The Theory of Vying for Dominance in Dynamic Network Formation

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Abstract

In many networks, a few highly central nodes have outsized impacts on the structure of the network and generate a large amount of value, but what determines which nodes become central? We hypothesize that the timing of entry into the network can also play a critical role. In this paper, we present a new dynamic model of network formation with history dependence, growth, and forward looking strategic agents. These features increase the importance of importance of the dynamics and generate novel strategic behaviors such as "vying for dominance," making a larger number of connections than is myopically beneficial in expectation of receiving more connections from future players. Due to the strategic richness, the model rapidly becomes intractable as the size of the network increases. However, if we restrict the model such that players must connect to one of the most central nodes as they join the network, we can restore tractability, and we find that all players either vying for dominance or playing myopically. Furthermore, if we assume players use a novelty seeking tie-breaking rule, players vie for dominance periodically, the solution is characterized by periodic vying for dominance separated by periods of low connection myopic play. Because vying becomes more expensive as the network grows, the time between vying agents increases exponentially over time.

1 Introduction

The structure of the networks underlying an economic system can have substantial impacts on the outcomes of that system. The theoretical literature provides many examples where network structure plays a large role in determining the behavior of agents in economically important settings such as trading networks[1], coordination games,[2] public goods provision,[3] and markets with network externalities[4]. Empirically, the structure of real world networks has been found to have significant impact on many interesting features of the systems in which they are found.[5] Centrality is a particularly valuable and important component of network structure, providing information, bargaining power, and influence.[6]

The importance of network structure network structure and centrality leads us to consider how nodes become central when networks form. We hypothesize that when nodes enter the network plays a large role in determining how central they eventually become. Intuitively, a node that enters the network very late will not become central; existing nodes will already dominate the network to such an extent that competing with them is not profitable. A node that enters the network early, on the other hand, may be unable to secure sufficient centrality to deter later competition, due to the lack of available connections.

Figure 1 illustrates the formation of a network composed of animation softwares. The networks begins with Photoshop and Poser unconnected. Maya enters the network and establishes connections

[1] Corominas-Bosch (2004); Kranton and Minehart (2001); and Blume et al. (2009)

[2] Calvó-Armengol et al. (2015); Apt et al. (2016); and McCubbins and Weller (2012)

[3] Allouch (2015); Bramoulle and Kranton (2007); and Carpenter et al. (2012)

[4] Candogan et al. (2012)

[5] For several reviews of the literature on the importance of network structure in empirical settings see: Bala and Goyal (2000); Jackson (2003); Jackson and Wolinsky (1996); andCarrillo and Gaduh (2012)

[6] For theoretical evidence see Kranton and Minehart (2001); Blume et al. (2009); Apt et al. (2016); Chen and Teng (2016). For empirical evidence see Pollack et al. (2015); Sarigöl et al. (2014); Powell et al. (1996); Rossi et al. (2015)

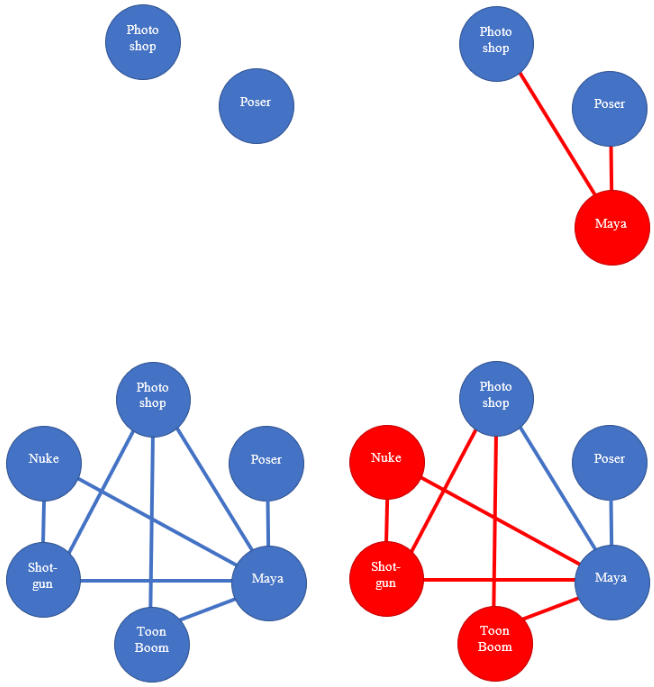

Figure 1: The formation of a network of compatibilities in animation software starting at the top left and going clock-wise.

(representing compatibilities) with the two existing nodes. Additional nodes–representing Toon Boom, Shotgun, and Nuke–join the network and connect to Maya. In the final network, Maya has a very dominant position with connections to all five other nodes.

In this example we see a number of phenomena which should be explained: preferential attachment and a non-monotonic relationship between entry timing and position. Preferential attachment is a property whereby having a relatively high centrality causes a node's centrality to increase more quickly over time. This phenomenon we first observed in taxonomic networks by Yule (1925), but it has since been found to be a common feature of networks arising in transaction networks,[7] social networks,[8] online link networks,[9] scientific collaboration networks,[10] and citation networks.[11] If preferential attachment was the only force at play, however, we would expect the earliest nodes to have the most connections rather than one that entered near the middle.

We provide a possible explanation for how Maya's dominance in the final network relates to the order of joining and preferential attachment. Poser could have made a connection to Photoshop, but that would not have provided high enough centrality to deter Maya's following suit. After making two connections, Maya gained has the highest centrality in the network (dominance), which made it appealing to later nodes to connect with. We call this type of behavior *vying for dominance.*

Definition: *Vying for dominance* is a move causing the player to become one of the dominant nodes immediately after his move.

These later players could not gain a centrality advantage over Maya, because doing so would have required making a cost prohibitive number of connections. Instead, they make a single connection to the most central node, becoming moderately central at low cost. We call this type of behavior a myopic move.

[7]Kondor et al. (2014)
[8]House et al. (2015)
[9]Eiron and McCurley (2003) and Albert-László et al. (2000) although the evidence is somewhat indirect
[10]Newman (2004)
[11]Wang et al. (2008)

Definition: A *myopic* move is a move which would be optimal if the game ended immediately after that move. In the games we discuss in this paper, this means making one connection to one of the most central (or *dominant*) nodes.

These later myopic moves give value to the earlier vying for dominance. A player who vies for dominance can effectively free ride on the connections from later players.

This explanation has a number of realistic features which would be desirable in a model of network formation: history dependence, growth, and forward looking strategic agents. Most networks of interest display history with the evolution of the network depending on its current state.[12] In addition, many networks grow with new nodes being added over time.[13] Having forward looking strategic agents is what allows for strategically rich behavior like that of Maya in the example. In addition to being realistic, these features also increase the importance of the dynamics and allow us to model the impact of entry timing on eventual node centrality.

In this paper, formalize the intuition of the example by presenting a new dynamic network formation with history. As we show in an example in Section 4.1, this model predicts myopic actions and vying for dominance. With this model we do three main thing of interest: we provide basic results and discuss extreme parameter values; we provide more complete results for a more tractable restricted game; and we discuss modifications and extensions to the base model .

In Section 5 we establish network efficiency and solutions results. Here the paper serves as a companion to Neligh (2017), giving proofs and corollaries for the basic results. In general when the cost of connections is high, the minimally connected network is efficient and tends to form. Conversely when the cost of connections is low, the complete network is efficient and tends to form. However, the thresholds do not line up well, so inefficient over-connection and inefficient under-connection are both possible.

We also show that in intermediate parameter ranges, the model predicts strategically rich behavior such as vying for dominance.

The basic model becomes quickly intractable as the number of nodes increases. But, surprisingly, we find that a simple and plausible restriction on the strategies simplifies the problem dramatically, and allows us to characterize the SPE of the game for any arbitrary finite number of nodes. Precisely, we find that if players are required to connect to at least one dominant node as they join the network, then in equilibrium all players will either connect to a subset of the dominant nodes (analogous to the myopic move) or all nodes (vying for dominance). We discuss this restricted game in Part II.

If we also put additional structure on the tie-break rule in the restricted game, we can characterize the dynamic sequence with which the two strategies are chosen. If we assume that players resolve indifferences in favor of connecting to newer node, the solution is characterized by individuals vying for dominance separated by periods of myopic play. Furthermore, the length of the periods of myopic behavior increase exponentially as the game goes on. For a vying move to be profitable, it must result in enough future connections to pay for the additional immediate costs of making many links. As the network grows, vying becomes more expensive, so more myopic actions are needed to support each vie for dominance.

Appendix Section 8 examines a number of modifications and extensions to the model. First we address what happens to the results in the base game when we allow players to make zero connections. This extension creates additional parameter regions of interest in which the empty network forms and is efficient. Because this formulation allows for multiple connected components, we also provide a result on how large various components of the network can be. Next we discuss allowing for heterogeneity in connection benefits, and allowing players to make additional connections after they first connect to the network. In general the main results still hold, although there are some minor caveats and several propositions must be rewritten to work in the new environments. Appendix Section briefly 9 considers how extensions and generalizations impact the major results from the restricted game.

[12]For evidence on history dependence see Puffert (2002) and Puffert (2004)

[13]See Mislove et al. (2013) for an example.

2 Lit Review

There has been a great deal of work on network formation in the past, but previous models did not have dynamics that were as "strong" in some sense as those in our model. These models either lacked dynamics entirely, used agents that were not fully forward looking, or used special setups in which the set of possible solutions depends only on static features of the networks such as stability or efficiency.

Early research on network formation did not include optimizing agents. Two papers in the field are Yule (1925), mentioned previously with their preferential attachment, and Erdös and Rényi (1960), with their small world networks. Both of these papers are dynamic but connections occur based on mechanical processes with no utility or rationality laying a role. As mentioned in the introduction, Yule (1925) introduced the concept of preferential attachment whereby nodes with many current connections tend to gain more connections in the future. This concept is an important factor contributing to vying for dominance.

Economic models of network formation have traditionally been stability based with few dynamic features. For example Jackson and Wolinsky (1996) propose a model of cooperative network formation[14] in which a network is stable if and only if every player who is part of a connection wants to keep that connection and no two players who are not connected want to connect. Note that this stability concept is cooperative because players need to agree to make connections.

Bala and Goyal (2000) developed a model of non-cooperative network formation, leading to another large branch in the literature.[15]. In non-cooperative models of network formation, a network is stable if and only if every person who is sponsoring a connection wants to maintain that connection and no player wants to sponsor a new connection. This model is non-cooperative in the sense that players can make connections unilaterally. It should be noted that Bala and Goyal (2000) did discuss a dynamic version of their model, but they did not allow for forward looking strategic agents.

A great deal of work has been done introducing the concept of farsighted stability into the domain of cooperative network formation in work by Page et al. (2003), Dutta et al. (2005), and Herings et al. (2009). This work does, to some extent introduce a form of implied dynamics and simple foresight into the model, but the models do not have a formal dynamic structure, and the agents are not forward looking in the subgame perfect sense.

Many recent models include dynamics in a more direct and formal manner[16]. However the agents in these models are not forward looking. In Watts (2001), for example, players are assumed to update their connections myopically without regard to future consequences. In Kim and Jo (2009) players only receive payment as they are joining the network, so future periods are irrelevant to the current mover.

There is one set of related models which includes both non-trivial dynamics and froward looking strategic agents. [17] However, these models usually employ special setups in which the feasibility of achieving a particular network depends only on static features of that network. For example Song and van der Schaar (2015) also propose a model with a repeated game-like structure, so all networks which produce more than min-max payoffs for all players are feasible solutions. In Mutuswami and Winter (2002) and Currarini and Morelli (2000) all efficient networks can be formed using centralized mechanisms.

There are are a few dynamic network formation models in which solutions do not depend on static features of the networks, although additional simplifying assumptions are generally used in these cases. In the model of Aumann and Myerson (1988), the payoff function used guarantees that only complete connected components can form. In other words, all nodes in a "group" must be connected to all other nodes in that group. Only the number of nodes in a particular group matters, because only one structure is possible for a given group size. This allows the network formation model to be reduced to a more standard model of dynamic coalition formation. The model of Chowdhury (2008) is one of the most similar to our own. Both models include sequential link formation and forward-looking strategic agents. In addition, there is the possibility in Chowdhury (2008) for early movers to make myopically

[14] For an in depth look at this type of network see the book by Jackson (2008)

[15] Note that they did consider dynamics in their original paper, although the solution concept was inherently non-dynamic. For a more in depth look at this type of network see the book Goyal (2007)

[16] Such as Watts (2001); Kim and Jo (2009); and Vazquez (2003)

[17] See Mutuswami and Winter (2002); Currarini and Morelli (2000); and Song and van der Schaar (2015)

sub-optimal moves in hopes of gaining future connections, which can be thought of as loosely similar to the vying for dominance behavior of our model. However, Chowdhury (2008) assumes that each node can only sponsor one connection, and thus rules out by assumption the possibility of competing for centrality by making multiple connections.

The general features of our model should be familiar, borrowing elements from existing models and puts them in a dynamic framework. We employ a sequential mechanism in which players can form links unilaterally as in Bala and Goyal (2000) but which employs a utility function based on the payoffs in Jackson and Wolinsky (1996). The growing set of agents and history dependence are similar Yule (1925).

The combination of strategic agents and history dependence is an important one, because it allows for a much greater depth and range of complex behaviors than either feature can generate by itself. Players makes complex decisions which influence the incentives of later players such as vying for dominance. Small changes near the beginning of the game can have dramatic impacts on later play. Whereas, in the models previously mentioned small changes in play tend to influence which players end up in which positions in the finished network (such as which node will be the center of a star network, in this model early play can dramatically influence the entire network structure. This leads to a number of mathematically interesting results and improves the realism of the model. The addition of growing set of agents to the model is important, because the set of agents is not stable in many interesting networks.[18] New firms and consumers are always joining the market. Furthermore, creating new ties or destroying old ones can be costly. Manufacturers do not immediately change their suppliers in response to small shifts in demand. Our model captures these interesting dynamic features which do not fit well with previous models.

Part I
The Base Model

3 Model

We now present the basic network growth model which we will be using for the remainder of this Part of the paper.

There is a group of players, each represented by a node. New players/nodes join the network one at a time. As players join the network, they choose which existing nodes to connect to. They must connect to at least one existing node. Once the last player has joined and made their choice the game ends, and players receive points based on number of connections made and position in the final network. Centrality is beneficial but making connections is costly

We now present the model more formally. Lower case letters refer to indices and non-parameter scalar values. Model parameters and networks are referred to by upper case letters. Edges between two nodes are referred to using tuples of the form (index 1,index 2). Bold lower case letters refer to sets of nodes, sets of edges, and vectors.

There is a set of players represented by nodes indexed $j \in \{1, ..., J\}$. Networks are represented as $G = \{\mathbf{n}(G); \mathbf{x}(G)\}$ where $\mathbf{n}(G)$ is a set of nodes, and $\mathbf{x}(G)$ is a set of edges represents by pairs of nodes. The networks are also indexed by time as G_t where $t \in \{1, 2, ...J\}$. Note that there is one time period for every player/node, so indices are largely interchangeable. The game begins with the initial network containing only Node 1 $G_1 = \{1; \emptyset\}$.

A strategy for player j maps every possible network state they can face, G_{t-1}, to a distribution over sets of connections. Each set of connections $\mathbf{h}_t$ must be non-empty and contain only connections between Node t and existing nodes in G_{t-1}. Player t is choosing which existing nodes to connect to.

After player t makes their move, the network evolves according to the following rule:

$$G_t = G_{t-1} \cup \{t; \mathbf{h}_t\}$$

[18]For an example seeMislove et al. (2013)

In other words, the new network is created by adding a node representing the new player and all of the connections made by that player to the existing network.

The game concludes after Player J makes his choice, generating the final network G_J.

Once the game has concluded, each player gets a payoff according to the following utility function.

$$u_i(\mathbf{h}_i, G_J) = Y - C|\mathbf{h}_i| + B\zeta_i(G_J, \delta)$$

$Y \in \mathbb{R}$ is a constant base payoff. $C|\mathbf{h}_i|$ is the cost of connections by individual i who purchased the set of connections $\mathbf{h}_i$. $C \in \mathbb{R}^+$ is the constant cost of connections. $B\zeta_i(G_J, \delta)$ is the benefit from centrality. $B \in \mathbb{R}^+$ is a constant multiplier, and $\zeta(G_J, \delta) = \sum_{j \neq i} \delta^{d_{ij}(G_J)-1}$ is a standard measure of closeness centrality. Decomposing $\sum_{j \neq i} \delta^{d_{ij}(G_J)-1}$, $\delta \in (0, 1)$ is a geometric discount factor. $d_{ij}(G_t)$ is the minimum distance between Node i and Node j in edges under network G_t. The minus one in the exponent adjusts the term such that we do not have to normalize B and C with respect to δ.

This type of geometrically payoff function is fairly standard and is used in both Watts (2001) and Jackson and Wolinsky (1996).[19] This type of network payoff is most relevant for systems in which some beneficial opportunity or information lands at a random node and then disseminates throughout the network with value decaying over time. It can also be applied as a useful approximation in any system where more central nodes gain more benefits, as this measure of centrality is highly correlated with other measures of centrality, especially in networks with low diameter.[20]

4 Solutions

We take Subgame Perfect Equilibria (SPE) as our solution concept of choice, because it captures the idea of forwards looking strategic agents the best. We use the standard definition of subgame perfect equilibrium: a strategy profile in which every action chosen with some positive probability after a given action history is optimal for the subgame beginning with that decision. Because this is a finite game of perfect information, existence of a SPE is assured. The solution to the game is not always unique; multiplicity of equilibria derives from the manner in which players resolve indifferences: the tie-breaking rule.

Definition. Tie-Breaking Rule: a tie-breaking rule refers to some rule by which players resolve indifferences in the construction of a SPE.

A player's tie-breaking rule can be thought of as a possibly stochastic mapping from action histories to strict orderings of moves. The selected strict ordering is used to transform the current actor's weak preference ordering on moves into a strict one (thereby determining that player's move). Note that the indifferences in this game are due to structural symmetries and similarities inherent in network formation and are not related to off path behavior. As such refinements like sequential equilibrium will not remove the main source of multiplicity.

Note that because indifference resolution is the only source of multiplicity in this game, a solution to the game can be fully characterized by the set of parameters and the tie-breaking rules employed by all players.

4.1 An Example

We present an example with four nodes in order to help build intuition and demonstrate how beliefs about the behavior of future players influence the behavior of earlier players. Nodes 1 and 2 have no decisions to make, so we shall ignore them. Assume $C > B(1-\delta)$. We solve the SPE through backwards as normal.

Player 4 will always connect to exactly one of the most central nodes. Note that making one connection to one of the most central nodes is always the best one connection move that the last

[19] Their payoff function is has $Y = 0$ and $B = \delta$, but otherwise is identical.

[20] For an examination of correlation in measures of centrality in real world networks, see Valente et al. (2008). The diameter of a network is the largest minimum distance between two nodes. We will discuss the importance of low diameter networks in more detail in Part II

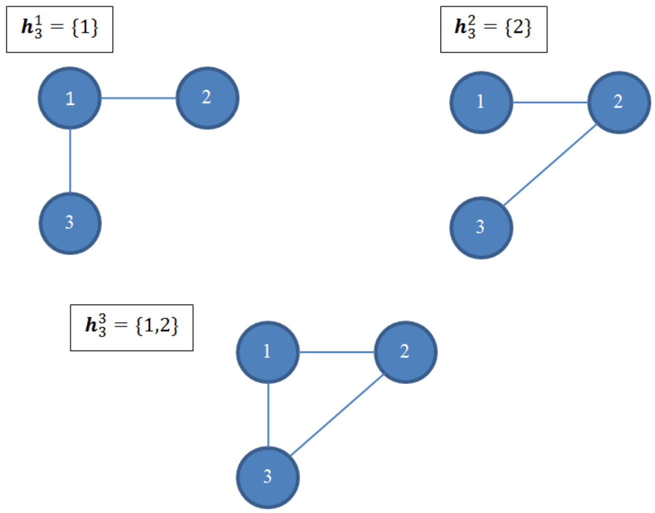

Figure 2: Immediate outcomes of the moves of Player 3

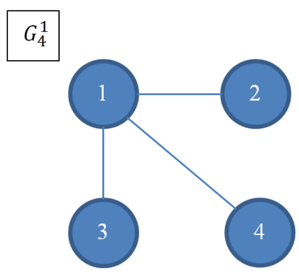

Figure 3: Final result of player 3 choosing move $\mathbf{h}_3^1$.

player can make. This comes from the fact that the centrality of the last player connecting to one node is *delta* times the centrality of the node connected to plus one.

Connecting to one of the most central nodes always results in Player 4 being a distance of two from the other nodes. Each additional connection would increase his cost by C with a benefit of at most $B(1-\delta)$, the benefit of decreasing the distance to a single node from two to one. Connecting to any number nodes that are not most central is strictly worse than connecting to one of the most central nodes.

It seems then, that there is something important about the set of most connected nodes.

Definition. Dominant Nodes: The set of dominant nodes $\mathbf{d}(G_t)$ is defined by

$$\mathbf{d}(G_t) = \{\arg\max_{i \in G_t} \sum_{j \neq i} \delta^{d_{ij}(G_J)-1}\}$$

In other words $\mathbf{d}(G_t)$ is the set most central nodes in G_t.$\mathbf{d}$ is always non-empty since the network always contains a finite number of nodes with real valued centrality. $\mathbf{d}$ may or may not be single valued.

We now consider the move of Player 3. Player 3 has three options for $\mathbf{h}_3$, $\mathbf{h}_3^1 = \{1\}, \mathbf{h}_3^2 = \{2\}, \mathbf{h}_3^3 = \{1,2\}$. Each one leads to a different G_3 as illustrated in Figure 2

After $\mathbf{h}_3^1$, the only possible result would be G_4^1, since there is only one element of $\mathbf{d}(G_3)$ and Player 4 will connect to exactly one element of $\mathbf{d}(G_3)$ (see Figure 3).

We will ignore $\mathbf{h}_3^2$ since it is symmetric to $\mathbf{h}_3^1$. Depending on Player 4's tie-breaking rule, move $\mathbf{h}_3^3$ could eventually lead to G_4^3 , G_4^4, or G_4^5 (see Figure 4). This means that Player 3's choice depends on

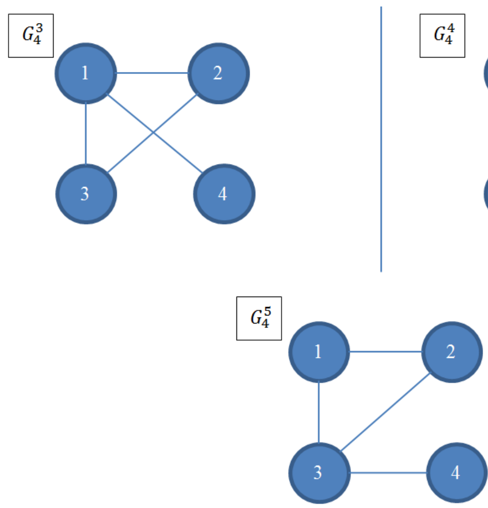

Figure 4: Possible final outcomes if player 3 chooses move $\mathbf{h}_3^3$.

his beliefs about Player 4's tie-breaking rule.

If Player 3 believes that choosing $\mathbf{h}_3^3$ will lead to G_4^3 or G_4^4, Player 3 will not choose that move, because the move costs C more than the other moves and results in being only one jump closer to one other node at the end of the game. This change in distance confers a benefit of at most $B(1-\delta)$ which is less than C.

If instead Player 3 believes that $\mathbf{h}_3^3$ will lead to G_3^5, the decision is slightly more complicated. $\mathbf{h}_3^3$ costs C more than the other two choices, but now the maximum benefit of $2B(1-\delta)$ could be enough to compensate that loss. This type of forward looking strategic consideration is an important feature of the general game as well. If move $\mathbf{h}_3^3$ leads to outcome G_3^5 with a probability of $P(G_3^5)$, then Player 3 will choose $\mathbf{h}_3^3$ as long as $B(1-\delta)(1+P(G_3^5)) > C$.

In this example, only those who have many connections already can potentially receive a connection from Player 4.[21] The benefits of connecting to dominant nodes generate a form of preferential attachment, which has important strategic ramifications.

Player 3 makes more connections than is myopically beneficial in order to gain the possibility of future connections. This behavior is the "vying for dominance" mentioned in the introduction.

This type of behavior will be very important throughout the discussion of the game. To some extent, the existence of vying for dominance behavior can be thought of as endogenizing the results of Akerlof and Holden (2016) in which players players bid early on for a resource which allows them to become the center of a network that is formed later, but in this case the investment itself is part of the network formation process. Players can invest in connections early on in order to receive more connections later.

The early mover advantage also arises naturally in this example although in a very different way. Consider what happens when Player 4 resolves indifferences in a uniform random manner and $1B(1-\delta) < C < \frac{4}{3}B(1-\delta)$. In this parameter range, Player 3 will make the extra connection in order to get the one third chance at a future connection from Player 4. The expected payoffs for each player are then

Player 1: $u_1 = \frac{7}{3}B + \frac{2}{3}\delta B$
Player 2: $u_2 = \frac{7}{3}B + \frac{2}{3}\delta B - C$
Player 3: $u_3 = \frac{7}{3} + \frac{2}{3}\delta B - 2C +$
Player 4: $u_4 = B + 2\delta B - C$

The payoffs are ranked as follows $u_1 > u_2 > u_3 > u_4$. While early mover advantage is not guaranteed in the game, it does tend to arise quite frequently. Earlier nodes have more opportunities

[21] Because only the most connected nodes can receive a connection from Player 4, the effect is more exaggerated than what is seen in traditional preferential attachment models Yule (1925)

to receive direct connections from future players, and that translates into higher payoffs.

5 Results

We now provide some results about which networks are efficient and which networks can form in different parameter regions. These results are intuitive and are generally quite robust to small changes in the assumptions of the model.[22] These results also provide a basis for some of the more novel results later in the paper.

5.1 Efficiency Results

One important question is how efficient are the networks formed by this process? Results on efficiency can provide important information for understanding the desirability of the solutions explored in other sections. When $\frac{C}{B} > (1-\delta)$ most of the outcomes are not Pareto ranked, but we can find a most efficient network in the following sense

Definition: We say an outcome network G_J is efficient if it generates the highest possible sum of utilities of all feasible outcome networks for given parameters.

Proposition 1:

- If $\frac{C}{B} < 2(1-\delta)$, then the efficient network is the complete network.
- If $\frac{C}{B} > 2(1-\delta)$, then the efficient network is the star network (on Node 1 or Node 2)
- If $\frac{C}{B} = 2(1-\delta)$, then all feasible networks which contain stars are efficient

Proof of Proposition 1: To begin, note that all networks have at least $J-1$ connections, because all players after the first must make at least one connect. Next, note that the most efficient network with any fixed number of connections $N \geq J-1$ must contain a star (if the network has fewer than $J-1$ connections it cannot contain a star). A network with N connections containing a star has $2N$ minimum paths of length one, and all other minimum paths are of length two. It is impossible to have more minimum paths of length one than $2N$ minimum paths of length one, so no configuration can produce higher centrality benefits. All networks with N connections produce a total connection cost of $C * N$

Given that the efficient network must contain a star, every additional increases the total centrality benefit of the network by $2B(1-\delta)$, because each connection moves two nodes from a distance of two to a distance of one. The social cost of connections is C, so when $2B(1-\delta) < C$, the feasible network containing a star with the minimum number of connections is socially optimal. The two networks that fit that criterion are the star network centered on Node 1 and the star network centered on Node 2. Similarly, when $2B(1-\delta) > C$, the feasible network containing a star with the maximum number of connections is socially optimal. The network which fits this criterion is the complete network. $\square$

Note that, while the sequential nature of the game does impose limits on the set of feasible networks, it does not impose strong limits on the structure of the set work in the following sense: given any connected network of un-indexed nodes, we can find an indexing under which the network can feasibly be formed. To find such an indexing, simply pick an arbitrary node as Node 1 and then pick nodes to index in order with the only requirement being that the node you pick must be directly connected to at least one already indexed node. Once all the nodes are indexed in this manner, you have an index which can feasibly produce the given network structure.

If each player takes a strategy whereby they connect to all older nodes that the node corresponding to their index is connected to in the indexed version of the original network, they will reproduce that network perfectly. Due to our indexing method, no player will ever be required to make an empty and hence illegal move.

[22]See Part II and the Appendix Section 8 for examples of ways that the results can be generalized

This result is quite similar to the result on efficiency from Jackson and Wolinsky (1996) with a few key differences. First, the empty network is never efficient, because it is never feasible in this game. The second key difference comes from the fact that the cost of each connection is only paid once by the player which makes it. In the cooperative game, the connection is costly to both parties. As such connections must be twice as costly in our game before they become socially inefficient. The efficiency threshold in Jackson and Wolinsky (1996) is $C/B = 1 - \delta$.

5.2 Solutions

Having established efficiency, we now examine the types of networks that can form in different parameter regions.

Proposition 2: If $\frac{C}{B} < (1 - \delta)$ then the complete network is the unique network which can form in SPE's of the game.

Proof of Proposition 2: We work by backwards induction.

Player J will always connect to every other player. To see this note that if J is not connected to every player, adding a new connection will increase his utility by at least $B(1 - \delta) - C > 0$.

If all future players will connect to every available player, so will player $J - k$. Suppose not, then at the end of the game he will be a distance one from all nodes he connects to (and all nodes that come after him) and a distance of at least two from from the nodes he doesn't connect to. This means that player $J - k$ can always increase his utility by connecting to an addition node if more connections are possible, since the gain, $B(1 - \delta)$, is greater than the cost, C. Therefore player $J - k$ will connect to all available nodes when he joins the network.

By induction, all players will connect to all available nodes when they join the network. This leads to the formation of a complete network. $\square$

This proposition is similar to a result from Jackson and Wolinsky (1996). In many network formation models, the complete network tends to form when the cost of connections is low.

Note on Proposition 2: This proposition is tight in a strong sense. If $\frac{C}{B} > (1 - \delta)$, then the complete network is no longer a possible outcome of any SPE at all. To see this consider the move of Player J. This player cannot connect to all other nodes in any SPE of the game. If he was connected to all other nodes but one, then the additional benefit from connecting to the last node would be $B(1 - \delta)$. This marginal benefit is less than the marginal cost of C. Since Player J must choose a myopically optimal move in any SPE, he cannot choose to connect to all nodes.

Proposition 3: If $\frac{C}{B} > (J - 1) - \frac{1-\delta^{J-3}}{1-\delta}$, then the star networks centered on Node 1 and Node 2 are the only networks which can be formed in SPE's of the game.

Proof of Proposition 3: The greatest benefit that a player can receive for making a one additional connection is the benefit Player 3 gets when, if Player 3 makes one connection, all future players connect in a chain moving away from Player 3, but when Player 3 makes two connections, all future Players connect directly to Player 3. A chain leading away from Node i is the worst possible network for Node i and a star centered on Node i is tied for the best. If Player 3's choice in this case selects between these two outcomes, Player 3's gain from making the extra connection is the difference $B\big((J-1) - \sum_{i=1}^{J-1} \delta^{i-1}\big)$

Simplifying and rearranging we get $B\bigg((J - 1) - \frac{1-\delta^{J-3}}{1-\delta}\bigg)$. If this is the maximum possible benefit that any player can get from making an additional connection under any strategy profile. We now employ backwards induction to arrive at the result.

Player J will connect to an $i \in \mathbf{d}(G_{J-1})$. To see this, note that once Player J has made one connection, the benefit of each additional connection is always going to be less than the maximum possible benefit from a connection, which is less than C. The best node to make a single connection to in this case is a dominant node by the definition of dominant nodes and assumes utility function.

Now consider Player $J - 1$. This player will also prefer every outcome where he make only one connection to every outcome where he must make multiple connections. If Player $J - 1$ is going to make one connection, it is optimal for him to connect to a dominant node, knowing that Player J will then connect to that same dominant node. If a player connects to one dominant node, that node will be the only dominant node in the next period.

Now consider the move of Player $J-k \geq 3$. By similar logic, Player $J-k$ will always connect only to one node. Knowing that all future players will each connect to a dominant node, Player $J-k$ will also connect to a single dominant node.

Player 3 will connect to either Node 2 or Node 1 since both are dominant nodes in G_2. Players 1 and 2 do not have any choices. Therefore all nodes will connect either to Node 1 or Node 2. $\square$

This proposition is a major deviation from previous literature. The ability of earlier moves to effect the incentives of later players means that the potential benefits of additional connections are much higher than in a one shot model.

Notes on Proposition 3: First, note that as δ approaches one, the condition for the proposition approaches $\frac{C}{B} > 1$. Conversely, as δ approaches zero, the condition approaches $\frac{C}{B} > J-2$. Also, note that the right hand side of the condition, is increasing in J, so the condition is more restrictive in large networks. Intuitively, this means that it is easier to generate non-star networks when the number of players is large and when the geometric discount factor is large.

Corollary 3.1: Proposition 3 is tight as long as δ is small in the following sense: If $\frac{C}{B} < J-2$ then if δ is sufficiently small there exists a SPE of the game which does not always generate a star network.

Proof of Corollary 3.1: Assume $\frac{C}{B} \geq 1$, because if the alternative is true and δ is sufficiently small, then the result is handled by Proposition 2. We being with the following conjecture

Lemma 3.1.1- there exists a Player j, who satisfies the following conditions:
(1) $(J-1)B-(j-1)C \geq B-C$
and
(2) $(J-2)B-(j-1)C \leq B-C$

If $\frac{C}{B} < J-2$ by the assumption that $\frac{C}{B} > 1$ we know $\exists \hat{j} \in [3, J)$ satisfying (1).

Call the the last $\hat{j}$ satisfying (1) player j. We have $\frac{J-2}{j} > \frac{C}{B}$ from the fact that j satisfies (1) and $\frac{J-2}{j-1} < \frac{C}{B}$ from the fact that Player $j+1$ does not. Now assume that (2) does not hold for j, so $\frac{J-3}{j} > \frac{C}{B}$. The two conditions $\frac{J-3}{j} > \frac{C}{B}$ and $\frac{J-2}{j-1} < \frac{C}{B}$ combine to give us $\frac{j-1}{j} > \frac{J-2}{J-3}$ which contradicts $j < J$. As such, (2) must hold for j. $\square$

Consider the move of Player j from Lemma 3.1.1 in the equilibrium in which players break ties in favor of connecting to newer nodes. If Player j connects to all existing players, he will receive connections from all future nodes. We can show by backwards induction that no future player will strictly prefer to make multiple connections if Player j does has connected to all existing nodes if δ is sufficiently small.

Player J will connect to the newest dominant node under this tie breaking scheme. Given that all players after Player $J-k$ will connect to the newest dominant node, Player $J-k$ will either connect to the newest dominant node or become the newest dominant node. All other moves are strictly worse as long as δ is small, because the benefit of any move which does not generate future free connections converges to $N(B-C)$ as δ goes to zero where N is the number of connections made. Given $C > B$, $N(B-C)$ is decreasing in N

The minimum number of connections required to become a dominant node is non-decreasing as the game progresses when δ is small. A player must connect to at least as many nodes as the current dominant node is connected to in order to become dominant if δ is near zero.

Because Player $J-k$ is moving after Player j already connected to all nodes, the best possible gain (and the lowest possible costs) that a Player $J-k > j$ can get from becoming a dominant node is gained by Player $J-k = j+1$ connecting to all nodes that Player j connected to.By doing this, Player $j+1$ gets $(J-2)B-(j-1)C+\delta B$. By making a single connection to Player j, a Player $J-k > j$ can get $B-C+g(\delta, G_{J-k})B$ where $g(\delta, G_{J-k})B$ is some function representing gains from indirect connections to other nodes. Notably $g(\delta, G_{J-k}) \geq \delta$. As such, thanks to condition (2), Player $J-k > j$ will always connect to a single dominant node rather than becoming a dominant node

It is better for Player j to connect to all nodes and gain direct connection from all future players instead of connecting to a single dominant node that all future players will connect to if $(J-1)B-$

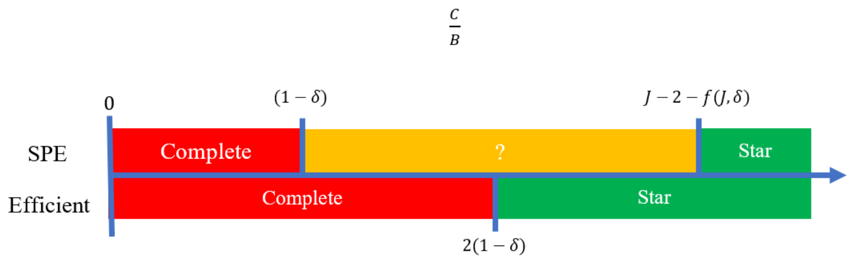

Figure 5: Visualization of parameter regions of interest. Note that the $(J-2-f(J,\delta))$ line can be either to the left or the right of the $2(1-\delta)$ line depending on parameters. $f(J,\delta)=\frac{\delta-\delta^{J-3}}{1-\delta}$.

$(j-1)C > B+(J-2)\delta B - C$. This condition always holds if (1) holds and δ is sufficiently small. Therefore if δ is sufficiently small, then the SPE of the game in which players break ties in favor of connecting to the newest node can not form a star network with certainty. Otherwise Player j would deviate. $\square$

Corollary 3.2: If δ is sufficiently small and $\frac{C}{B} > J-i+1$ then all players after Player i will connect to the same dominant node.

Proof of Corollary 3.2: Consider the move of Player $i+1$. The maximum future benefit that Player $i+1$ can gain from making an extra connection is less than the benefit of moving all future nodes from an being an infinite distance away to being directly connected and all current nodes from an infinite distance away to a distance of two away, $(J-i)B+(i-1)\delta B$.[23] The total benefit from an extra connection is then less than $B(J-i+1+g(\delta,G_i)+(i-1)\delta B)$. This means that if $\frac{C}{B} > J-i+1$ and δ is sufficiently small, then it will never be beneficial for Player $i+1$ to make multiple connections.

Similar logic applies to all players after Player $i+1$ as well, who all have strictly lower possible future benefits. Therefore, no player after Player i will make multiple connections.

Given that no players after i will make multiple connections, we can show by backwards induction that it is the optimal for all players after Player i to connect to the same dominant node. Player J will connect to a single dominant node since that is myopically optimal. If all future players will make a single connection to a dominant node regardless of Player $J-k$'s move, and Player $J-k$ cannot profitably make more than one connection, then it is optimal for him to connect to a single dominant node regardless of the current network state.

If Player $J-k$ connects to a dominant node, it will become the only dominant node. As such all future players will connect to it, which provides as much benefit as Player $J-k$ can receive from future nodes without receiving direct connections from those players. Receiving direct connections is not possible, given that all future players will connect to a dominant node, and it is impossible for any Player after Player 2 to become dominant by making one connection.

This logic can apply to any $J-k > i$, so after Player i all players connect to a single dominant node.[24] $\square$

6 Summary of Results

The results of the previous sections are summarized in Figure 5.

There are parameter regions where the star network is formed as the unique SPE outcome and regions where the complete network is formed as the unique SPE outcome as well as regions where both networks are efficient. Furthermore, there is an intermediate region where we cannot guarantee either the star or the complete network. The size of this unknown region is increasing in the number of players, meaning that larger networks will more often have non-degenerate structures, because in

[23] Player i will never be directly connected to any nodes he is not already directly connected to.

[24] If there are multiple dominant nodes before this type of play begins, the first single connection player will select which one becomes the unique dominant nodes based on their tie-breaking rule.

larger networks early moves can have larger potential future benefits. Note that the complete network cannot form in the yellow region, the complete network cannot form. The star network may form in the yellow region, but it is not guaranteed to be a solution, and as discussed in Corollary 3.1, it is never the unique SPE outcome as long as δ is small.

Because the threshold determining efficiency and solutions do not align, it is possible that both inefficiently under-connected and inefficiently over-connected networks can arise in subgame perfect equilibria.Inefficient under-connection arises when $(1-\delta) < \frac{C}{B} < 2(1-\delta)$, and inefficient over-connection can arise when $(J-1) - \frac{1-\delta^{J-3}}{1-\delta} > \frac{C}{B} > 2(1-\delta)$. This feature means that it is imperative that anyone regulating this type of system pay very close attention to the fundamentals of the system in order to know what type of adjustments are appropriate.

We can briefly contrast these results to similar ones found in Jackson and Wolinsky (1996) and Watts (2001) in order to establish which features drive several features of these results.

The fact that we are requiring non-empty connections eliminates the potential for an empty network. As such parameter regions where the empty network might be the unique solution or the efficient network in Jackson and Wolinsky (1996) are rolled into regions corresponding to the star network.

The fact that we are using non-cooperative network formation shifts the efficiency threshold from $\frac{C}{B} = 1-\delta$ to $\frac{C}{B} = 2(1-\delta)$. Since the region where the complete network is guaranteed does not shift, this change allows for the possibility of inefficient under-connection.

While there is an intermediate parameter region in Jackson and Wolinsky (1996) where non-degenerate networks can form, our model generates very different types of networks and behavior. The stable networks in Jackson and Wolinsky (1996) are always locally efficient in the sense that changing them by adding or removing one connection will decrease welfare. Inefficiency in their model is driven by the difference between global and local optimum. In our model, on the other hand, local optimality is not guaranteed. Inefficiency is instead driven by the existence of positive and negative externalities. Nodes create positive externalities whenever they connect to another node. Vying for dominance, however, can produce negative externalities by taking future connections away from players who would otherwise be receive them.

The region in which non-degenerate networks can form very large. It is possible to have networks in this our model wherein players make multiple connections for much higher values of C/B especially when δ is small. While Jackson and Wolinsky (1996) do not establish a bound above which only empty networks can form, this bound would need to be less than $C/B = 1+\delta(J-2)$, because $B+\delta B(J-2)$ is the largest possible benefit a player can receive from maintaining a connection in their model. Contrast that with our model in which players can theoretically gain upwards of $(J-2)B(1-\delta)$ from a single connection.

6.1 Restoring Efficiency with a Centralized Tax/Subsidy Mechanism

While the game can generate inefficiency in its natural operation, achieving efficiency through a centralized mechanism is actually fairly trivial. If $\frac{C}{B} < 2(1-\delta)$ then the planner can add a subsidy of $C - B(1-\delta) + \epsilon$ per connection. The effective cost of connections is reduced below $B(1-\delta)$ so the complete network forms. Since a total of $J(J-1)/2$ connections will be made in SPE for this effective cost level, the subsidy can be offset by a total lump sum tax of $(C - B(1-\delta))\frac{J(J-1)}{2}$ divided in some way between participants. If the average number of connections made without the subsidy is K the total welfare gain from such a scheme will always be at least $(2B(1-\delta) - C)(\frac{J(J-1)}{2} - K)$

If $\frac{C}{B} > 2(1-\delta)$, the planner should instead impose a tax of $(J-1)B - C$ per connection. The effective cost of connections would then be increased to $(J-1)B$ which is high enough that only the star network will form. A total of $J-1$ connections will be made, leading to a total revenue of $((J-1)B - C)(J-1)$. If the average number of connections made without the tax is K then the efficiency gain from the tax is at least to $(C - 2B(1-\delta))(K-(J-1))$.

6.2 Tractability

The question naturally arises of what happens in the non-degenerate region (yellow) region. This question is, in general, very difficult to answer. It is important to note that the equilibria in the yellow region are not, in general, unique and tie-breaking rules play a role in determining the outcome.

While we can solve the game for small networks (up to around 6 nodes in size), solving larger games is very difficult. Brute force backwards induction in a J node network would require looking at the payoffs associated with $(J-1)!2^{J^2-J}$ possible networks. It may be possible that some simplifications may be made which reduce the computational complexity, but in general, the range and sophistication of strategic behaviors grows rapidly in complexity with the size of the network. Once networks get larger than five nodes, more we start to see players using types of behaviors that are neither taking a myopic action nor vying for dominance. A six node example is provided in the next section to show the types of strategic complexity that can arise in larger networks. We can take a number of approaches to exploring the game in the unknown region, all of which will be explored at various points in the paper.

In Part II we consider a more tractable modified version of the game. In the restricted game, players are required to connect to at least one dominant node as part of their move. Surprisingly, this restriction is enough to get us tractability and allow us to pin down precisely the critical times at which nodes should vie and become dominant. We also provide another tractable version of the game in which players join and leave the network continually in the Appendix. This type of Continual Game allows us to employ some of the techniques used to study repeated games.

6.3 Strategic Behaviors Example

This example demonstrates two counter-intuitive properties which can make finding solutions difficult in larger networks. The first property is that nodes do not always move myopically or vie for dominance. While in the small example discussed before, these are always the best options, in larger networks, more complex behaviors can arise. The second property is that it is not always possible to support a star network by using a tie-breaking rule which favors older nodes. Again. in the previous example, if players break ties in favor of earlier nodes, only the star network can form, but this is not always the case.

Example: Consider what happens in the case where $J = 6, B = 1, \delta = 0.05, C = 1.1$, and all players break ties in favor of connecting to the set of nodes with the greatest total age.

Player 6 will connect to the oldest dominant node, since the gain from a second connection can never compensate for the cost, and he breaks ties in favor of older nodes.

Player 5 will connect to all existing nodes if doing so will make him the only dominant node. Otherwise, he will connect to a single dominant node as well, because if he is not the only dominant node, he will not receive a connection from Player 6. If Player 5 cannot receive a future connection, a myopic move is his best choice. Player 5 can only become the sole dominant node if he is facing a chain or a box.

Player 4 knows this, so he will create the chain if available, since doing so requires only one connection and gets him a future connection from Player 5. Creating the box also provides a future connection but requires Player 4 to purchase one more connection, which is not worth the cost. No other moves provide a future connection for him. By the tie-breaking method, Player 4 will always connect to the oldest node of any three node chain he faces to create the longer chain (which will always be Node 1 or Node 2).

If he can not forming a chain (since he is facing a complete network), Player 4 will connect to the oldest dominant node, which can never be Node 3.

Player 3, then will just connect to one node, since doing so will have minimal cost and provide one future connection from Player 5. Connecting to two nodes will provide no future connections. As such, by tie-breaking rule, Player 3 will connect to Node 1, Player 4 will connect to Node 2, Player 5 will connect to all existing nodes, and Player 6 will connect to Node 5 (see 6).

Note that Player 4 in this case is not choosing a myopic move or vying for dominance. In addition, the star network does not form even though players are using a tie-breaking rule which favors older nodes. In Part II we examine a slightly modified version of the game in which these phenomena do not arise.

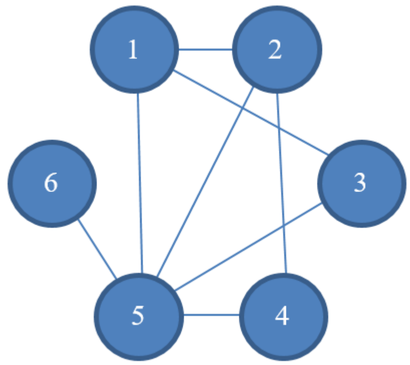

Figure 6: Graph generated by stability seeking tie-breaking rule when $B = 1, \delta = 0.05, C = 1.1$.

Part II
The Dominant Node Restricted Game

In this section we present a model which is a more tractable variant of the base game.

This model could be used to represent a system in which the diameter of the network is intentionally kept very small or in which there is strong incentive to connect directly to the current dominant node. As we shall show, the dynamics of this game are very similar to the dynamics of small networks for the unrestricted game. Even in systems that do not satisfy the restriction, it is likely that the simpler dynamics of this game can teach us about the behaviors of cognitively limited agents who are unable to solve the more complex version of the game or systems with many myopically motivated agents who do not care about future play.

6.4 Restriction

In order to allow the model to be tractable we make the following assumption

Dominant Node Restriction (DNR): We require that $\forall \mathbf{h}_t, G_t \exists i$ such that $(i, t) \in \mathbf{h}_t$ and $i \in \mathbf{d}(G_t)$. In other words, every player must connect to at least one dominant node as they join the network.

The power of this assumption comes from the fact that it guarantees that there will always be one dominant node connected to all other nodes, which simplifies the nature of vying and myopic moves substantially.

This assumption is primarily valuable in how is improves the tractability of the game without eliminating important strategic features. The restriction may be more realistic in environments in which agents are cognitively limited or in which there are strong incentives to maintain a low diameter in the network. The restriction keeps the diameter of the network to two. A group that wished to guarantee that their network maintained a low diameter may choose to enforce this rule on new entrants. One could also the restriction as being related to accessibility with Players having to essentially "enter" the network through the dominant node before connecting to others.

It should be noted that in small networks such as the example in Section 4.1, players will naturally choose moves which satisfy the restriction without being required to do so. Players generally want to connect to better connected nodes at least in a myopic sense. Results only diverge for larger networks where more complex strategic concerns can come into play.

6.5 A Note on Parameters

One important feature of the DNR is that it guarantees that nodes are never more than a distance of two away. This can be shown through a simple induction. If all nodes are currently connected to the most connected nodes, and the current player must connect to at least one of the most connected nodes, then at the end of the current turn, all players will still be connected to the most connected nodes. Thus, players will always either be a distance of two from each other or directly connected. The directly connected players confer a benefit of B while the distance two players provide δB. As such we can rewrite the utility function as

$$u_i = Y - C \cdot |\mathbf{h}_i| + B(1-\delta) \cdot \sum_{j \neq i} (d_{ij}(G_J) = 1) + \delta B(J-1)$$

We can then define $\tilde{Y} = Y + \delta B(J-1)$ since $\delta B(J-1)$ is fixed and $\beta = B(1-\delta)$. We then rewrite the utility function as:

$$u_i = \tilde{Y} - C \cdot |\mathbf{h}_i| + \beta \cdot \sum_{j \neq i} (d_{ij}(G_J) = 1)$$

We will primarily be using this formulation in this section.

6.6 Analogous Results from the Base Game

Most of the results from the base game still apply essentially without modification with the reasoning behind them being unchanged. We provide the modified propositions below.

Proposition DNR 1:

- If $C/\beta < 2$, then the complete network is the efficient equilibrium that can be formed in the DNR game.
- If $C/\beta > 2$, then the star network (on Node 1 or Node 2) is the efficient network that can be formed in the DNR game.
- If $C/\beta = 2$, then all feasible networks are efficient

Note that all feasible networks in the DNR contain a star.

Proposition DNR 2: If $C/\beta < 1$ then the complete network is only network which can be formed by SPE of the DNR game.

Proposition DNR 3 If $C/\beta > (J-1)$, then the star networks centered on Node 1 and Node 2 are the only networks which can be formed in SPE of the DNR game.

So far we have not gained much of anything from the DNR that we did not have before. The real gains in terms of tractability come in the previously unknown region where $1 < C/\beta < J-1$

6.7 Markov Perfection

In general, the SPE's of the game are not well behaved for intermediate values of C/β due to the possibility of sunspots. However, by refining our solution concept, we can get rid of most types of sunspot behavior . To that end, we will be using Markov Perfect Equilibrium as our primary solution concept for the remainder of this Part of the paper.

Simply stated, Markov Perfect Equilibrium (MPE) requires that each player in equilibrium conditions their moves on the "coarsest" possible partition of game histories, with the added requirement that the partition on which players are conditioning cannot be any coarser than the partition where all histories which lead to the same set of available moves are in the same element of the partition.[25] In our game we do not have to worry about the definition of "coarseness", because it is actually possible to construct an equilibrium where each move depends only the set of available moves.[26]

[25] Maskin and Tirole (2001)

[26] It is actually possible to create equilibria using even coarser partitions of this game, but the standard definition of Markov Perfection we are using does not allow this.

It should be noted that most intuitive tie-breaking rules generate a MPE of the game. For example, breaking ties in favor of moves with low total age, breaking ties in favor of moves with high total age, and breaking ties uniformly at random all generate MPE's of this game.

The following result allows uses the solution concept of Markov Perfect Equilibria in order to reduce the set of moves that players will make in equilibrium and therefore the set of possible networks that can be formed. The remaining rational moves fall into two easy to interpret categories with natural parallels in real world systems. This result is also critical to solving of the restricted game and result will also allow us to characterize several specific solutions which we will do later in the section.

Proposition DNR 4: If $C \neq \beta$ then, in all Markov Perfect Equilibria of the DNR game, $\forall t$ $\mathbf{h}_t(G_t) \subseteq \mathbf{d}(G_t)$ (a Type A Move) or $\mathbf{h}_t(G_t) = \{1, 2, ..., t-1\}$ (Move B).

In other words every node will either only connect to a subset of the dominant nodes or connect to every node in the network. Players will not, for example, connect to one dominant node and one non-dominant node. In most situations, the vast majority of feasible moves involve connecting to a single dominant node and then some other subset of non-dominant nodes, so this reduction in rational moves is quite substantial.

Proof of Proposition DNR 4: The case where $C < \beta$ has already been covered, so we will assume $C > \beta$. We begin by showing that Player J must always choose a Type A Move, and that his choice among Type A Moves depends on the network only through the set of dominant nodes. We then show that if all future players choose either a Type A Move or Move B, and that the choices among these moves do not depend on the current network except through the current set of dominant nodes, then it is optimal for the current node to do the same (and no other moves are optimal). Note that there is a one to one correspondence between the set of dominant nodes and the set of possible actions, so conditioning on one is the same as conditioning on the other.

<u>Player J</u> We begin the backwards induction by considering the move of Player J. We want to show that Player J will always only connect to $b_{J-1} \in \mathbf{d}(G_{J-1})$ (Type A Move). Player J will always connect to some b_{J-1}, by the DNR. Any additional connection can only provide Player J benefit of B. The marginal utility of each additional connection would be $B - C < 0$. All other possible moves are strictly worse than $\mathbf{h}_J = b_{J-1} \in \mathbf{d}(G_{J-1})$ (Type A Move).

Player J is indifferent between all singleton Type A Moves, as such, by Markov Perfection, he can only condition his move on the set of dominant nodes $\mathbf{d}(G_{J-1})$

<u>Player $J-k$</u> In examining the move of Player $J-k$'s move, we will assume the following conjecture.

Conjecture DNR 4.1- All future players play a Type A Move or Move B, and their behavior depends only on the set $\mathbf{d}(G_{t-1})$.

Note that in this conjecture, we are not requiring that the sequence of future moves includes both types of move. If the future sequence of moves must include only Type A Moves, the conjecture is still satisfied.

If Player J is the only remaining player then this conjecture is satisfied since we have already shown that Player J's behavior meets this criterion. If we can show that under Conjecture MPE 1 Player $J-k$ will also only choose a Type A Move or Move B and his choice will only depend the on the network state through $\mathbf{d}(G_{J-k-1})$, then we are done by induction.

We prove this using two lemmas

Lemma 4.1.1- Given conjecture DNR 4.1 A Type A Move or Move B will always be a strictly better option than any other type of move. This proves that Player $J-k$ will always choose Move A or Move B.

Lemma 4.1.2- Given conjecture DNR 4.1 The expected value and feasibility of the Type A Moves and Move B depends only on $\mathbf{d}(G_{J-k-1})$

Proof of Lemma 4.1.1: The lemma follows from the fact that the game is future dependent in the sense that the utility that Player $J-k$ receives depends only on the myopic utility of his action (net loss from making current connections) and the behavior of future players (gain from future connections) and the fact that these components are additively separable. Since the probability of any set of future actions depends only on $\mathbf{d}(G_{J-k})$, any optimal action by Player $J-k$ must be the myopically best way of achieving the resulting $\mathbf{d}(G_{J-k})$.

Let us then consider the different possible moves. Every dominant node that Player $J-k$ connects to is going to remain a dominant node. No other node that Player $J-k$ connects to will become a dominant node. Player $J-k$ can only become a dominant node by connecting to all players, since the current dominant node is connected to all players. Since new connections are myopically harmful, Player $J-k$ wants to choose the move with the fewest possible connections to achieve a given $\mathbf{d}(G_{J-k})$. That means Player $J-k$ will connect to all nodes (Move B, if he wants to add his own node to $\mathbf{d}(G_{J-k})$) or just another a subset of the current dominant nodes (a Type A Move, if he wants that subset to be in $\mathbf{d}(G_{J-k})$). No other moves can achieve different $\mathbf{d}(G_{J-k})$ or the same $\mathbf{d}(G_{J-k})$ with fewer connections. $\square$

Proof of Lemma 4.1.2: We first show that the feasibility of each Type A Move and move B depends only on $\mathbf{d}(G_{J-k-1})$. The set of feasible Type A Moves is the set of subsets of $\mathbf{d}(G_{J-k-1})$ and Move B is always possible. The expected value of each of these moves can again be decomposed into myopic and future portions. The myopic utilities from each move are fixed since they depend only on the number of connections made. The future benefits from all Type A Moves are similarly fixed, since each Type A Move will always produce the same $\mathbf{d}(G_{J-k})$ with $\mathbf{d}(G_{J-k})$ equal to the nodes Player $J-k$ connected to regardless of the current network structure conditional on that move being a Type A Move. Whether or not a given move is a Type A Move depends only on $\mathbf{d}(G_{J-k})$ by definition. The future benefits of Move B depend only on $\mathbf{d}(G_{J-k-1})$ since after Move B $\mathbf{d}(G_{J-k}) = \mathbf{d}(G_{J-k-1}) \cap J-k$. $\square$

These results, combined with the requirements of MPE, prove that Player $J-k$'s choice among Type A Moves and Move B only depends on $\mathbf{d}(G_{J-k-1})$. Because Player $J-k$ can optimally condition his action on $\mathbf{d}(G_{J-k-1})$, and there is a direct one to one mapping from $\mathbf{d}(G_{J-k-1})$ to available actions, Player $J-k$ must condition his move on $\mathbf{d}(G_{J-k-1})$ in any MPE. All that remains is proving the lemmas $\square$

Note that if $C = \beta$ then all players randomizing among all feasible moves constitutes Markov Perfect Equilibria. Such an equilibrium generates all feasible networks with positive probability.

While the exact impact of Proposition DNR 4 on the state space is hard to determine, since the number of dominant nodes at each step is highly path dependent, this result is very strong in the sense that it eliminates a tremendous number of possible moves. Player i who is facing a network with k dominant nodes will have $2^i - i + k$ possible moves. Proposition DNR 4 eliminates all but $2^k + 1$ of those moves.[27]

Consider an example in which Player 5 is moving. All existing nodes are connected only to Node 1. Node 1 is the only dominant node in this case, so Player 5 can connect to any combination of nodes as long as he connects to Node 1. In this case there are seven such combinations, but due to Proposition DNR 4, Player 5 will only ever choose one of two moves: he will connect to Node 1 only, or he will connect to all existing nodes. The reduction in the set of potential moves can be even greater for later nodes.

6.7.1 Interpretation of Proposition DNR 4

This result has a number of interesting consequence. It guarantees that the “rich get richer” in the sense that only the most connected nodes can ever become relatively more connected than another node. Every time a player chooses Move B every previous node gains a connection. Every time a Player chooses a Type A Move some subset of the dominant nodes gain connections. If Player i gains a connection when Player j does not, Player i must be a dominant node, because whenever nodes are being selective about who they connect to, they will want to connect to dominant nodes only. Connecting to dominant nodes provides the greatest myopic benefit and provides the only method of manipulating future play.

Given this result it is also easy to see how early mover advantage in payoffs can easily arise, although it is not guaranteed. A Player i who chooses to take a Type A Move by making one connection will

[27]Unless $k = i$ in which case it does not eliminate any moves.

gain $\beta - C + \beta \sum_{j=i+1}^{J}(\mathbf{h}_j = \{1, 2, 3, ..., j-1\})$. In other words this player makes their myopic payoff plus β for every future player making a B move. An earlier player will have more Move B's following their move, meaning they will be better off. Also note that by optimality, players who choose to make Move B will always do better in expectation than a player in the same position choosing a Type A Move. As such, the early mover advantage of Type A players implies that Players choosing Move B will also generally have an early mover advantage in the sens that there is a lower bound on the payoffs of Move B players which is higher near the start of the network.

We will say more about early move advantage when we discuss an example in Section 6.8.

6.7.2 Importance and Limitations of Markov Perfection

Markov perfection prevents certain types of unusual sunspot equilibria where players condition the way that they resolve indifferences on payoff irrelevant features of the game history. Consider an example where Player 6 will never connect to Player 5 unless Player 4 connects only to Player 1 and Player 2 (an alternative move that is neither a Type A Move nor Move B). If Player 4 makes that specific move, Player 6 will always connect to node 5 if node 5 is red. Player 6 is indifferent, so he can freely condition his moves this way. Other Players will always resolve indifferences in favor of connecting to Player 1. Assume $C = \beta + \epsilon$.

Most moves Player 4 can choose will lead to no future connections. However, if Player 4 makes the specified move, he can change the incentives of Player 5 such that is is now in Player 5's interest to choose Move B. This means that Player 4 will get an extra future connection from this alternative move relative to all other moves, which makes it worth the extra cost relative to just connecting to one dominant node. This unusual move does not influence the payoffs of any move for Players 5 and 6. A move that is payoff irrelevant can change the behavior of future players by changing the way that other players even further down the line resolve indifferences.

It may seem unusual that Type A Moves involve connecting to a subset of dominant nodes rather than a single dominant node. The reason for this is two fold. First, Markov Perfection does not prevent sunspots which depend on the set of dominant nodes. This is due to the fact that under the definition used in Maskin and Tirole (2001), the partition of histories on which players may condition their moves can be no coarser than the partition which determines the set of available moves to the player. In this game the set of dominant nodes maps one to one with the set of available moves. [28]

Even if we eliminate all sunspots, however, there can still be unusual combination of tie-breaking behaviors which lead to a non-singleton Type A Moves being optimal. An example is provided in the Appendix (Section 10.1). The example is fairly complex, but the basic idea is that sometimes it is a good idea to keep more dominant nodes in the set of dominant nodes to prevent a particular node from vying for dominance, since that node's vying for dominance discourages several nodes from vying for dominance in the future.

6.8 Exponential Slowdown and Novelty Seeking Tie-Breaking Rules

We will now use Proposition DNR 4 in order to pin down the exact critical times in which nodes should vie for dominance, In this game, even Markov Perfection is not enough to give us uniqueness. How players resolve indifferences can have a substantial impact on the network that forms even within the class of Markov Perfect tie-breaking strategies.For example, if all players break indifferences in favor of connecting to older nodes (a Markov Perfect tie-breaking rule), the only possible network that can form is the star network centered on Node 1.[29] The dynamics of this network are not particularly interesting. Dynamics are more interesting when players break ties at random (another Markov Perfect tie-breaking rule), but they are not particularly clean. We analyze instead what happens when players

[28] Eliminating this minimally coarse partition requirement is also not particularly satisfying, because doing so eliminates random tie-breaking. The equilibrium in which all players break ties in favor of connecting to the oldest dominant node is defined by the partition determined by the current oldest dominant node. That partition is coarser than the partition defined by the set of dominant nodes, which is the partition upon which players condition in in the random tie-breaking rule equilibrium.

[29] Note that this is not the case in the base game. See 6.3 for the counter-example

break ties in favor of newer nodes, because the dynamics of the resulting MPE's are both clean and non-trivia.

Definition: We define the novelty seeking tie-breaking rule as follows. When a player is indifferent between Type A moves, they will always choose the move with the lowest total age. Total age of a move is the sum of the ages of all nodes connected to in that move.

This tie-breaking rule generates a MPE of the game in which there are periods of Type A play separated by instances of Move B play, and number of Type A Moves between Move B's follows a predictable pattern.

Proposition DNR 5: If we assume novelty seeking tie-breaking $\beta < C$, and we ignore the integer constraints of node indexing, then in the unique solution to the DNR Game, the time between Move B's grows exponentially.

Proof of Proposition DNR 5: We begin by characterizing the equilibrium in a manner that is slightly stricter that the equilibria described in Proposition MPE 1. We already satisfy the conditions of Proposition MPE 1, because if players always break ties according to the NSTB, then their moves depend only on the set of optimal moves, which, as we showed in the proof of Proposition MPE 1, is enough to guarantee a MPE of this game.

Lemma DNR 5.1- Under the NSTB approach all players will choose Move B or connect to the node in $\mathbf{d}(G_{t-1})$ that joined the network most recently which we denote to b^*_{t-1} (Move A'), and which they pick does not depend on the current network state.

Proof of Lemma DNR 5.1: As before, we show this by induction.

Player J: As we saw in the proof of Proposition MPE 1, player J will connect to a single $b_{J-1} \in \mathbf{d}(G_{J-1})$. Furthermore, player J will be indifferent between all members of $\mathbf{d}(G_{J-1})$, so by our assumption of novelty seeking tie-breaking strategies, player J will choose move A'. By connecting to any element of $\mathbf{d}(G_{J-1})$, Player J will be a distance 1 from the node he connects to and a distance 2 from everyone else leading to a payoff of $B - C$.

Player $J - k$: We begin by defining a induction conjecture.

Conjecture DNR 5.1- All future nodes will choose Move B or Move A' and this decision does not depend on the current network.

Now we want to show that, given Conjecture DNR 5.1, Player $J - k$ will choose Move B or Move A', and this decision does not depend on the current network. We do this by showing that it is weakly optimal for player $J - k$ to choose move A' or strictly optimal for player $J - k$ to choose move B, and which possibility is true does not depend on the current network. Showing this will guarantee our result, since when move A' is weakly optimal, it will always be selected by novelty seeking tie-breaking.

Under Conjecture DNR 5.1, a player will get a fixed number of future connections for any move that does not put him in $\mathbf{d}(G_{J-k})$ since all future B movers will connect to him and all future A' movers will not. It is immediate that if node j is not in $\mathbf{d}(G_j)$ he will never be in $\mathbf{d}(G_t)$, since the dominant nodes are always connected to all other nodes, and Player j cannot become connected to the nodes he is currently not connected to. We have also established that the only move that will result in node j being part of $\mathbf{d}(G_j)$ is move B.

By the DNR a player must connect to at least one $b_{t-1} \in \mathbf{d}(G_{t-1})$. Adding any connection to a move will always make the move worse as long as that change does not make that move into move B, because doing so increases the connection cost by C and decreases network loss by only B without changing future moves. As such the best move must always be connecting to a single b_{t-1} or move B. Since the set of future connections is fixed for all non-B moves and move A' involves connecting to a single b_{t-1}, move A' is always tied for maximum utility for all non-B moves. All that remains is to show that whether move B is strictly better does not depend on the current network. Since we have already shown that the payoff for move A' does not depend on the network, we must now show that move B payoffs are similarly fixed.

Since the connection cost of move B is already fixed and the network loss from previous nodes is as well we must show that the set of future direct connections to $J - k$ under move B does not depend on the network. By Conjecture DNR 5.1, we know that the order of future A' and B moves is predetermined. Consider the player after $J-k$ if he uses move B. If the next player is a Move A' player, they will connect to node $J-k$ by the novelty seeking preferences assumption.

If player $J-k$ chooses move B every move A' player will connect to player $J-k$ until another player uses move B. We will use this feature again in the second half of this proof. After that, all move B players will connect to $J-k$ and all move A' players will not (since they will be connecting to the most recent Move B player instead). The sequence of future connections if player $J-k$ uses move B, then, does not depend on the current network.

By induction all players will use move A' or move B and this choice does not depend on the network. □

Exponential Growth: Having characterized the equilibrium in general terms, we now determine which players will choose move A' and which will choose move B.

Consider J sufficiently large as to ensure that all the node indices referenced are positive.

As before we work through backwards induction.

Player J: Begin by considering the move of Player J. His expected utility from move A' is $-C-\beta-2\beta(J-2)$, because he will be directly connected to a single node in $\mathbf{d}(G_{J-1})$ and two jumps from all other nodes. Player J's expected utility from move B is $-(J-1)C-(J-1)\beta$ which is always, because $C>\beta$.

Player $J-k$: We now consider Player $J-k$ such all nodes after $J-k$ will choose move A'. By Lemma DNR 5.1 we do not have to worry about Player $J-k$ influencing the choice of future players. If Player $J-k$ chooses Move B, Player $J-k+1$ will directly connect to Node $J-k$ since Node $J-k$ will be the newest node in $\mathbf{d}(G_J-k)$. All nodes after $J-k+1$ will connect to node $J-k$, because by then $J-k$ will be the only node in $\mathbf{d}(G_t)$.

Player $J-k$ will receive of $-C-\beta-(J-2)2\beta$ from move A' and $-(J-k-1)C-(J-k-1)\beta-k\beta$ from move B. In both cases the first term (of the form $...C$) represents the costs of connections. The second term (of the form $...\beta$) is the myopic gain from connections. The last term (also of the form $...\beta$) represents the gain from future connections.

Player $J-k$ will then choose move B if

$$(J-2)(1-\beta/C)<k$$

If this relation is not satisfied when $k=J-3$, then no nodes will select move B, and the resulting network will be a star on Node 2. We call the first node that satisfies the condition for move B optimality v_1. Call the set of all nodes choosing move B, $\mathbf{v}$.

Player v_j-k: Consider now a Player v_j-k who moves k moves before the next move B (v_j).

This time we will write the choice in terms of the gains and costs of Move B relative to Move A' to reduce extraneous terms. The extra cost (net gains from the immediate connections) is: $(v_j-k-2)(C-\beta)=\gamma_{v_j-k}$.

The relative benefit of Move B from future direct connections will be $(k-1)\beta=\rho_{v_j-k}$. Player v_j-k will choose move B if the gains are greater than the costs. As the time until the next platform increases and ρ_i increases, and as we move backwards towards the first move, γ_i decreases. This means that, given a fixed next Move B player, Player v_j, we can find the previous Move B player by going backwards until we find a node such that $\gamma_i<\rho_i$.

Define k_j as the lowest value of k such that $\gamma_{v_j-k_j}<\rho_{v_j-k_j}$, and then say $v_j-k_j=v_{j+1}$. See Figure 7 for a visualization of this relationship.

Now we ignore the integer constraints on node indices and say that a node will choose move B when the costs equal the benefits. In other words, we assume v_j satisfies$\gamma_{v_j}=\beta_{v_j}\forall j$. We can see the kind of errors that are introduced by this approximation by comparing the dotted line and the dashed line in Figure 7. The dotted line shows where the relative costs and benefits are equal, while the dashed line shows gives the actual index of Player v_{j+1}. The difference between the dashed line and the dotted line will always be less than one. We discuss the issue of approximation in more detail in the next section.

Recall that $\gamma_{v_j}=(C-\beta)v_j$ and $\rho_{v_j}=(k_{j-1}-1)\beta$ by construction.

Node v_{j+1} will have k_j fewer nodes to connect to for move B, so

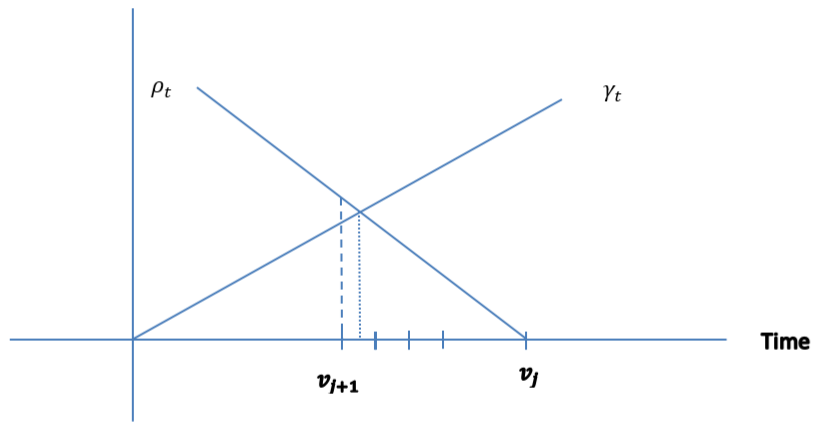

Figure 7: This figure demonstrates how, given v_j we may find v_{j+1} by going backwards in time until we find a node where optimality is satisfied.

$$\gamma_{j+1} = \gamma_j - (C - \beta) * k_j$$

Since $\rho_{v_j} = \gamma_{v_j}$ for each v_j this gives us

$$\rho_{j+1} = \rho_j - (C - \beta) * k_j$$

Which, after substitution and simplification gives us

$$k_j = k_{j-1}(\beta/C)$$

Combine this with the condition for node v_1 derived above, $k_1 = (J-2)(1-\beta/C)$, to get

$$k_j = (J-2)(1-\beta/C) * (\beta/C)^{j-1}$$

Recall that $\beta/C < 1$, so as j increases, k_j shrinks exponentially. $\square$

Figure 8 shows graphically how this geometric relationship arises.

6.9 Quality of Approximation

It is important to determine just how large the errors introduced by our decision to ignore integer constraints might be. Define the true Move B players between the true $\bar{v}_j$ and the approximate v_j derived by ignoring integer node indexing constraints will always be less than one. We also define $\bar{k}_j$ by $\bar{v}_j = \bar{v}_{j-1} - \bar{k}_j$ Begin by noting that $\bar{v}_1$ will be less than 1 away from the approximate v_1, because $v_1 = J - (J-2)\frac{1-\beta}{C}$, and $\bar{v}_1 = J - roundup((J-2)\frac{1-\beta}{C}, 1)$(see Figure 7 for a visualization of the relationship between the true value and the approximation).

In other words$|\bar{v}_1 - v_1| < 1$. In addition, note that $v_1 \geq \bar{v}$

Now take as given a true value $\bar{v}_j$ with the approximation v_j and assume $v_j \geq \bar{v}_j$. Then we have $(k_{j+1} - 1)\beta = (v_j - k_{j+1} - 2)(C - \beta)$ or $k_{j+1} = (v_j - 2)(1 - \frac{\beta}{C}) + \frac{\beta}{C}$. Similarly, $\bar{k}_{j+1} = roundup((\bar{v}_j - 2)(1 - \frac{\beta}{C}) + \frac{\beta}{C}, 1)$. As such we must have $\bar{k}_{j+1} \leq k_{j+1} + 1$.

In addition, because $v_{j+1} = v_j - k_{j+1}$ and $\bar{v}_{j+1} = \bar{v}_j - \bar{k}_{j+1}$ we have $v_{j+1} = v_j - (v_j - 2)(1 - \frac{\beta}{C}) + \frac{\beta}{C}$ and $\bar{v}_{j+1} = \bar{v}_j - roundup((\bar{v}_j - 2)(1 - \frac{\beta}{C}) + \frac{\beta}{C}, 1)$ which combined with $v_j \geq \bar{v}_j$ implies $v_{j+1} \geq \bar{v}_{j+1}$

These two results imply that $|\bar{v}_j - v_j| < 1 + |\bar{v}_{j-1} - v_{j-1}|$ by induction.

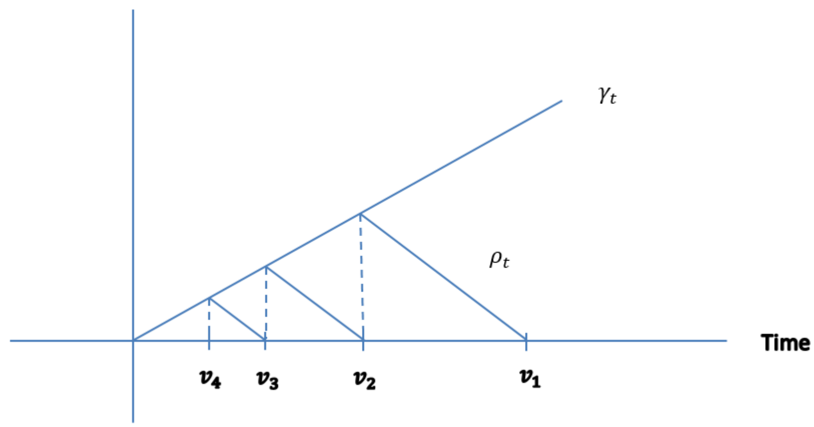

Figure 8: This figure shows how the behavior of the costs and benefits of move B determines when such moves occur.

This combined with the result from v_1 gives us $|v_j - \tilde{v}_j| < j$ which means that the result of Proposition NSTB 1 is strong in the later stages of the game, but it may be an inaccurate description of what happens near the beginning of the game.

7 Conclusion

In this paper we explore how entry timing can influence the structure of a network, and in particular, which nodes become central. To this end, we introduce and analyze a model of network growth with history dependence and strategic agents. The model combines a number of features which are endemic to real world systems in a novel way. The unique combination of features opens up a wide range of interesting dynamics and behaviors including "vying for dominance" in which a player invests heavily in connections, facing a myopic loss, in order to try to supplant the incumbent dominant node and receive connections from future players.

For this new game we proved several results analogous to those in the traditional network formation literature.We also showed that the strategic nature of the game creates a parameter region where the simple results do not hold. In this region, results are often intractable. To deal with this problem, without eliminating the interesting dynamics of the game, we proposed the dominant node restricted model. Under the DNR model, we proved that Players will always connect to a a subset of the dominant nodes or connect to all nodes in hopes of becoming dominant themselves. We examined the solution to the DNR defined by novelty seeking tie-breaking and found that the number of players between those who vie for dominance increases exponentially as the game progresses. As the network grows, it becomes more expensive to vie for dominance, and so the rewards for each new vying player must grow as well.

Some related work has been done which was not included in the main body of the text. We covered several extensions to the both the base game and the DNR game in the Appendix Sections 8 and 9 looking at what happens when certain key assumptions of the model are loosened. We examine how results change when we do not require players to make a connection, connections have heterogeneous costs, players may own multiple nodes, or when we loosen the rigid move order in the game.

There remains a great deal of work to be done, however. We still have very little understanding of how SPE's of the base game function for large networks in the intermediate parameter region. In particular, there is more work to be done exploring the complex strategic interactions that go beyond vying for dominance as seen in Section6.3.In addition, there is a great deal more to explore in the variants of this game. Continual games with nodes being constantly added to the network seem to be an excellent approach to modeling many real world systems.

Finally, in its current form, this game cannot be easily applied to many real world settings, because the size of the existing networks would make the game intractable for those systems. Simplifications and modifications are needed before this model of network growth can be applied to specific real world settings.

Part III
Appendices

8 Extensions and Generalizations of the Base Game

In this section we cover a number of extensions to the base game and their consequences. In general, the major qualitative nature of the results is fairly robust to generalizations, but the details of solutions to the game can often change a great deal. We are only going to cover the effects of the generalizations on SPE results, because the impacts of the generalizations on the NE results are more complicated and go beyond the scope of this paper.

8.1 No Connection Requirement

Many readers are likely curious about what happens when we do not require players to make and connections. The answer looks fairly similar to what we see with the base game, although there are some added wrinkles.

Proposition A 1:

- If $\frac{C}{B} < 2(1-\delta)$, then the complete network is the most efficient equilibrium.
- If $2 + (J-2)\delta > \frac{C}{B} > 2(1-\delta)$, then the star network is the most efficient network.
- If $\frac{C}{B} > 2 + (J-2)\delta$, then the empty network is the most connected network

Proof of Proposition A 2:

First, consider the case when $\frac{C}{B} < 2(1-\delta)$. Following the same logic as the proof of Proposition EF, it must be the case that the complete network is the most efficient network.

Recall from the proof of Proposition EF that the most efficient connected network is the star when $\frac{C}{B} > 2(1-\delta)$. We can apply the same logic to show that if $\frac{C}{B} > 2(1-\delta)$, then the efficiency of any network can be weakly improved by converting every connected component of the network into a star network. Let $V(G)$ be a vector of length J containing the sizes of each connected component of G in descending order with zeroes for when there are no more connected components. The total welfare generated by a subset of size k is

$$f(k) = (k-1)(2B - C + (k-2)\delta B)$$

.

We are trying find $V(G)$ to maximize the sum $\sum_{i=1}^{J} f(V_i)$, subject to the constraint that $V_i(G)$ is a whole number and $\sum_{i=1}^{J} V_i(G) = J$.

Note that $f(k)$s is strictly convex in k and has a positive derivative as long as $2B-(2k-3)\delta B > C$. Since $2B - (2k-3)\delta B >$ is increasing in k, and $2B - (2k-3)\delta B) > C$ will always hold when $k = J$ fail to hold when $k = 0$ it must be that those are the two local optima of total surplus. Geometrically, $f(k)$ is U-shaped, so the maximum must be on one of the ends.

The total welfare from a completely disconnected network is 0, so we just need to check whether $(J-1)(2B - C + (J-2)\delta B) > 0$ or $2 + (J-2)\delta > \frac{C}{B}$. Since we have assumed that $2 + (J-2)\delta > \frac{C}{B}$ we have the result. $\square$

Proposition A 2: If $\frac{C}{B} < (1-\delta)$ then the complete network is the only network which can be formed in SPE's of the game.

Proof of Proposition A 2: Identical to the proof of Proposition SPE 1.

Proposition A 3: If $\frac{C}{B} > B - C + (J-2)\delta B$, then the empty network is the only network which can be formed in SPE's of the game.

Proof of A 3: Consider Player J. The most points that Player J can from a single connection is $B - C + (J-2)\delta B$ which he can receive by making one connection to a node connected directly to all other nodes. When C exceeds this amount, Player J will never make a connection regardless of network shape. Knowing that Player J will not connect, Player $J-1$ can make no more than $B - C + (J-3)\delta B$ points from making a move with a non-empty set of connections. Player $J-1$ will then not make any connections.

Knowing that all future players will make no connections, Player $J-k$ will make at most $B - C + (J-k-2)\delta B$ points by making a connection. Therefore by induction, no player will make any connections. □

We also get an entirely new result related to connected components of the solution network.

Proposition A 4: The second largest connected component (a connected component of a graph is a subgraph such that all nodes in the subgraph are connected by paths to all other nodes in the subgraph, and any node which is connected to a node in the subgraph is in the subgraph) of a network formed by a SPE must contain fewer than k nodes where k is the lowest integer such that. $1 + \sum_{i=1}^{k-1} \delta^{roundup(i/2)} > \frac{C}{B}$

Proof of Proposition A 4: Consider Player J, who only cares about the myopic benefits of his move. When Player J connects to a connected component nodes the minimum benefit he can gain comes when the subset of nodes are connected in a chain. In that case Player J makes at least $B + B\sum_{i=1}^{k-1} \delta^{roundup(i/2)}$ from connecting. If this value is greater than C, then such a connection will be made.

As such, any connected component with k nodes or more will be connected to node J which means it is impossible for the game to conclude with more than one such subset. □

Corollary A 4.1: When $\frac{C}{B} < 1$ only connected networks can be formed with positive probability be SPE's of the game

Proof of Corollary A 4.1: When $\frac{C}{B} < 1$, by Proposition A3, the second largest connected component must contain fewer than one nodes. Therefore, the second largest connected component must be empty, so the graph must be connected. □

8.2 Heterogeneity and Joint Node Ownership

In general, heterogeneity in the benefits of connection does not have a substantial impact on the results. If we replace payoffs in the base game with

$$u_i = Y - C \cdot |\mathbf{h}_i| + \sum_{j \neq i} B_{ij} \delta^{d_{ij}(G_J)-1}$$

Then we still have the results of Propositions 1, 2, and 3 as long as the B_{ij}'s all satisfy the restrictions on B in the base game.

Another natural generalization for the base game involves allowing a single agent to control multiple nodes. There is little in general said about this type of modification other than the fact it introduces heterogeneity of benefits in that when a player makes a connection to one of their earlier nodes, they get twice the benefit. Proposition 1 still holds if we allow for joint node ownership. If the effective B_{ij}'s satisfy the requirements for B in Propositions2 and 3, then those propositions still hold. The logic behind the proofs remains the same.

8.3 Multiple Moves

There is another question which cannot be addressed quite so easily: what happens if we relax the restriction that each player only moves as they join the network? Let's say that each Player i who has already joined the network has a p_{ij} chance of getting the opportunity to move again after each new Player j joins the network. Moving again means that a player has the opportunity to make a connection to any node in the network that they are not currently connected to. If multiple players are moving again, they do so in random order. The random chance and ordering are drawn and revealed immediately after Player j's move. Proposition EF still hold since the payoffs have not changed and the set of feasible networks has not changed. Proposition SPE 2 also holds by the same logic as before. No player wants to make any extra conditions when costs are so high that they could not possibly benefit from doing so.

Proposition SPE 1 is slightly more complicated. If the move order was deterministic (with $p_{ij} \in 0,1$), it would still hold, since each player would always want to make every connection that no later player will be able to make. If some Player i knows that Player j will have an opportunity to make connection (i, j) later in the game, they will hold off, knowing that Player j will make the connection when the time comes. If we allow for randomness, however, the proposition can fall apart. Consider what happens when $\frac{C}{B} = 1 - \delta - \epsilon$ and there is some Player i with $1 > p_{iJ} > 0$. Assume that all other p's are zero. In this case if ϵ is sufficiently small and p_{iJ} is sufficiently high, Player J will prefer to not connect to Player i. If Player J makes the connection, he receives a benefit of $B - C = \epsilon$ from that connection. If he does not connect, Player i may get to move, and in that case Player i will choose to connect to Player J. Therefore, Player J's expected benefit from not moving is $p_{iJ}B - \epsilon$, which can easily be greater than zero.

If Player J chooses not to connect to Node i and gets unlucky in this case, then the complete network will not form. The complete network will always have a positive chance of forming, however, since any player who knows for certain that they are the last player who can make a specific connection will always do so.

9 Extensions Generalizations and Special Examples of the DNR Model

In this section we examine the extensions and generalizations as they are applied to the DNR model.

We again introduce heterogeneity of benefits by replacing the base game payoff function with the one given in Section 8.2. Say $\beta_{ij} = B_{ij}/(1 - \delta)$. Heterogeneity does much the same in the DNR model as it did in the Base model. Propositions DNR 1, DNR 2, and DNR 3 still hold if all of the β_{ij}'s satisfy the requirements for β. There is, however, one other result which does change slightly. Proposition DNR 4 holds but we must modify our definition of Move A. We define a modified Move A for Player i as connecting only to a subset of the dominant nodes in G_{i-1} plus connecting to any non-dominant nodes such that $\beta_{ij} < 1$. Move A players are still just essentially making a myopic move, but now a myopic move can involve multiple connections.

Allowing for multiple moves and joint node ownership effects the main results for the DNR in the same way it effects the main results of the base game. It is notable that allowing for multiple moves can potentially decrease the gains from vying for dominance, because can potentially allow for more nodes to compete with a newly dominant node. Allowing for multiple moves does not change the results when players use the Novelty Seeking Tie-Breaking Rule, however.

10 Counter Examples

In this appendix we provide a number of worked counter-examples which illustrate several points from the main body of the text.

10.1 Abusing Multiple Dominant Nodes Example

This example shows why, even in MPE's with no sunspot tie-breaking, players sometimes connect to non-singleton subsets of $\mathbf{d}(G_{t-1})$. In this example we have $J = 8, B = 1, C = 1.19$. Player 4 will be optimally choosing to connect to two elements of $\mathbf{d}(G_{t-1})$. Ties are broken in the following manner:

- Player 8 favors Player 6 over Player 7 and Player 7 over everyone else, Otherwise ties are broken at random
- Player 7 breaks ties randomly but never favors node 5

- Player 6 favors Player 5 and otherwise is random
- Player 5 breaks ties randomly
- Player 4 never breaks ties in favor of Player 3
- All other indifferences are resolved randomly, although this is largely unimportant

We solve the game by backwards induction, giving each player's strategy and then a proof of why is is optimal given the strategies of later players.

Player 8 will always chooses Move A connecting to one node

Proof: Always true of Player J

Player 7 will choose move B unless Node 6 is dominant. If node 6 is dominant, he will choose Move A connecting to a single node. Choice of singleton Move A is payoff irrelevant for Player 7.

Proof: It is profitable for Player 7 to choose move B relative to move A when Node 6 is not dominant if $7\beta - 6C > B - C$, which is true since $(J-2)/(J-3) = 6/5 > C/\beta = 1.19$. Move B is always bad when Player 6 is dominant, since Player 7 will never get a connection from Player 8 in that case. We can see that the choice of singleton Move A is payoff irrelevant, because given any such move, Player 8 will not connect to Player 7.

Player 6 will choose Move B if the number of dominant nodes other than node 5 is less than two. Otherwise he will choose a singleton Move A. Again choice of singleton Move A is payoff irrelevant

Proof: If Player 6 becomes dominant then both Player 7 and Player 8 will connect to the same dominant node chosen at random, but never to Node 5 by tie-breaking. Therefore the expected gain from choosing Move B when k nodes other than Node 5 are dominant would be $4\beta + 2\beta/(k+1) - 4C$. Note that $4\beta + 2\beta/(k+1) - 4C > 0$ when $1 + 1/(2k+2) > C/\beta$. Given our parameters this is equivalent to $k < 1.63$. As such, Move B is profitable when $k = 1$ but unprofitable when $k = 2$. Note that if Player 6 picks single connection Move A, Players 7 and 8 will always pick Move B and single connection Move A respectively meaning that Player 7's choice of singleton Move A is not payoff relevant.

Player 5 will become dominant if the number of dominant nodes is greater than or equal to 2. Otherwise he will choose a singleton Move A.

Proof: To see this note that if there are at least two nodes that are dominant, then when Player 5 picks move B, Player 6 will choose Move A and connect to Node 5. Player 7 will choose Move B and receive a connection from Player 8.

If Player 5 were to choose Move A instead, then Player 6 would choose Move B and then Player 7 and 8 would choose Move A. Move B is then preferred if $6\beta - 4C > 2\beta - C$ or $4/3 > \frac{C}{B}$, which it is for the given parameters.

If the number of dominant nodes is less than 2 then Player 5 will never get a future connection after choosing Move B, so move B is never optimal, and he must choose a singleton Move A.

Player 4 will connect to two dominant nodes if there are at least two dominant nodes (Move A^*),. Otherwise, Player 4 connects to a single dominant node.

Proof: If there are two or more dominant nodes and Player 4 chooses Move B then the resulting moves will take the pattern BABA, with neither Move A player connecting to Player 4. This is the same pattern the future moves take if Player 4 connects to only two dominant nodes, so connecting to two nodes is preferred. If Player 4 connects to one dominant node the future moves will follow the pattern ABAA. Player 4 will then make two connections if $4\beta - 2C > 2\beta - C$ or $2 > C/\beta$ which it is for the given parameters.

If there are not two dominant nodes then all moves lead to the future move pattern ABAA, so connecting to a single dominant node is optimal.

Player 3 Player 3 makes 2 connections

Proof: If Player 3 makes two connections the future move pattern is ABABA, whereas if Player 3 makes one connection, the future move pattern is AABAA. Player 3 will never receive a connection from any of the Move A Players . Since $C/\beta < 2$, it is optimal for Player 3.

So in the end we see the following move pattern (BB)BA^*BAB. The first two Players do not actually make choices, but they do end up as dominant nodes, so we consider them to be using move B. The special A^* move is when Player 4 makes two connections in order to prevent the pattern

(BB)BAABAA, which would generate less payoff. Essentially, Player 4 picks between the following two move sequences and chooses the latter

Player	1	2	3	4	5	6	7	8
Sequence 1	B	B	B	A	A	B	A	A
Sequence 2	B	B	B	A^*	B	A	B	A

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