Supplementary Appendix to The Theory of Vying for Dominance in Dynamic Network Formation

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Abstract

In Neligh (2017) we explore a game of dynamic network formation with forward-looking strategic agents. We find that SPEs of the game are not generally tractable for large networks, although the game is tractable in small networks and in simplified versions of the game. In this note, instead of looking for SPNE's of the game, we look for NE in which players behave consistently across network states. We establish necessary and sufficient conditions for the existence of such NE's and provide an algorithm to determine if the conditions hold. The algorithm can also provide sufficient conditions for the existence of certain classes of SPE's as well as fully characterizing the set of SPE's when the geometric discount factor is zero.

1 Nash Equilibrium Without Subgame Perfection

If we are willing to drop the requirement of subgame perfection, then we can say a little bit more about the solutions for the game. In this sections we will establish necessary and sufficient conditions for an outcome to be consistent with play from a Nash Equilibrium (NE) satisfying several conditions. Assume $(J-2)-f(J,\delta) > \frac{C}{B} > (1-\delta)$ in this sections as the other cases have already been addressed more thoroughly.

If we do not require subgame perfection, then we can employ strategies with unenforceable punishment strategies. Any strategy profile in which each player makes at least more than their min-max utility at each decision node that is visited with positive probability can be supported by a NE of the game. In this section, bold uppercase and bold Greek letters will be used to refer to matrices.

1.1 The Question

One fairly natural question is, can an observer determine whether a given network can be the result of players employing strategy profiles that constitute a NE? This question has both predictive and empirical merit. Predicatively, we can determine which networks we should not see forming. Empirically, we can observe a network and potentially reject the hypothesis that players are playing a NE.

It is difficult to work with the general class of NE's in this game, because the set of strategy profiles is so large. We can, however, establish necessary and sufficient conditions for the observed network to be consistent with play from a NE satisfying the following conditions.

Condition NE 1: Players do not mix between making different numbers of connections when indifferent.

Condition NE 2: Players do not condition the number of moves they make on the current network structure at any decision node that is visited with positive probability.

With these two conditions guarantee that players will always be making the same number of connections every time the network formation process is run using the strategy profiles from a given NE.

Before we begin, we provide two definitions that are critical to the propositions and proofs in this section.

Punishment Utility: We define $Upun_i(G_{i-1})$ as the best possible utility that Player i can get given G_{i-1}, knowing that all later players will connect as far as possible from them. In the case of this game the most effective punishment that someone can suffer involves all remaining players making connections as far from that player as possible.

Myopic Utility: We define $Omyopic_i(G_i)$ as the utility of Player i would have received if the game had ended immediately after his move.

Note that both of these values are effectively observable, because they can be inferred from the final network G

Proposition NE 1: Given that the randomization procedure is commonly known, a given outcome network, G, can be supported by a NE satisfying Conditions NE 1 and NE 2 as long as there exists a probabilistic allocation of connections between players such that each player makes the number of connections that they made in the observed network and every player makes more than their min-max payoff in expectation.

The existence of such an allocation is equivalent to an upper triangular JxJ matrix $\boldsymbol{\alpha}$ with real valued entries between 0 and 1 satisfying $\boldsymbol{\alpha}^t\mathbf{1} \geq \mathbf{v}^2(G)$ and $\boldsymbol{\alpha}\mathbf{1} = \mathbf{v}^1(G)$. If x_i is the number of connections Player i made in excess of one, then $\mathbf{v}_i^1 = x_i(G) + 1$ and $\mathbf{v}_i^2 = \dfrac{Upun_i(G_{i-1}) - Omyopic_i}{B(1-\delta)}$. We suppress the dependence of the $\mathbf{v}$ vectors on G to economize on notation.

Proof of Proposition NE 1:

From an outcome network we can deduce x_i (the number of connections beyond the first made by Player j), $Upun_i(G_{i-1})$ (the maximum payoff achievable by Player i facing network G_{i-1} given that the punishment strategy will be inflicted on Player i), and $Omyopic_i$ (the myopic payoff made by Player i)

Say that we are trying to support a strategy profile which can be represented by the matrix $\mathbf{a}$ where $\alpha(j,i)$ is the probability of Player j's connecting to node i at all decision nodes that are visited with positive probability. Players do not condition their probabilities of making connections on current network structure. Players make their decisions based on a previously determined outcome of the public randomization. Whenever Player j does not connect according to the values $\alpha(j,i)$ he is subject to the punishment strategy of other players.

It must be the case that $\boldsymbol{\alpha}(j,i) \in [0,1]$. By the structure of the game it must be that if $j \leq i$ then $\alpha(j,i) = 0$. By Assumptions NE 1 and NE 2 we know that $\sum_{i=1}^{J} a(j,i) = x_j + 1$ is fixed. Given these constraints, any strategy profile in players make connections according to the prescribed probabilities and each player makes at least $Upun_i(G_{i-1})$ at every decision node visited with positive probability can be supported as a NE.

Since every additional direct connection from a future player gives a player at least $B(1-\delta)$ additional points, and punished players receive no future connections, we can guarantee that every player is making more than $Upun_i(G_{i-1})$ in expectation if $Omyopic_i + B(1-\delta)\sum_{j=1}^{J} \mathbf{a}(j,i) \geq Upun_i(G_{i-1})$, or $\sum_{j=1}^{J} \alpha(j,i) \geq (Upun_i(G_{i-1}) - Omyopic_i)/(B - B\delta)$.

In other words we just need to find $\boldsymbol{\alpha}(j,i)$ satisfying the following requirements

Initial Requirements

1. $0 \leq \boldsymbol{\alpha}(j,i) \leq 1 \forall i$ by single connection property of the graph.
2. $\boldsymbol{\alpha}(j,i) = 0 \forall i \geq j$ by the fact that nodes can only make connections to earlier nodes.
3. $\sum_{i=1}^{J} \boldsymbol{\alpha}(j,i) = x_j + 1$ by the fact that number of connections made must match the data.
4. $\sum_{j=1}^{J} \boldsymbol{\alpha}(j,i) \geq \dfrac{Upun_i(G_{i-1}) - Omyopic_i}{B(1-\delta)}$ By punishment condition.

Simply rewriting these restrictions in matrix form gives the requirement in the proposition. $\square$.

1.1.1 The Algorithm

Now we present an algorithm which can check for the existence of the matrix $\boldsymbol{\alpha}$ described above. However, before we go on, we will transform the problem a bit in order to make later steps simple.

To facilitate this we introduce complimentary conditions on a matrix $\mathbf{A}$. Essentially we are rewriting conditions in terms of the connections received from rather than connections provided to.

Dual Requirements

1. $\mathbf{A}(i,j) = 0 \forall j \leq i$, because each node cannot receive more than one connection from another node.
2. $0 \leq \mathbf{A}(i,j) \leq 1 \forall j$, because nodes cannot receive connections from older nodes
3. $\sum_{i=1}^{J} \mathbf{A}(i,j) \leq x_j + 1$ because nodes cannot provide more connections than observed.
4. $\sum_{j=1}^{J} \mathbf{A}(i,j) = \dfrac{Upun_i(G_{i-1}) - Omyopic_i}{B(1-\delta)}$ which is the dual punishment condition

We now show that a valid allocation α exists satisfying the initial requirements (IR) if and only if a dual allocation $\mathbf{A}$ exists which satisfies the dual requirements (DR). Consider an arbitrary such $\mathbf{A}$. We define $\tilde{\alpha}$ by We define $\tilde{\alpha}(i,j) = A(j,i)$. We immediately have $\tilde{\alpha}(i,j)$ satisfying IR1 and IR2 by DR1 and DR2. I there exists a j such that $\tilde{\alpha}(i,j)$ does not satisfy j the IR3 requirement is not satisfied with equality we can then add mass to arbitrary $\tilde{\alpha}(i,j)$ without violating IR1 or IR2 until equality is reached. Doing this will only increase the LHS of IR4, so, since IR4 already held with equality that constraint will continue hold.

It is always possible to do this transformation without violating IR1 and IR2, because the inferred $\mathbf{v}_j^1$ will be less than or equal to $j-1$ for any input network, and $\sum_{i=1}^{J} 1(i<j) = j-1$.

The transformed $\tilde{\alpha}$ will also satisfy IR4, because the untransformed $\tilde{\alpha}$ already satisfies it, and the transformation only increases the LHS value. The fact that the untransformed $\tilde{\alpha}$ satisfies IR4 is immediate from the fact that $\mathbf{A}$ satisfies DR4.

Now we must show that the existence of α implies the existence of the $\mathbf{A}$. To do this simply invert the process. Define a $\tilde{\mathbf{A}}$ by $\tilde{\mathbf{A}}(i,j) = \mathbf{a}(j,i)$. Again, we immediately get $\tilde{\mathbf{A}}$ satisfying DR1 and DR2 from α's satisfying IR1 and IR2.

If $\tilde{\mathbf{A}}$ does not satisfy the j the DR4, we can transform it by simply decreasing arbitrary $\tilde{\mathbf{A}}(i,j)$'s. Such a transformation can always be performed without violating DR1 and DR2 because the inferred $\mathbf{v}_j^2$ must be greater than or equal to zero.

The transformed $\tilde{\mathbf{A}}$ must also satisfy DR3, because the untransformed version satisfies DR3, and the transformation only lowers the LHS of the DR3 requirements. The fact that the untransformed $\tilde{\mathbf{A}}$ satisfies DR3 comes immediately α's satisfying IR3.

In other words, the question of whether or not a given matrix can be the result of a Nash Equilibrium given assumptions NE 1 and NE 2 can be rephrased as given a pair of vectors $\mathbf{v}^1, \mathbf{v}^2$ where $\mathbf{v}_j^1 = N_j + 1$ and $\mathbf{v}_j^2 = N_j(\dfrac{C}{B} - 1) + \delta * \Delta g_j(\delta)$, can we find a triangular matrix $\mathbf{A}$ with entries between 0 and 1 which satisfies $\mathbf{A}^t\mathbf{1} \leq \mathbf{v}^1$ and $\mathbf{A1} = \mathbf{v}^2$

Lemma: For a given pair of vectors $\mathbf{v}^1$ and $\mathbf{v}^2$ of length J, the Algorithm to Check for a Triangular Probability Matrix (ACTPM) will produce a triangular matrix $\mathbf{A}$ with entries between 0 and 1 which satisfies $\mathbf{A}^t\mathbf{1} \leq \mathbf{v}^1$ and $\mathbf{A1} = \mathbf{v}^2$ if such a matrix exists.

Algorithm to Check for Triangular Probability Matrix

Begin with an empty allocation $\mathbf{A}(i,j) = 0 \, \forall \, i,j$

For $j = J, J-1, ..., 2, 1$ execute the following steps

1. In this repetition we will be allocating $\mathbf{A}_J$ which is the vector of entries $\mathbf{A}(j,i)$ where $i \in 1, 2, 3, ..., J$
2. Allocate that V_j^2 by assigning $\mathbf{A}$ in order to minimize $\sum_{i=j}^{J}(\mathbf{v}_i^1 - \sum_{l=j}^{J} \mathbf{A}(i,l)^2$ subject to DR1, DR 2, and $\sum_{i=1}^{J} \mathbf{A}(j,i) = \mathbf{v}_j^2$. In other words, minimizing a sum of a strictly convex function of the space remaining on the active inequality constraints.
3. If such an allocation cannot be generated without exceeding the size of a bin or violating DR1 or DR2, then the algorithm fails.

Intuitively this algorithm adds the nodes one at a time. As each node is added, you must assign that node probabilities of receiving connections from future nodes. Each node must be assigned enough connections to meet their equality constraint without violating the inequality constraint that no node can be assigned to give more connections in expectation than it was observed to give. We require that the assignment always reduce the amount of slack on the loosest inequality constraint first, as doing so is optimal in terms of allowing allocations to exist as we shall show.

Proof of Lemma

We must show is that, if the algorithmic allocation fails, then no feasible allocation exists which satisfies the requirements.

Any algorithm which satisfies the dual requirements can be backwards constructed as we shall explain. Let $\mathbf{A}^j$ be an interim allocation, which is like a standard allocation except $\mathbf{A}^j(i,l) = 0 \forall i \leq j \forall l$. We say that a interim allocation is valid if it meets DR1 and DR2 and the DR3's and DR4's associated with $i > j$. Note that any valid allocation $\mathbf{A}$ can be converted into an interim allocation $\mathbf{A}^j$ at any point simply by setting the correct entries in $\mathbf{A}$ to zero.

As a consequence, it is also possible if a valid interim allocation $\mathbf{A}^{j-1}$ exists, to construct a valid interim allocation $\mathbf{A}^j$ by setting row j of $\mathbf{A}^{j-1}$ to zero. One can then reconstruct $\mathbf{A}^{j-1}$ from $\mathbf{A}^j$ by choosing the appropriate values for row j.

This means that any valid $\mathbf{A}$ can be constructed from the all zeroes $\mathbf{A}^J$ by introducing constraint pairs DR3 and DR4 along with another valid row of $\mathbf{A}$ one at a time.

The question then arises, when is it possible to create a valid $\mathbf{A}^{j-1}$ from a given $\mathbf{A}^j$? It is possible only when sufficient space exists in the in the active inequality constraints to satisfy the new equality constraint while not violating the DR1 and DR2. In other words, it is possible to produce a valid interim allocation when $\sum_{j=1}^{J} F(\mathbf{A}, j) \geq \mathbf{v}_j^1$ where $F(\mathbf{A}, j) = \max(1, \mathbf{v}_j^1 - \sum_{i=1}^{j-1} \mathbf{A}(i,j))$.

To prove the lemma, we show that the algorithmic allocation achieves maximum feasible $\sum_{j=i}^{J} F(\mathbf{A}, j)$ of all allocations at each step. This in turn proves that the algorithmic allocation produces the maximum feasible $\sum_{j=i}^{J} F(\mathbf{A}, j)$ of all valid allocations at each stage of its backwards construction including the final one.

This means that if any backwards construction process can create a valid $\mathbf{A}$ matrix, then the ACTPM can as well.

Call the algorithmic allocation matrix $\bar{\mathbf{A}}$. By DR4, the sum $\sum_{j=i+1}^{J} f(\mathbf{A}, j)$ where $f(\mathbf{A}, j) = (N_j - \sum_{l=1}^{j-1} \mathbf{A}(l,j))$ will be the same for any valid continuation allocation. The algorithmic allocation also always produces the same fixed $\sum_{j=i+1}^{J} f(\mathbf{A}, j)$ as any valid allocation when it produces an allocation because we require DR4 to be met at each step..

Define $f_{\mathbf{A}}(1)$ as the highest value $f(\mathbf{A}, j)$ of given $\mathbf{A}$ for any j. Define $f_{\mathbf{A}}(k)$ as the k the highest value $f(\mathbf{A}, j)$ take on for any j. The vector $f_{\mathbf{A}}$ contains $f(\mathbf{A}, j)$ for all j in weakly decreasing order. Say there is another feasible allocation $\mathbf{A}$. Define $\Delta f_{\mathbf{A}} = f_{\bar{\mathbf{A}}} - f_{\mathbf{A}}$.

Ultimately, we want to show that $\sum_{j=1}^{J} F(\mathbf{A}, j) \geq \sum_{j=1}^{J} F(\bar{\mathbf{A}}, j))$. By Karamata's inequality, the condition $\sum_{i=1}^{j} \Delta f_{\mathbf{A}}(i) \leq 0 \forall j \leq J$ implies that result [1]. Therefore if we prove $\sum_{i=1}^{j} \Delta f_{\mathbf{A}}(i) \leq 0 \forall j \leq J$ we are done.

First we prove $\Delta f_{\mathbf{A}}(1) \leq 0$ or $f_{\bar{\mathbf{A}}}(1) \leq f_{\mathbf{A}}(1)$. This is not an essential step, but it does help illustrate what is happening when we generalize. To see this assume that the reverse is true and $f_{\bar{\mathbf{A}}}(1) > f_{\mathbf{A}}(1)$. Say that the quantity $f_{\bar{\mathbf{A}}}(1)$ corresponds to the space remaining on the $j_{\bar{\mathbf{A}}}^1$ the constraint. In other words $f_{\bar{\mathbf{A}}}(1) = f(\bar{\mathbf{A}}, j_{\bar{\mathbf{A}}}^1)$. Note that if two constraints ever have the same remaining space during the algorithm, they will have the same remaining space from then on by construction.

When we say that an inequality constraint j (or set of constraints $\mathbf{S}$) receives weight during a given step i that just means that during that backwards construction step for $\mathbf{A}$, we are assigning some positive value to $\mathbf{A}(i,j)$ (or to some $\mathbf{A}(i.j), i \in S$) during that step.

At the end of the backwards construction process either one DR3 constraint has more space left than any other or several are tied for first. If only one has the most space, then it must have received maximum (one or $\mathbf{v}_j^2$, whichever is smaller) weight every step that it was active in the backwards construction process. As such, it is impossible that $f_{\bar{\mathbf{A}}}(1) > f_{\mathbf{A}}(1)$ in this case, because that would

[1] Karamata (1932)

imply that $f(\mathbf{A}, j^1_{\bar{\mathbf{A}}}) \geq f(\bar{\mathbf{A}}, j^1_{\bar{\mathbf{A}}})$, contradicting the fact that the $j^1_{\bar{\mathbf{A}}}$ the constraint received maximum possible weight in the backwards construction of $\bar{\mathbf{A}}$.

Now consider what happens when several constraints are tied for most space remaining under $\bar{\mathbf{A}}$. Call the set of indices for these tied constraints $\mathbf{S}_1$. If $f_{\bar{\mathbf{A}}}(1) > f_{\mathbf{A}}(1)$, it must be the case that all of these constraints receive more under $\mathbf{A}$ than $\bar{\mathbf{A}}$. If this were not the case then one of the constraints would need to receive less weight under $\mathbf{A}$ than $\bar{\mathbf{A}}$, and that constraint would have more than $f_{\bar{\mathbf{A}}}(1)$ space remaining.

This implies that the total weight assigned to all of the tied for first constraints must be higher under the alternate allocation than the algorithmic allocation. Note that in the algorithmic allocation, relative ranking of space remaining on constraints is preserved in he sense that if $f(\bar{\mathbf{A}}^m, j) \geq f(\bar{\mathbf{A}}^m, j)$ then $f(\bar{\mathbf{A}}^l, j) \geq f(\bar{\mathbf{A}}^l, j) \forall l \leq m$. This plus the fact that if two constraints ever have the same space remaining implies that they always have the same space remaining leads us to conclude that at no point during the backwards construction process could any weight have been applied to constraints outside $\mathbf{S}_1$ set which could otherwise have been applied to constraint inside $\mathbf{S}_1$. Therefore it must be the case that $f_{\bar{\mathbf{A}}}(1) \leq f_{\mathbf{A}}(1)$.

Now we use a similar approach to prove $\sum_{i=1}^{j} \Delta f_{\mathbf{A}}(i) \leq 0 \forall j \leq |\mathbf{S}_1|$. Again the proof is not strictly necessary, but it does illustrate more about the workings of the general proof.

The most weight that can be added to any subset of constraints $\mathbf{S}$ during any step j is $\max(\mathbf{v}^2_j, |\mathbf{S}|)$. It must be the case that $\max(\mathbf{v}^2_j, |\mathbf{S}^j_1|)$ weight is added to constraints in $\mathbf{S}^j_1$ every step, since by construction if any weight is added to a constraint j and constraint i does not have a full one weight added to it during step m, it must be the case that $f(\bar{\mathbf{A}}^{m+1}, j) \geq f(\bar{\mathbf{A}}^{m+1}, j)$. Thus by preservation of relative ranking, any constraint that received weight in preference to one in $\mathbf{S}^j_1$ must also be in $\mathbf{S}^j_1$.

Now, since the amount of weight given to the constraints in $\mathbf{S}_1$ is weakly lower under $\mathbf{A}$ than $\bar{\mathbf{A}}$, and since under $\bar{\mathbf{A}}$, $f(\bar{\mathbf{A}}, i) = f(\bar{\mathbf{A}}, j) \forall i, j \in \mathbf{S}_1$, it must be the case that $\sum_{i \in \mathbf{S}_1}^{or:j} \big(f(\bar{\mathbf{A}}, i) - f(\mathbf{A}, i)\big) \leq 0 \forall j \leq |\mathbf{S}_1|$ where $\sum_{i \in \mathbf{S}_1}^{or:j}$ denotes the sum of the j smallest realizations of the summation from the set of possible elements given $i \in \mathbf{S}_1$. By construction, $\sum_{i \in \mathbf{S}_1}^{or:j} \big(f(\bar{\mathbf{A}}, i) - f(\mathbf{A}, i)\big) \geq \sum_{i=1}^{j} \Delta f_{\mathbf{A}}(i) \forall j \leq |\mathbf{S}_1|$.

Now consider the set of constraints tied for n the most space remaining. Again by relative rank persistence, constraints in $\mathbf{S}_l$ where $l > n$ never receive weight in preference to constraints in $\mathbf{S}_n$ during the backwards construction of $\bar{\mathbf{A}}$. This means that $\mathbf{S}_n$ cannot receive M more total weight under $\mathbf{A}$ than under $\bar{\mathbf{A}}$ unless $\mathbf{S}_{<n} = \cup_{j<k} \mathbf{S}_j$ receives at least M less weight.

In constraints in the set $\mathbf{S}_n$ receive M more weight under $\mathbf{A}$ than $\bar{\mathbf{A}}$, we know $\sum_{i \in \mathbf{S}_{<n}} f(\bar{\mathbf{A}}, i) - f(\mathbf{A}, i) \leq -M$. Because $\sum_{i \in \mathbf{S}_{<n}} f(\mathbf{A}, i) \leq \sum_{i=1}^{|\mathbf{S}_{<n}|} f_{\mathbf{A}}(i)$, this implies that $\sum_{i=1}^{|\mathbf{S}_{<n}|} \Delta f_{\mathbf{A}}(i) \leq -M$.

If $\mathbf{S}_n$ receives M more weight under $\mathbf{A}$ then since $f(\bar{\mathbf{A}}, i)$ is the same $\forall i \in \mathbf{S}_n$ it must be that $\sum_{i \in \mathbf{S}_n}^{or:j} (f(\bar{\mathbf{A}}, i) - f(\mathbf{A}, i)) \leq M \forall j \leq |\mathbf{S}_n|$. Furthermore, $\sum_{i \in \mathbf{S}_n}^{or:j} (f(\bar{\mathbf{A}}, i) - f(\mathbf{A}, i)) \geq \sum_{i=|\mathbf{S}_{<n}|+1}^{j} \Delta f_{\mathbf{A}}(i) \forall j \leq |\mathbf{S}_n|$, so $\sum_{i=1}^{j} \Delta f_{\mathbf{A}}(i) \leq 0 \forall j \leq |\mathbf{S}_{\leq n}|$.

Since this is true for all n, we have$\sum_{i=1}^{j} \Delta f_{\mathbf{A}}(i) \leq 0 \forall j \leq J$ $\square$

1.1.2 The Necessary condition

In the above sections we proved that, for a given network, the existence of a matrix satisfying certain conditions derived from that network is sufficient for the existence of a NE with satisfying certain conditions which could generate said network with positive probability. We also provided an algorithm to check for the existence of such a matrix. We now show provide a necessary condition for a given network to supportable through NE play satisfying when δ is small.

This necessary condition takes a very similar form to the sufficient condition discussed above, and in fact the two converge to a single necessary and sufficient condition when $\delta = 0$

Proposition NE 2: Under Assumption NE 1 and Assumption NE 2, and δ sufficiently small for the given ϵ a given outcome network G cannot be supported by a NE unless there exists an upper triangular JxJ matrix α with real valued entries between 0 and 1 satisfying $\alpha^t \mathbf{1} \geq \tilde{\mathbf{v}}^2(G)$ and $\alpha \mathbf{1} = \mathbf{v}^1(G)$.

Say x_i is the number of connections Player i made in excess of one, $Omyopic_i$ is the observed myopic payout received by i , and $Upun_i(G_{i-1})$ is the punishment utility for Player i which can be inferred from the observed G_{i-1} . Then $\mathbf{v}^1_i = x_i + 1$ and $\tilde{\mathbf{v}}^2_i = \dfrac{1}{B(1-\delta)}\big(Upun_i(G_{i-1}) - Omyopic_i - \delta B(J-j)\big) - \epsilon$.

Proof of Proposition NE 2: Assume that there exists a strategy profile $\mathcal{S}$ which supports the observed network G. In any Nash equilibrium, all players must be making more than their min-max payoff in expectation at every decision node. The maximum expected payoff that a Player i who will receive k_i connections in expectation can receive after making the move observed in network G is $Omyopic_i(G) + \delta B(J-j) + k_i(G)B(1-\delta)$, where the first term is the myopic payoff, the second term is the maximum payoff from non-connecting nodes, and the last term is the additional gain from direct connections. It must then be the case that $Omyopic_i(G) + \delta B(J-j) + k_i B(1-\delta) \geq Upun_i(G_{i-1} \forall i$. We can then define $k_i(G)$ as the minimum k_i such that the inequality is satisfied.

Note that the fixed number of connections made by each player and the low δ mean that we can get ensure that all possible $Omyopic_i(\tilde{G})$'s that a player can manifest are arbitrarily close together for all $\tilde{G}$ generated with positive probability by $\mathcal{S}$. This in turn allows us to a assume that all of the $k_i(\tilde{G})$'s, for $\tilde{G}$ generated with positive probability by $\mathcal{S}$, will always be arbitrarily close together and arbitrarily close to the $k_i(G)$ inferred from the observed network. This means that if a Player i ever receives less than $k_i(G) - \epsilon$ connections in expectation after a given decision node, then there exists a $\bar{\delta}$ corresponding to that ϵ such that for $\delta < \bar{\delta}$, Player i is making less than their min-max payoff.

Thus if δ is sufficiently small relative to ϵ, it must be the case that all players receive at least $k_i(G) - \epsilon$ connections in expectation at each decision node in $\mathcal{S}$. Then, by Bayes plausibility, the matrix α ex-ante connection probabilities under $\mathcal{S}$ must satisfy the following requirements:

1. $0 \leq \alpha(j,i) \leq 1 \forall i$
2. $\alpha(j,i) = 0 \forall i \geq j$
3. $\sum_{i=1}^{J} \alpha(j,i) = x_j + 1$
4. $\sum_{j=1}^{J} \alpha(j,i) \geq \dfrac{1}{B(1-\delta)}\big(Upun_i(G_{i-1}) - Omyopic_i - \delta B(J-j)\big) - \epsilon$

Which is precisely the set of requirements in the statement. $\square$

Notes on Proposition NE 2: Because this proposition takes a similar form to Proposition NE 1, the ACTPM can be employed to check for existence.

The inclusion of the ϵ and the sufficiently small δ requirement can cause concern when one is not sure how small of a δ is needed for a given ϵ. In general, the necessary condition holds when fairly tightly, with an ϵ very close to zero, when δ is small enough that direct connections are substantially more important than indirect connections when determining payoff. Note that we do not need δ to be small if the system has some way of guaranteeing that the network remains small in diameter. Small diameter networks tend to have payoffs which depend primarily on number of connections made rather broad network structure.

1.1.3 Value of the NE results

These conditions are most useful in contexts where plausibility of threats is not a major concern, because we are dealing with standard NE rather than SPE. This can be the case when players are not sophisticated or when norms are highly valued and enforced independent of individual disutilty from the punishments. If possible, this type of subgame imperfect play could be beneficial to the system, because there are generally NE supported by the types of strategy profiles described in the proof of Proposition NE 1 which produce more connected networks than the most connected outcomes of SPE's. When more connections improve welfare, the gain from choosing a non-subgame perfect equilibrium could be substantial. The sufficient condition is particularly useful when NE's are the primary focus, because it does not depend as heavily on a low δ parameter.

While the usefulness of the conditions and algorithm in this section are not immediately obvious to readers more concerned with subgame perfection, they do have important implications. Most obviously, a necessary condition for NE satisfying conditions NE 1 and NE 2 is also a necessary condition for SPE satisfying those conditions. The necessary condition can provide a good upper bound on the number of connections that those types of SPE's can produce for given parameter values. The condition will reject potential outcomes in which groups of players jointly make implausibly high number of connections.

Empirically, the necessary condition can be used to check for rationality and belief consistency under the assumption that players stick to equilibria of the kind discussed. If two players both make so many connections that there is no way for later connections to make up both of their myopic losses, then it is likely that either they have different beliefs about the connection behavior of future players, or they are irrational. Checking the necessary condition will catch these kinds of inconsistencies.

As mentioned before, when δ goes to zero the necessary and sufficient conditions converge. In addition, when $\delta = 0$, the joint condition becomes a necessary and sufficient condition the existence of a SPE supporting the observed outcome under assumptions NE 1 and NE 2. The proof is omitted due to its similarities with the preceding proofs.

References

Karamata, J. (1932). Sur une inégalité relative aux fonctions convexes. *Publ. Math. Univ. Belgrade*, 1:145–148.

Neligh, N. (2017). The theory of vying for dominance in dynamic network formation. Working Paper.