Observational Learning with Ordered States

Navin Kartik    SangMok Lee    Daniel Rappoport

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Sequential observational learning model (Banerjee '92; BHW '92)

unknown state $\omega \in \Omega$

each $n = 1, 2, \ldots$ takes action $a_n \in A$ finite

using private signal and history of actions

homogenous prefs $u(a, \omega)$

Many extensions, variations

Q: does society eventually learn $\omega$?

A: Unbounded vs. bounded beliefs/signals (Smith & Sørensen '00)

Given any prior,

~ can private beliefs $\rightarrow$ certainty about every $\omega$?

~ are private beliefs bounded away from 0 about every $\omega$?
Motivation (2)

- Unbounded beliefs $\implies$ learning for all prefs
  Bounded beliefs $\implies$ nonlearning for all (nontrival) prefs

- Essentially exhaustive with two states (most papers)

- But with multiple states, a large gap

- Suppose $\Omega = \{1, 2, 3\}$ and signals $\mathcal{N}(\omega, 1)$

  Neither unbounded nor bounded!
    $\implies$ can become certain about 1 or 3 but not 2

So is there learning? Say with $u(a, \omega) = -(a - \omega)^2$
This Paper

• Prefs satisfying single-crossing differences (SCD)
  widely-used property (Milgrom & Shannon ’94)
  but not previously for learning
  satisfied by quadratic loss

• Information satisfying directionally unbounded beliefs (DUB)
  new property
  → can get certainty about each state vs. lower/upper sets
  weaker than unbounded beliefs
  satisfied by normal information

• Main result
  SCD & DUB are a minimal pair of sufficient conditions for learning
Literature

Most related (will elaborate later)

- Smith & Sørensen '00
- Arieli & Mueller-Frank '19

Other mechanisms for learning

- Infinite actions with responsive prefs Lee '93; Ali '18
- Suitably heterogenous preferences Goeree, Palfrey, Rogers '06
- Prices/congestion costs Avery & Zemsky '98; Eyster, Galeotti, Kartik, Rabin '14

Things we don’t tackle

- Partial observation of history Acemoglu, Dahleh, Lobel, Ozdaglar '11
- Speed of convergence Rosenberg & Vielle '19
Model
Environment

- Countable set of states $\Omega \subset \mathbb{R}$ (finite or infinite)
  so states are ordered

- Signal set $S \subset \mathbb{R}$, either countable or interval
  order not needed for main result
  but is when we invoke MLRP

- Action set $\mathcal{A}$; countable choice set $\mathcal{A} \subseteq \mathcal{A}$

- Signal structure $f(s|\omega)$
  assume no signal can exclude any state: $f(\cdot) > 0$
  technical: $\forall s \ f(s|\cdot)$ is bounded
The Game

- State $\omega$ drawn from pmf $\mu_0 \in \Delta \Omega$; unobservable

- Agents 1, 2, ... sequentially select actions
  - agent $n$ chooses $a_n \in A$ at date $n \in \mathbb{N}$
  - after observing indep private signal $s_n \sim f(\cdot | \omega)$
  - and action history $h^n \equiv (a_1, \ldots, a_{n-1}) \in A^{n-1}$

- Strategy $\sigma_n : S \times A^{n-1} \rightarrow \Delta A$

- All agents have vNM utility $u : A \times \Omega \rightarrow \mathbb{R}$

- Bayes Nash equilibria (or refinements)
Learning (1)

- Given prior $\mu_0$, info structure $f$, and strategies $(\sigma_n)$, every history induces a public belief $\mu(h^n) \in \Delta \Omega$

- Let $\tilde{\mu}_n$ denote corresponding r.v.

- $\langle \tilde{\mu}_n \rangle$ is a martingale that $\to_{a.s.} \tilde{\mu}^*$

- Intuitively, learning if, a.s., $\tilde{\mu}^*$ allows agents to make correct decisions
Learning (2)

For $\mu \in \Delta \Omega$, let $c(\mu) \equiv \arg\max_{a \in A} \mathbb{E}_\mu[u(a, \omega)]$ (omitting dependence of $c$ on $A$)

Let $Q$ be beliefs with adequate knowledge: $Q \equiv \{\mu : \bigcap_{\omega \in \text{Supp} \mu} c(\omega) \neq \emptyset\}$

→ no gain to learning anything further

Definition

Fix prefs $u$ and info structure $f$.

1. There is adequate learning if for every choice set, every prior, and every equilibrium, $\Pr(\tilde{\mu}^* \in Q) = 1$.

2. There is inadequate learning if for some choice set and prior, in every equilibrium $\Pr(\tilde{\mu}^* \in Q) < 1$.

(1): asympt. take correct actions

(2): asympt. sometimes take incorrect actions, for some choice set and prior

For what $(u, f)$ is there adequate learning?
SCD Preferences
and
DUB Information
Single-Crossing Differences

\( h : \mathbb{R} \rightarrow \mathbb{R}\setminus\{0\} \) is single crossing if either

\[ \forall x < x': \ h(x) > 0 \implies h(x') \geq 0; \]

or

\[ \forall x < x': \ h(x) < 0 \implies h(x') \leq 0. \]

Definition

Utility \( u : \mathcal{A} \times \Omega \rightarrow \mathbb{R} \) has single-crossing differences (SCD) if

\[ \forall a, a' : \ u(a, \omega) - u(a', \omega) \text{ is single crossing in } \omega. \]

\sim \text{ Milgrom & Shannon '94 / Athey '01, but w/o order on } \mathcal{A} \text{ (KLR '19)}

implied by supermodularity

SCD \iff interval choice:

\[ \forall \text{ choice sets and } \omega_1 < \omega_2 < \omega_3, \ \{a\} = c(\omega_1) \cap c(\omega_3) \implies a \in c(\omega_2) \]
Directionally Unbounded Beliefs

Definition

Signal structure $f(s|\omega)$ has directionally unbounded beliefs (DUB) if $\forall \omega$:

(i) $\exists (\bar{s}_i)$ s.t. $\forall \omega' < \omega$, $\lim_{i \to \infty} \frac{f(\bar{s}_i|\omega')}{f(\bar{s}_i|\omega)} = 0$; and

(ii) $\exists (s_i)$ s.t. $\forall \omega' > \omega$, $\lim_{i \to \infty} \frac{f(s_i|\omega')}{f(s_i|\omega)} = 0$.

+ uniform boundedness condition for $\Omega$ infinite

(i) $\iff$ can simultaneously distinguish $\omega$ from all lower states

   given any prior $\mu$ with $\mu(\omega) > 0$, can rule out $\{\omega' : \omega' < \omega\}$

(ii) $\iff$ can simultaneously distinguish $\omega$ from all higher states

   given any prior $\mu$ with $\mu(\omega) > 0$, can rule out $\{\omega' : \omega' < \omega\}$

May not sim distinguish $\omega$ from both lower and higher states!
Main Result
Learning with SCD & DUB

Theorem

1. SCD prefs & DUB info $\implies$ adequate learning.
2. If prefs fail SCD, then $\exists$ DUB info with inadequate learning.
3. If info fails DUB and $|\Omega| < \infty$, then $\exists$ SCD prefs with inadeq learning.

A key fact for the proof

- Given $(u, f, A)$, say that $\mu \in \Delta \Omega$ is stationary if a.s. $c(\mu_s) = c(\mu)$ ($\mu_s$ is posterior; assume unique choices)

- Adeq learning $\iff$ all stationary beliefs have adeq knowledge ($\mu \in Q$)
  $\implies$ if $\mu \notin Q$ is stationary, consider the prior being $\mu$
  $\iff$ all limits beliefs are stationary (Arieli & Muller-Frank ’19)

- Nb: adeq knowledge means no value of any info;
  stationary means no value of info from $f(s|\omega)$
Learning with SCD & DUB

Theorem

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2. If prefs fail SCD, then $\exists$ DUB info with inadequate learning.
3. If info fails DUB and $|\Omega| < \infty$, then $\exists$ SCD prefs with inadeq learning.

Intuition for part 1:

- In general, $\mu \notin Q$ may be stationary $\therefore$ mismatch between prefs and info
- SCD $\implies$ if $\mu \notin Q$ then $\exists \omega^* \in \text{Supp } \mu$ s.t.
  
  $u(c(\omega^*), \omega^*) > u(c(\mu), \omega^*)$ \text{ and }
  
  $u(c(\omega^*), \omega) \geq u(c(\mu), \omega) \ \forall \omega > \omega^*$ or $\forall \omega < \omega^*$

- DUB $\implies$ $\exists$ signals that rule out $\{\omega : \omega < \omega^*\}$ and $\{\omega : \omega > \omega^*\}$
  
  $\rightarrow c(\mu)$ not chosen after those signals

- So SCD + DUB $\implies$ any $\mu \notin Q$ is not stationary

(faulty intuition)
Learning with SCD & DUB

Theorem

1. SCD prefs & DUB info \( \Rightarrow \) adequate learning.

2. If prefs fail SCD, then \( \exists \) DUB info with inadequate learning.

3. If info fails DUB and \( |\Omega| < \infty \), then \( \exists \) SCD prefs with inadeq learning.

Intuition for part 1:
Learning with SCD & DUB

**Theorem**

1. SCD prefs & DUB info \(\implies\) adequate learning.
2. If prefs fail SCD, then \(\exists\) DUB info with inadequate learning.
3. If info fails DUB and \(|\Omega| < \infty\), then \(\exists\) SCD prefs with inadequate learning.

Intuition for part 2:

In fact: given non-SCD prefs, any DUB and MLRP info \(\implies\) inadequate learning.
Learning with SCD & DUB

Theorem

1. SCD prefs & DUB info $\implies$ adequate learning.

2. If prefs fail SCD, then $\exists$ DUB info with inadequate learning.

3. If info fails DUB and $|\Omega| < \infty$, then $\exists$ SCD prefs with inadequate learning.

Intuition for part 3:

Trickier case in right panel: what if certainty possible about extreme states? May need to restrict prior’s support
Learning with SCD & DUB

Theorem

1. SCD prefs & DUB info $\implies$ adequate learning.

2. If prefs fail SCD, then $\exists$ DUB info with inadequate learning.

3. If info fails DUB and $|\Omega| < \infty$, then $\exists$ SCD prefs with inadeq learning.

Intuition for part 3:

In fact: Assume MLRP and not DUB.

$\exists$ SCD prefs s.t. there is inadeq learning for any full support prior.
Discussion
DUB in Location Families

Location family: \( S = \mathbb{R} \) and for some density \( g \), \( f(s|\omega) = g(s - \omega) \)

- e.g., Normal info
- Intuitively, DUB requires a thin tail of standard density \( g \)

\( g \) strictly subexponential: \( \exists p > 1 \) s.t. \( g(x) < \exp[-|x|^p] \) for large \( |x| \)

Proposition

In a location family, DUB holds if \( g \) is strictly subexponential.

- If \( g \) is exponential then \( g(s - \omega')/g(s - \omega) = \exp(\omega' - \omega) \) is indep of \( s \)
- An even thicker tail (superexp) makes extreme signals uninformative
- So Laplace, Cauchy, Student-t distrs fail DUB
Unbounded Beliefs

Restrict to finite $\Omega$, for simplicity

Unbounded beliefs: $\forall \omega \exists (s_i) \text{ s.t. } \forall \omega', \omega \neq \omega, \lim_{i \to \infty} \frac{f(s_i|\omega')}{f(s_i|\omega)} = 0$.

(Smith & Sørensen '00; “totally unbounded” in Arieli & Mueller-Frank '19)

- Each $\omega$ can be simultaneously distinguished from all others
- DUB weaker $\therefore$ for each $\omega$, separately distinguish upper and lower sets
- Unbounded beliefs $\iff$ adeq learning for all preferences
- But unbounded beliefs very demanding with more than two states

Proposition

Assume $|\Omega| > 2$. MLRP $\implies$ not unbounded beliefs.

- So, with multiple states, must restrict prefs to obtain learning
Pairwise Unbounded Beliefs

Pairwise unbounded beliefs: \( \forall \omega \exists \omega' \neq \omega \exists (s_i) \text{ s.t. } \lim_{i \to \infty} \frac{f(s_i|\omega')}{f(s_i|\omega)} = 0. \)

(Arieli & Mueller-Frank ’19)

- Each \( \omega \) can be distinguished from every other, but not simultaneously
- DUB is stronger: simultaneously distinguish each \( \omega \) from its upper set and its lower set
- Pairwise UB is not sufficient for adeq learning under SCD
- But pairwise UB is necessary for adeq learning over any “minimally-rich” class of preferences

Proposition

Assume MLRP. Pairwise UB \( \iff \) DUB.

- So, given MLRP, DUB is unavoidable for adeq learning
On Bounded Beliefs

Bounded beliefs: $\forall \omega, \omega', \frac{f(s|\omega')}{f(s|\omega)}$ is bounded above in $s$

- A natural notion, generalizing two-state case
- But stronger than just ruling out certainty about any state
- Negation of pairwise unbounded beliefs (for every pair)
- So incompatible with DUB
- Guarantees inadequate learning for all nontrivial prefs
Conclusion
Conclusion

Recap:
- Std condition, unbounded beliefs, very demanding with > 2 states
- Study learning under economic pref restriction with ordered states
- New informational condition: DUB
  - weaker than unbounded beliefs
  - rules out bounded beliefs
- DUB info and SCD prefs are minimal pair of suff conditions for adequate learning

Future directions:
- Extend to other obsv learning environments (e.g., partial histories)
- Speed of convergence?
- Is DUB useful in other contexts?
Thank you!
Faulty Intuition for Sufficiency

- Take $\mu$ with inadequate knowledge
  1. SCD implies different optimal actions at extreme states of $\text{Supp } \mu$
  2. DUB implies potential certainty about extreme states of $\text{Supp } \mu$
  3. $\mu$ is non-stationary

- Is learning about SCD or different optimal actions at extreme states?

- Our result applies to infinite states and weak SCD environments where above logic fails
  → Not about responsive preferences
Faulty Intuition for Sufficiency

- Let $\Omega = \mathbb{Z}$ and $A = \mathbb{Z} \cup \{a^*\}$

$$u(a, \omega) = \begin{cases} 
1 & \text{if } a = \omega \\
0 & \text{if } a \notin \{\omega, a^*\} \\
1 - \varepsilon & \text{if } a = a^* 
\end{cases}$$

- For small $\varepsilon > 0$, $a^*$ is a safe action but suboptimal in every state

- Different optimal action in every state, but $u$ violates SCD

- Suppose $s \sim \mathcal{N}(\omega, 1)$. For any full support prior
  - signals cannot provide certainty about any state
  - for small enough $\varepsilon > 0$ the prior is stationary