Observational Learning with Ordered States

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Motivation (1)

- Sequential observational learning model (Banerjee '92; BHW '92)

  unknown state $\omega \in \Omega$

  each $n = 1, 2, \ldots$ takes action $a_n \in A$ finite

  using private signal and history of actions

  homogenous prefs $u(a, \omega)$

- Many extensions, variations

- Q: does society eventually learn ω?

- A: Unbounded vs. bounded beliefs/signals (Smith & Sørensen '00)

  Given any prior,

  $\sim$ can private beliefs $\rightarrow$ certainty about every ω?

  $\sim$ are private beliefs bounded away from 0 about every ω?

Motivation (2)

- Unbounded beliefs $\implies$ learning for all prefs

  Bounded beliefs $\implies$ nonlearning for all (nontrival) prefs

- Essentially exhaustive with two states (most papers)

- But with multiple states, a large gap

- Suppose $\Omega = \{1, 2, 3\}$ and signals $\mathcal{N}(\omega, 1)$

  Neither unbounded nor bounded!

  $\rightarrow$ can become certain about 1 or 3 but not 2

  So is there learning? Say with $u(a, \omega) = -(a - \omega)^2$

This Paper

- Prefs satisfying single-crossing differences (SCD)

  widely-used property (Milgrom & Shannon '94)

  but not previously for learning

  satisfied by quadratic loss

- Information satisfying directionally unbounded beliefs (DUB)

  new property

  $\rightarrow$ can get certainty about each state vs. lower/upper sets

  weaker than unbounded beliefs

  satisfied by normal information

- Main result

  SCD & DUB are a minimal pair of sufficient conditions for learning

Literature

Most related (will elaborate later)

- Smith & Sørensen '00
- Arieli & Mueller-Frank '19

Other mechanisms for learning

- Infinite actions with responsive prefs Lee '93; Ali '18
- Suitably heterogenous preferences Goeree, Palfrey, Rogers '06
- Prices/congestion costs
  Avery & Zemsky '98; Eyster, Galeotti, Kartik, Rabin '14

Things we don't tackle

- Partial observation of history Acemoglu, Dahleh, Lobel, Ozdaglar '11
- Speed of convergence Rosenberg & Vielle '19

Model

Environment

- Countable set of states $\Omega \subset \mathbb{R}$ (finite or infinite)

  so states are ordered

- Signal set $S \subset \mathbb{R}$, either countable or interval

  order not needed for main result

  but is when we invoke MLRP

- Action set $\mathcal{A}$; countable choice set $A \subseteq \mathcal{A}$

- Signal structure $f(s|\omega)$

  assume no signal can exclude any state: $f(\cdot) > 0$

  technical: $\forall s \; f(s|\cdot)$ is bounded

The Game

- State ω drawn from pmf $\mu_0 \in \Delta\Omega$; unobservable
- Agents $1, 2, \ldots$ sequentially select actions

  agent n chooses $a_n \in A$ at date $n \in \mathbb{N}$

  after observing indep private signal $s_n \sim f(\cdot|\omega)$

  and action history $h^n \equiv (a_1, \ldots, a_{n-1}) \in A^{n-1}$
- Strategy $\sigma_n : S \times A^{n-1} \to \Delta A$
- All agents have vNM utility $u : \mathcal{A} \times \Omega \to \mathbb{R}$
- Bayes Nash equilibria (or refinements)

Learning (1)

- Given prior μ_0, info structure f, and strategies (σ_n), every history induces a public belief $\mu(h^n) \in \Delta\Omega$
- Let $\tilde{\mu}_n$ denote corresponding r.v.
- $\langle\tilde{\mu}_n\rangle$ is a martingale that $\to_{\text{a.s.}} \tilde{\mu}^*$
- Intuitively, learning if, a.s., $\tilde{\mu}^*$ allows agents to make correct decisions

Learning (2)

For $\mu \in \Delta\Omega$, let $c(\mu) \equiv \operatorname{argmax}_{a \in A} \mathbb{E}_\mu[u(a,\omega)]$ (omitting dependence of c on A)

Let Q be beliefs with adequate knowledge: $Q \equiv \{\mu : \bigcap_{\omega \in \operatorname{Supp}\mu} c(\omega) \neq \emptyset\}$

$\rightarrow$ no gain to learning anything further

Definition

Fix prefs u and info structure f.

1. There is adequate learning if for every choice set, every prior, and every equilibrium, $\Pr(\tilde{\mu}^* \in Q) = 1$.
2. There is inadequate learning if for some choice set and prior, in every equilibrium $\Pr(\tilde{\mu}^* \in Q) < 1$.

(1): asympt. take correct actions

(2): asympt. sometimes take incorrect actions, for some choice set and prior

For what (u, f) is there adequate learning?

SCD Preferences

and

DUB Information

Single-Crossing Differences

$h : \mathbb{R} \to \mathbb{R}\backslash\{0\}$ is single crossing if either

$$\forall x < x'\colon\ h(x) > 0 \implies h(x') \geq 0;$$

or

$$\forall x < x'\colon\ h(x) < 0 \implies h(x') \leq 0.$$

Definition

Utility $u : \mathcal{A} \times \Omega \to \mathbb{R}$ has single-crossing differences (SCD) if

$\forall a, a' :\ u(a,\omega) - u(a',\omega)$ is single crossing in ω.

$\sim$ Milgrom & Shannon '94 / Athey '01, but w/o order on $\mathcal{A}$ (KLR '19)

implied by supermodularity

SCD $\iff$ interval choice:

$\forall$ choice sets and $\omega_1 < \omega_2 < \omega_3$, $\{a\} = c(\omega_1) \cap c(\omega_3) \implies a \in c(\omega_2)$

Directionally Unbounded Beliefs

Definition

Signal structure $f(s|\omega)$ has directionally unbounded beliefs (DUB) if $\forall\omega$:

(i) $\exists\ (\overline{s}_i)$ s.t. $\forall\omega' < \omega$, $\lim_{i\to\infty} \frac{f(\overline{s}_i|\omega')}{f(\overline{s}_i|\omega)} = 0$; and

(ii) $\exists\ (\underline{s}_i)$ s.t. $\forall\omega' > \omega$, $\lim_{i\to\infty} \frac{f(\underline{s}_i|\omega')}{f(\underline{s}_i|\omega)} = 0$.

$+$ uniform boundedness condition for Ω infinite

(i) $\iff$ can simultaneously distinguish ω from all lower states

given any prior μ with $\mu(\omega) > 0$, can rule out $\{\omega' : \omega' < \omega\}$

(ii) $\iff$ can simultaneously distinguish ω from all higher states

given any prior μ with $\mu(\omega) > 0$, can rule out $\{\omega' : \omega' < \omega\}$

May not sim distinguish ω from both lower and higher states!

Main Result

Learning with SCD & DUB

Theorem

1. SCD prefs & DUB info $\implies$ adequate learning.
2. If prefs fail SCD, then $\exists$ DUB info with inadequate learning.
3. If info fails DUB and $|\Omega| < \infty$, then $\exists$ SCD prefs with inadeq learning.

A key fact for the proof

- Given (u, f, A), say that $\mu \in \Delta\Omega$ is stationary if a.s. $c(\mu_s) = c(\mu)$
  (μ_s is posterior; assume unique choices)
- Adeq learning $\iff$ all stationary beliefs have adeq knowledge $(\mu \in Q)$

  $\implies$ if $\mu \notin Q$ is stationary, consider the prior being μ

  $\impliedby$ all limits beliefs are stationary (Arieli & Muller-Frank '19)
- Nb: adeq knowledge means no value of **any** info;
  stationary means no value of info from $f(s|\omega)$

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Intuition for part 1:

- In general, $\mu \notin Q$ may be stationary $\because$ mismatch between prefs and info
- SCD $\implies$ if $\mu \notin Q$ then $\exists \omega^* \in \operatorname{Supp} \mu$ s.t.

$$u(c(\omega^*), \omega^*) > u(c(\mu), \omega^*) \quad \textbf{and}$$

$$u(c(\omega^*), \omega) \geq u(c(\mu), \omega) \quad \forall \omega > \omega^* \text{ or } \forall \omega < \omega^*$$

- DUB $\implies$ $\exists$ signals that rule out $\{\omega : \omega < \omega^*\}$ and $\{\omega : \omega > \omega^*\}$

  $\rightarrow c(\mu)$ not chosen after those signals

- So SCD + DUB $\implies$ any $\mu \notin Q$ is not stationary (faulty intuition)

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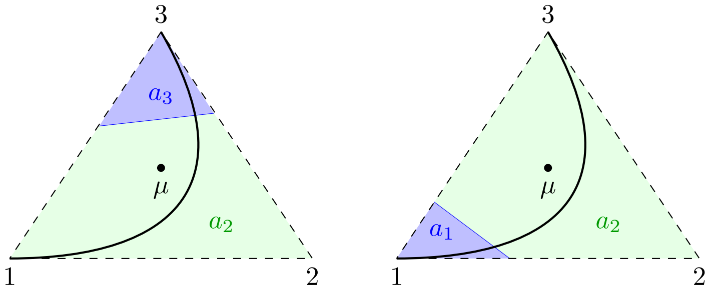
Intuition for part 1:



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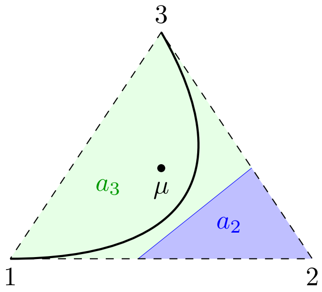
Intuition for part 2:



In fact: given non-SCD prefs, any DUB and MLRP info $\implies$ inadequate learning

Learning with SCD & DUB

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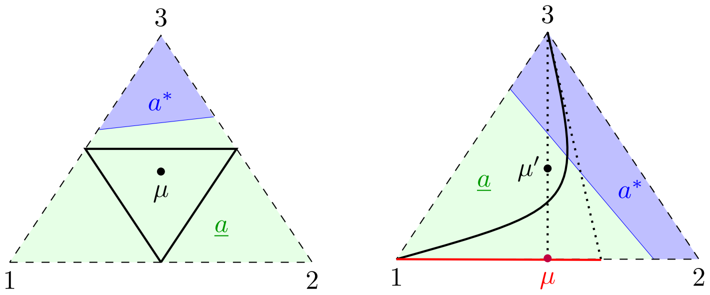
Intuition for part 3:



Trickier case in right panel: what if certainty possible about extreme states?

May need to restrict prior's support

Learning with SCD & DUB

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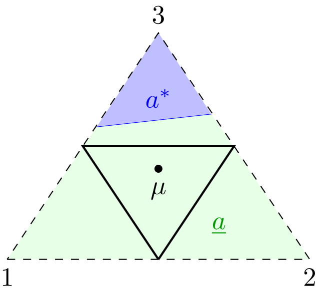

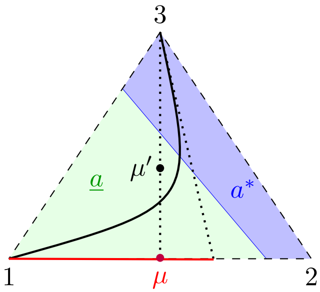

In fact: Assume MLRP and not DUB.
$\exists$ SCD prefs s.t. there is inadeq learning for any full support prior.

Discussion

DUB in Location Families

Location family: $S = \mathbb{R}$ and for some density g, $f(s|\omega) = g(s - \omega)$

- e.g., Normal info
- Intuitively, DUB requires a thin tail of standard density g

g strictly subexponential: $\exists p > 1$ s.t. $g(x) < \exp[-|x|^p]$ for large $|x|$

Proposition

In a location family, DUB holds if g is strictly subexponential.

- If g is exponential then $g(s - \omega')/g(s - \omega) = \exp(\omega' - \omega)$ is indep of s
- An even thicker tail (superexp) makes extreme signals uninformative
- So Laplace, Cauchy, Student-t distrs fail DUB

Unbounded Beliefs

Restrict to finite Ω, for simplicity

Unbounded beliefs: $\forall\omega\ \exists\ (s_i)$ s.t. $\forall\omega' \neq \omega$, $\lim_{i\to\infty} \frac{f(s_i|\omega')}{f(s_i|\omega)} = 0$.

(Smith & Sørensen '00; "totally unbounded" in Arieli & Mueller-Frank '19)

- Each ω can be simultaneously distinguished from all others
- DUB weaker $\because$ for each ω, separately distinguish upper and lower sets
- Unbounded beliefs $\iff$ adeq learning for all preferences
- But unbounded beliefs very demanding with more than two states

Proposition

Assume $|\Omega| > 2$. MLRP $\implies$ not unbounded beliefs.

- So, with multiple states, must restrict prefs to obtain learning

Pairwise Unbounded Beliefs

Pairwise unbounded beliefs: $\forall \omega \; \forall \omega' \neq \omega \; \exists \, (s_i)$ s.t. $\lim_{i \to \infty} \frac{f(s_i|\omega')}{f(s_i|\omega)} = 0.$

(Arieli & Mueller-Frank '19)

- Each ω can be distinguished from every other, but not simultaneously
- DUB is stronger: simultaneously distinguish each ω from its upper set and its lower set
- Pairwise UB is not sufficient for adeq learning under SCD
- But pairwise UB is necessary for adeq learning over any "minimally-rich" class of preferences

Proposition

Assume MLRP. Pairwise UB $\iff$ DUB.

- So, given MLRP, DUB is unavoidable for adeq learning

On Bounded Beliefs

Bounded beliefs: $\forall \omega, \omega'$, $\frac{f(s|\omega')}{f(s|\omega)}$ is bounded above in s

- A natural notion, generalizing two-state case
- But stronger than just ruling out certainty about any state
- Negation of pairwise unbounded beliefs (for every pair)
- So incompatible with DUB
- Guarantees inadequate learning for all nontrivial prefs

Conclusion

Conclusion

Recap:

- Std condition, unbounded beliefs, very demanding with > 2 states
- Study learning under economic pref restriction with ordered states
- New informational condition: DUB
  - $\rightarrow$ weaker than unbounded beliefs
  - $\rightarrow$ rules out bounded beliefs
- DUB info and SCD prefs are minimal pair
  of suff conditions for adequate learning

Future directions:

- Extend to other obsv learning environments (e.g., partial histories)
- Speed of convergence?
- Is DUB useful in other contexts?

Thank you!

Faulty Intuition for Sufficiency

(thm)

- Take μ with inadequate knowledge
  1. SCD implies different optimal actions at extreme states of $\operatorname{Supp}\mu$
  2. DUB implies potential certainty about extreme states of $\operatorname{Supp}\mu$
  3. μ is non-stationary

- Is learning about SCD or different optimal actions at extreme states?

- Our result applies to infinite states and weak SCD environments where above logic fails

  $\rightarrow$ Not about responsive preferences

Faulty Intuition for Sufficiency

(thm)

- Let $\Omega = \mathbb{Z}$ and $A = \mathbb{Z} \cup \{a^*\}$

$$u(a,\omega) = \begin{cases} 1 & \text{if } a = \omega \\ 0 & \text{if } a \notin \{\omega, a^*\} \\ 1-\varepsilon & \text{if } a = a^* \end{cases}$$

- For small $\varepsilon > 0$, a^* is a safe action but suboptimal in every state
- Different optimal action in every state, but u violates SCD
- Suppose $s \sim \mathcal{N}(\omega, 1)$. For any full support prior
  - signals cannot provide certainty about any state
  - for small enough $\varepsilon > 0$ the prior is stationary