bar{i}_{k-1} - 1]$, we have

$$u^S(a_k^{**}(n), t_k^{**}(n)) \leq u^S(a_k^{**}(n+1), t_k^{**}(n)). \tag{42}$$

Part (1) $(k-1)$ implies $\underline{i}_k \geq \underline{i}_{k-1} - 1$ and so $n \in [\underline{i}_{k-1}, \bar{i}_{k-1} - 1]$. Therefore $a_k^{**}(n) < a_k^{**}(n+1)$ by Part (2) $(k-1)$. Part (6) and Inequality (42) imply that

$$u^S(a_k(n), \theta) \leq u^S(a_k(n+1), \theta) \text{ for all } \theta < t_k^{**}(n) \text{ and all } a_k \in \mathcal{R}_k$$

with strict inequality for $a_k = a_k^{**}$. By Part (7), $a_k^{**} \in \mathcal{R}_k$. Therefore $t_k(n) \geq t_k^{**}(n)$ for all $t_k \in \mathcal{T}_k$.

When $n \geq \bar{i}_{k-1}$, $t_{k-1}^{**}(n) = 1$. By Part (8) $(k-1)$, $t(n) \geq t_{k-1}^{**}(n)$ for all $t \in \mathcal{T}_{k-1}$. Because $\mathcal{T}_k \subset \mathcal{T}_{k-1}$, it follows that $t_k(n) = 0$.

Part (9). t_k^* is a best response to a_k^*. Further, it follows from the second clause of Part (1) $(k-1)$ that $t_k^*(n)$ is the unique best response to a_k^* for $n \in [\underline{i}_{k-1}, \bar{i}_{k-1})$.

Next consider the case where $n \geq \bar{i}_{k-1}$. If, in addition, $n > \bar{i}_k$, then $a_k^{**}(n) \geq a^S(1)$ by Part (4), so all types weakly prefer $a_k^{**}(\bar{i}_k)$ to $a_k^{**}(n)$ and hence, by Part (6) they also prefer $a_k(\bar{i}_k)$ to $a_k(n)$ for all $a_k \in \mathcal{R}_k$. Hence it is not possible to dominate t_k^{**} for these n. Otherwise, it must be that $n = \bar{i}_k$. By the definition of t_k^{**} if $a_k^{**}(n-1) = a_k^{**}(n)$, then $t_k^{**}(n-1) = t_k^{**}(n)$. But, by definition of $\bar{i}_k$, $1 = t_k^{**}(\bar{i}_k) > t_k^{**}(\bar{i}_k - 1)$. We conclude that $n = \bar{i}_k$ implies that $a_k^{**}(n-1) < a_k^{**}(n)$; so types θ close to one strictly prefer $a_k^{**}(n)$ to $a_k^{**}(n+1)$ and hence t_k^{**} cannot be weakly dominated at n.

Hence if t_k weakly dominates t_k^{**} it must be that $t_k(n) > 0$ for $n \leq \underline{i}_k$, which implies that $t_k(\underline{i}_k) > 0$. If $\underline{i}_k \leq M - k - 1$, then $k < M$ and $t_k(\underline{i}_k) = 0$ for all $t_k \in \mathcal{T}_k$ by the definition of $\mathcal{T}_k$. In this case, $t_k(\underline{i}_k) > 0$ is impossible. If $\underline{i}_k > M - k - 1$, then, by the definition of t_k^{**}, $u^S(a_k^{**}(\underline{i}_k), 0) \leq u^S(a^R(0), 0)$. Also, $a_k^{**}(\underline{i}_k) \geq a^R(0)$. Consequently, $u^S(a_k^{**}(\underline{i}_k), \theta) \leq u^S(a^R(0), \theta)$ for all θ and so t_k with $t_k(\underline{i}_k) > 0$ cannot weakly dominate t_k^{**}.

□

The claim constructs an increasing sequence of Sender strategies and an increasing sequence of Receiver strategies that provide bounds for strategies remaining during the process of deletion. It is straightforward to confirm that these sequences converge to an equilibrium.

Claim 12. *The sequence $\{(a_k^{**}, t_k^{**})\}$ converges. The limit, $\{(a^{**}, t^{**})\}$ is a Nash equilibrium for the game.*

Proof. Consider the sequence $\{a_k^{**}\}$. It follows from Part (3) of Claim 11 that the sequence $\{a_k^{**}(n)\}$ is monotonically increasing and bounded. Hence it converges. Similarly, from Part (5), $\{t_k^{**}\}$ converges. Let $(a^{**}, t^{**}) = \lim_{k \to \infty}(a_k^{**}, t_k^{**})$. By the definition of the sequences $\{a_k^{**}\}$ and $\{t_k^{**}\}$, (a^{**}, t^{**}) is a Nash equilibrium. □

We add to the claim an observation that depends on boundary conditions:

Claim 13. *If $a^S(1) > a^R(1)$, then there exists $\delta > 0$ such that $t^{**}(M-1) \leq 1-\delta$.*

Proof. By assumption, there exists $\delta > 0$ such that any Sender with type $\theta \geq 1-\delta$ prefers $a^R(1)$ to any smaller action. It must be that $t_k^{**}(M-1) \leq 1-\delta$ for all k. Hence $t^{**}(M-1) \leq 1$.

□

Claim 14. *If $a^S(1) > a^R(1)$, $\{(a^{**}, t^{**})\}$ satisfies NITS and the Sender uses only the highest N^* messages with positive probability.*

Proof. From Claim 13, $t^{**}(M-1) < 1$, so that $\bar{i}^{**} \equiv lim_{k\to\infty}\bar{i}_k = M$. Claim 11 guarantees that t_k^{**} is strictly increasing for $n \in (\underline{i}^{**}, M]$. Because $\{(a^{**}, t^{**})\}$ is a Nash Equilibrium and all Nash Equilibria induce at most N^* actions, it must be that if $M > N^*$ then $\underline{i}^{**} > 0$.

It follows from Part 4 of Claim 11 that NITS is satisfied. □

This result is sufficient to conclude that $a^{**}(n) = a^*(n)$ for $n > \underline{i}^*$ if there is a unique NITS outcome. That is, when there is only one equilibrium that satisfies NITS, dominance selects that outcome.

We have described an iterative process that associates with any pair (a, t) another pair $F(a, t)$. Starting from a high initial condition, the process generates a decreasing sequence that converges to an equilibrium. Starting from a low initial condition, the process generates an increasing sequence that converges to an equilibrium. Any equilibrium outcome of the process is a fixed-point of the process and the process satisfies $F(a, t) \geq F(a', t')$ if $(a, t) \geq (a', t')$. Consequently the argument guarantees that the decreasing sequence converges to the biggest NITS equilibrium and the increasing sequence converges to the smallest NITS equilibrium. That is, if NITS is not unique, then the set of strategies that survive deletion of weakly dominated strategies contains at least two equilibrium outcomes. The Example in Appendix D illustrates this property.

In Appendix A (Claim 7) we demonstrated that the essential properties of (a^*, t^*) do not depend on the order of deletion of weakly dominated strategies. Similarly, the essential properties of (a^{**}, t^{**}) do not depend on the order of deletion of weakly dominated strategies.

Claim 15. *If $t \in \tilde{\mathcal{T}}^*$, then $t \geq t^{**}$. If $a \in \tilde{\mathcal{R}}^*$, then $a(n) \geq a^S(1)$ if $n \geq \bar{i}^{**}$ and $a(n) \geq a^{**}(n)$ if $n \leq \bar{i}^{**}$.*

References

[1] Nemanja Antić and Nicola Persico. Equilibrium selection through forward induction in cheap talk games. Technical report, Northwestern University, August 2020.

[2] Andreas Blume. Equilibrium refinements in sender-receiver games. *Journal of Economic Theory*, 64(1):66–77, October 1994.

[3] Andreas Blume. Neighborhood stability in sender–receiver games. *Games and Economic Behavior*, 13(1):2–25, 1996.

[4] Adam Brandenburger, Amanda Friedenberg, and H. Jerome Keisler. Admissibility in Games. *Econometrica*, 76(2):307–352, 2008.

[5] Ying Chen. Perturbed communication games with honest senders and naive receivers. *Joumal of Economic Theory*, 146(2):401–424, March 2011.

[6] Ying Chen, Navin Kartik, and Joel Sobel. Selecting cheap-talk equilibria. *Econometrica*, 76(1):117–136, January 2008.

[7] Vincent P. Crawford and Joel Sobel. Strategic information transmission. *Econometrica*, 50(6):1431–1451, November 1982.

[8] Francesc Dilmé. Robust information transmission. Technical report, University of Bonn, December 28, 2020.

[9] Joseph Farrell. Meaning and credibility in cheap-talk games. *Games and Economic Behavior*, 5(4):514–531, October 1993.

[10] Sidartha Gordon. On infinite cheap talk equilibria. Technical report, Université de Montréal, August 2010.

[11] Sidartha Gordon. Iteratively stable cheap talk equilibrium. Technical report, Université de Montréal, May 2011.

[12] Jerry R. Green and Nancy L. Stokey. A two-person game of information transmission. *Journal of Economic Theory*, 135(1):90–104, 2007.

[13] John Hillas and Dov Samet. Weak dominance: A mystery cracked. Technical report, Tel Aviv University, June 9 2014.

[14] Ehud Kalai and Dov Samet. Persistent equilibria in strategic games. *International Journal of Game Theory*, 13(3):129–144, 1984.

[15] Navin Kartik. Strategic communication with lying costs. *Review of Economic Studies*, 76(4):1359–1395, 2009.

[16] Navin Kartik and Joel Sobel. Effective communication in cheap-talk games. Technical report, UCSD, in preparation 2015.

[17] Elon Kohlberg and Jean-François Mertens. On the strategic stability of equilibria. *Econometrica*, 54(5):1003–1037, September 1986.

[18] Melody Pei-yu Lo and Wojciech Olszewski. Learning in cheap-talk games. Technical report, Northwestern University, 2018.

[19] Pei-yu Lo. Common knowledge of language and iterative admissibility in a sender-receiver game. Technical report, Brown University, June 2007.

[20] Pei-yu Lo. Language and coordination games. Technical report, University of Hong Kong, October 2009.

[21] Jeffrey Mensch. On the existence of monotone pure-strategy perfect Bayesian equilibrium in games with complementarities. Technical report, Hebrew University of Jerusalem, 2018.

[22] Paul Milgrom and John Roberts. Rationalizability, learning, and equilibrium in games with strategic complementarities. *Econometrica*, 58(6):1255–1277, November 1990.

[23] Wojciech Olszewski. Rich language and refinements of cheap-talk equilibria. *Journal of Economic Theory*, 128(1):164–186, 2006.

[24] Wojciech Olszewski. A result on convergence of sequences of iterations, with applications to best-response dynamics. Technical report, Northwestern University, 2019.

[25] Joel Sobel. Interative weak dominance and interval-dominance supermodular games. *Theoretical Economics*, 14(1):71–102, January 2019.

[26] Donald M. Topkis. Equilibrium points in nonzero-sum n-person submodular games. *Siam Journal of Control and Optimization*, 17:773–787, 1979.

[27] Eric Van Damme. *Stability and Perfection of Nash Equilibria.* Springer Verlag,, Berlin, 1987.

[28] Xavier Vives. Nash equilibrium with strategic complementarities. *Journal of Mathematical Economics*, 19(3):305–321, 1990.