Stackelberg Routing in Atomic Network Games

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Joint work with José Correa (UAI, Chile)

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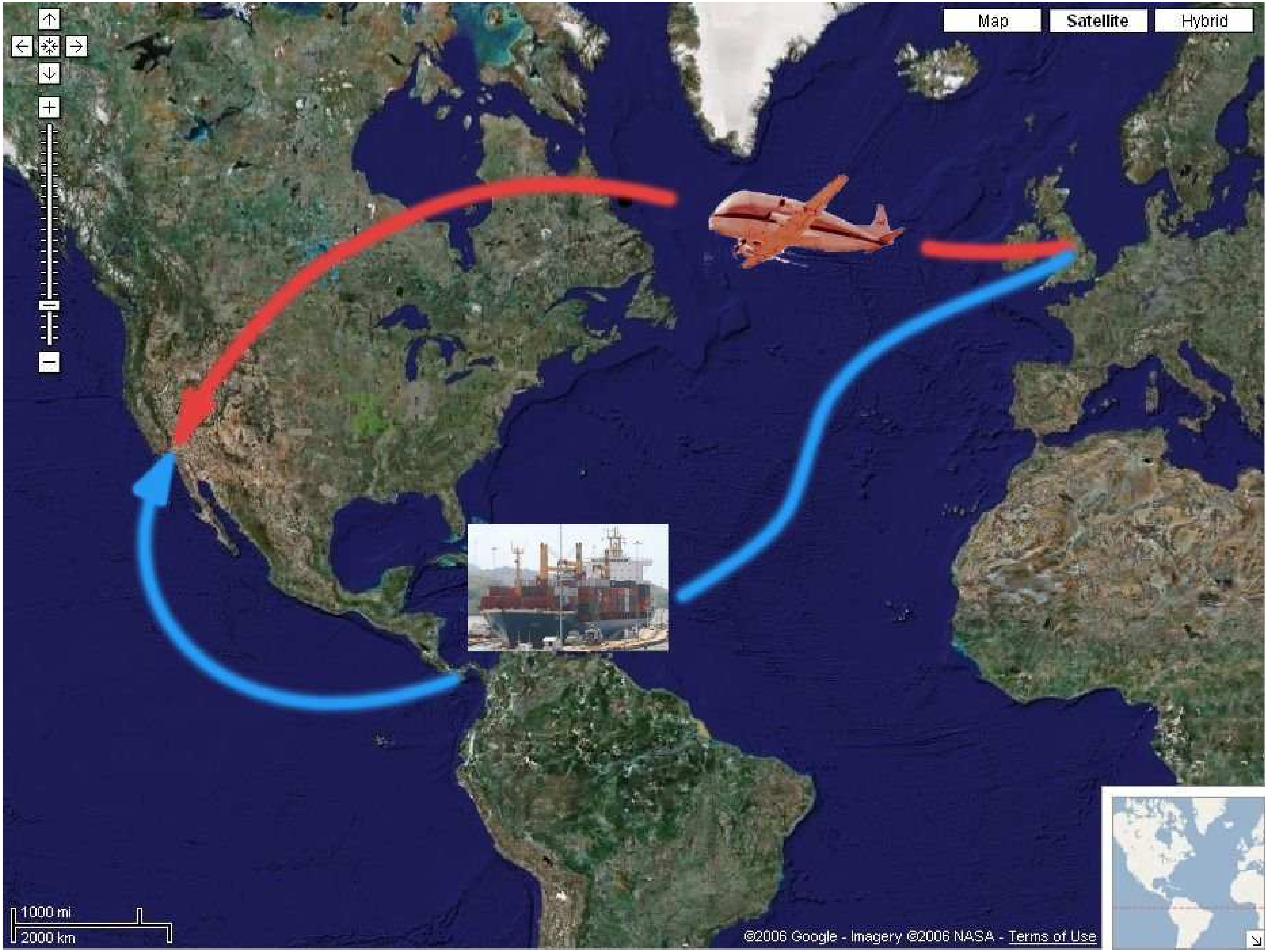

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Network Games

- Agents select routes to ship flow from origin to destination
- Agents want to maximize their utilities
- Utilities depend on others' decisions
- Most common solution concept: Nash equilibrium (**NE**)

All agents play best responses to others' decisions

- But for certain market structures, **NE** not a good model

Stackelberg Games

von Stackelberg'34

- Two types of agents: leader and follower
- If followers do not have market power, they cannot **individually** influence utility of leader
- Leader can **predict** followers' reactions instead of playing a best response to followers
- It's a **Stackelberg game**: leader optimizes subject to equilibrium constraints in 2^{nd} stage

This talk: How efficient is the system?

Outline

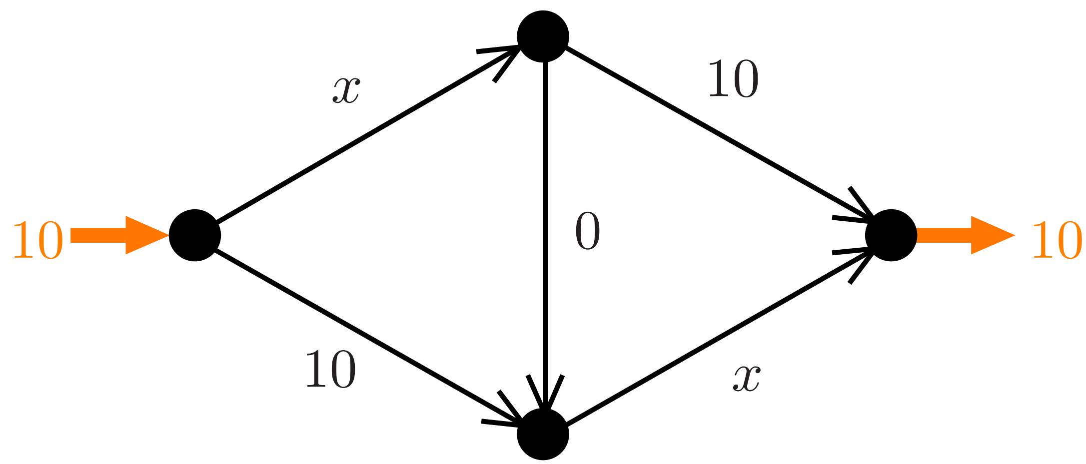
- Network Games and their Social Optima, Nash Equilibria, and Stackelberg Equilibria

- Strategies for the Leader

- Bounds on the Price of Anarchy

The Competitive Model

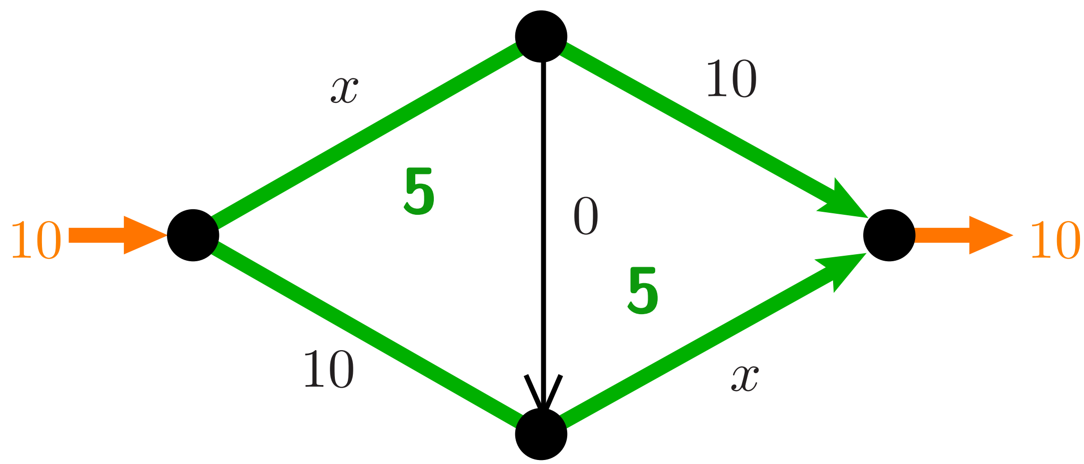
- Network $G = (N, A)$ with a fixed demand between pair (s, t)
- Resources have cost functions c_a depending on total demand x_a
- Cost on a path is $c_P(x) := \sum_{a \in P} c_a(x_a)$
- Example: Braess' Instance



Social Optimum

- Total Cost $C(x) := \sum_{a \in A} c_a(x_a)x_a$

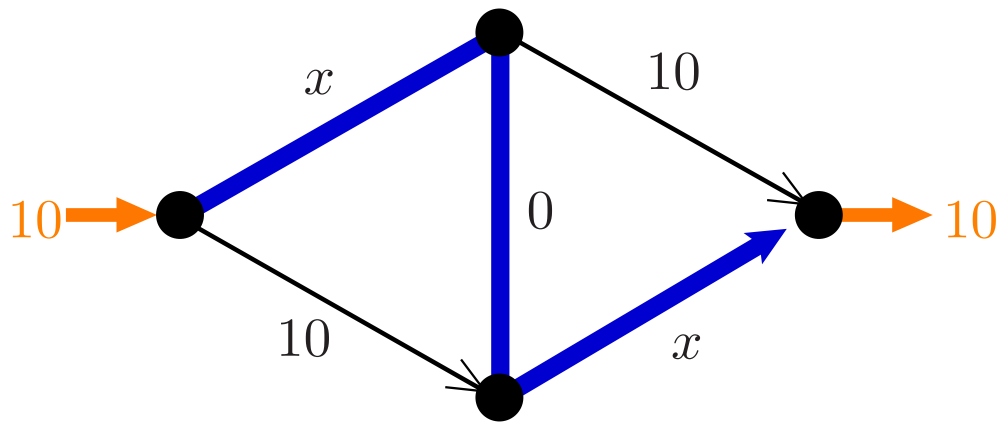
- A **SO** is a feasible flow x^{SO} that minimizes $C(\cdot)$



Nash Equilibrium

- Consider nonatomic players

- A **NE** is a flow x^{NE} such that nobody can switch to a path with smaller travel time — Wardrop'52



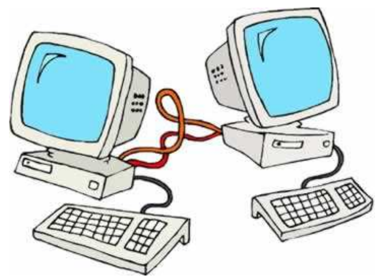
- **NE** characterized by a Variational Inequality: — Smith'79

$$\sum_{a \in A} c_a(x_a^{\mathsf{NE}}) x_a^{\mathsf{NE}} \leq \sum_{a \in A} c_a(x_a^{\mathsf{NE}}) x_a \text{ for all } x$$ — Dafermos'80

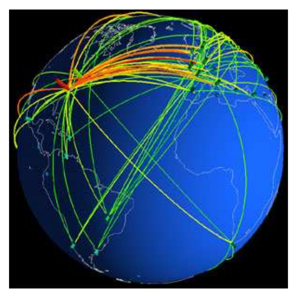
Market Structure

Two types of players:

a. One leader with market power that controls $d\%$ of flow

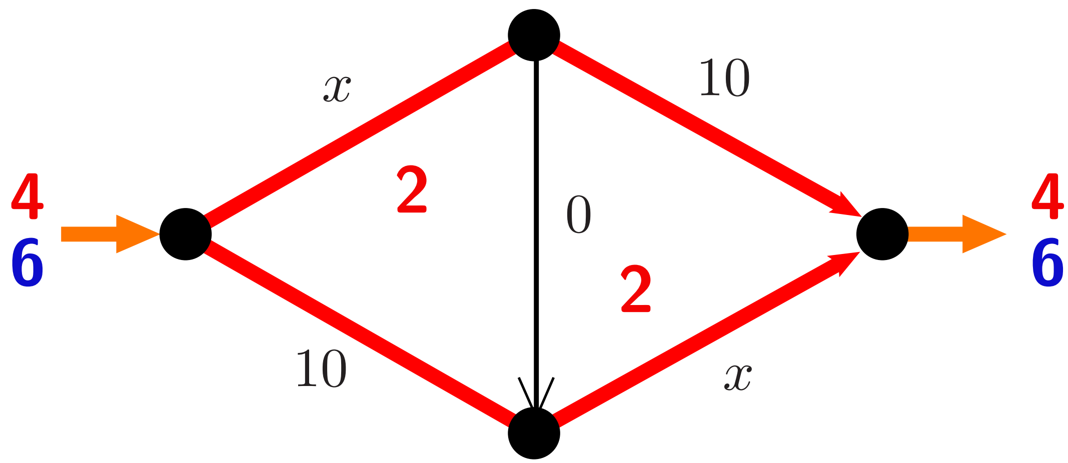




b. Followers without market power that control $(1 - d)\%$ of flow




Stackelberg Equilibrium

- Leader plays x, and followers collectively play y, which is a **NE** after x is loaded in the network



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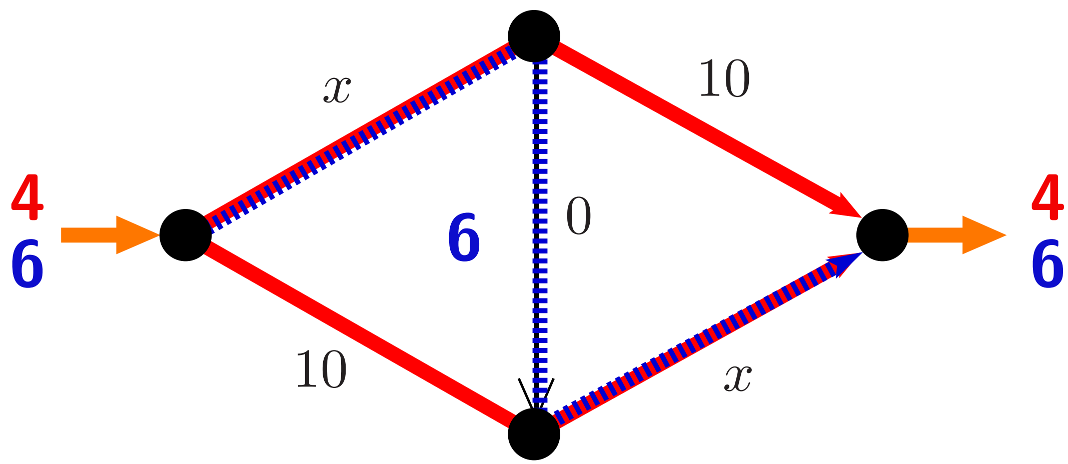


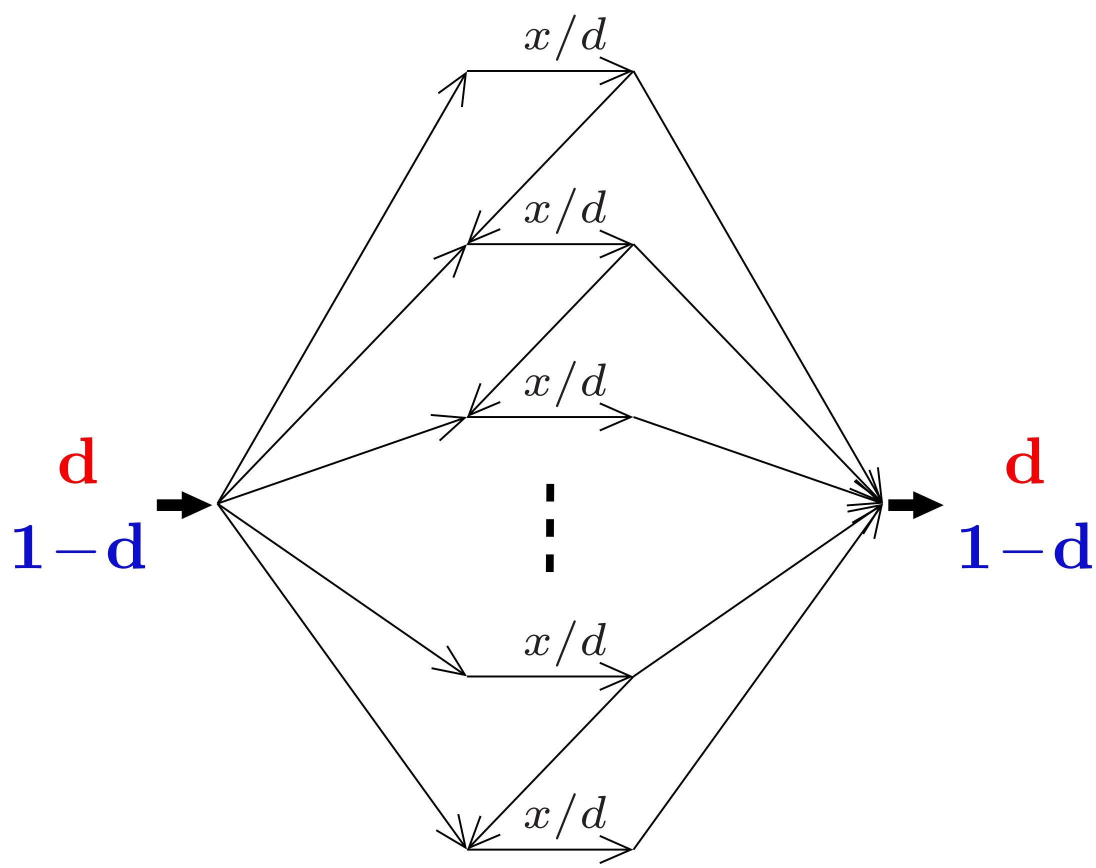
- Definition: a **SE** is a pair (x, y) that minimizes the leader's cost

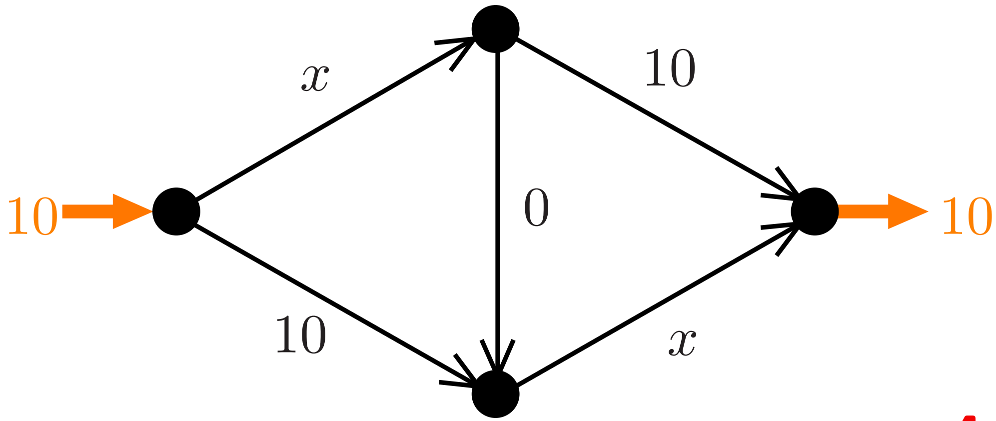
$$C^x(x+y) \;:=\; \sum_{a\in A} c_a(x_a+y_a)x_a$$

- Total Cost is $C(x+y) = \sum_{a\in A} c_a(x_a+y_a)(x_a+y_a)$

Strategies for Leader

Leader should be careful to not hurt itself (and followers)

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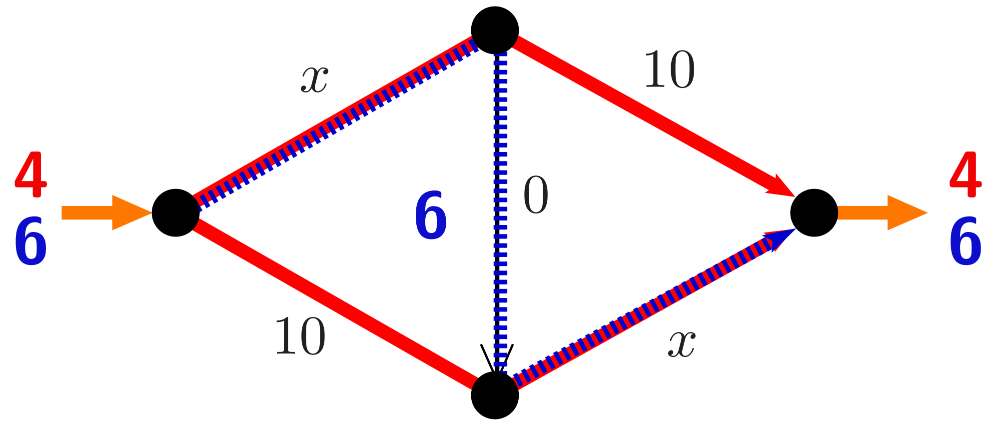
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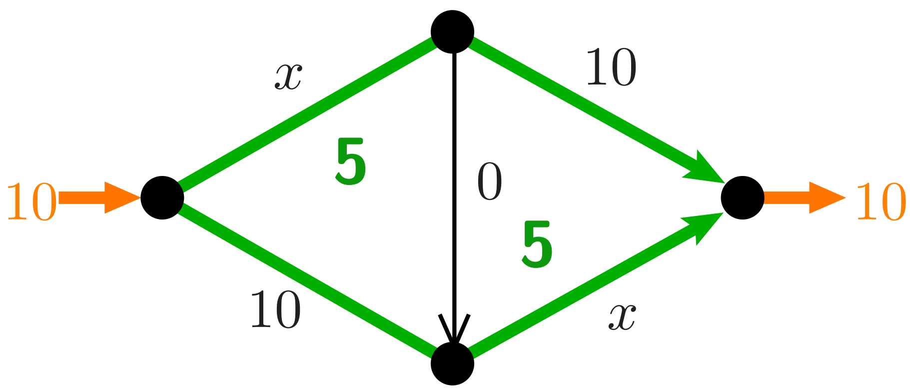
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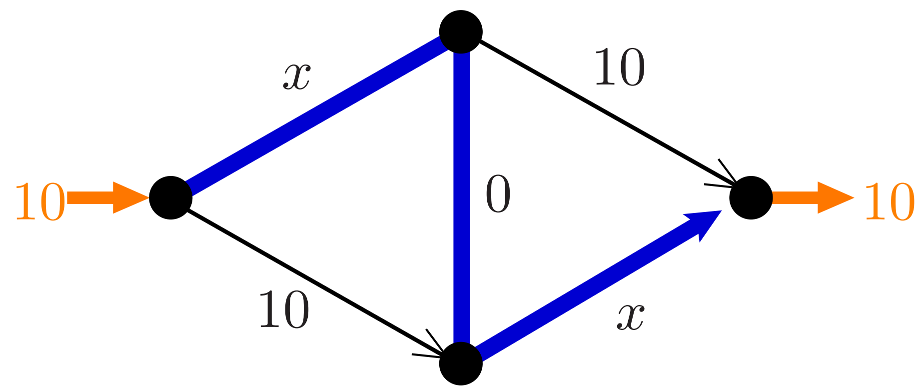
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Computing a SE

- Leader's problem is an MPEC (math program with equilibrium constraints)
- Many algorithms to solve MPECs exist — Luo et al.'96
- Leader's problem hard. Do heuristics work? — Roughgarden'04

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  Scale : $x = d\, x^{\mathsf{SO}}$

  LP : $x = \arg\max \left\{ \sum_{a \in A} z_a c_a(x_a^{\mathsf{SO}}) : z_a \leq x_a^{\mathsf{SO}} \text{ and } z \text{ feasible} \right\}$

- Both are "reasonable", i.e., satisfy $x \leq x^{\mathsf{SO}}$

Price of Anarchy

Price of Anarchy measures impact of lack of central coordination

Papadimitriou STOC'01

$$\textbf{POA} := \max_{\text{instances}} \frac{C(\textbf{SE})}{C(\textbf{SO})}$$

- For unrestricted cost functions, **POA** is unbounded
- We will assume a fixed set of cost functions $\mathcal{C}$, e.g., affine

Bounds on the Resulting Game

(see also Roughgarden'04, Karakostas and Kolliopoulos'06, and Swamy'07)

Theorem. For $x \leq x^{\mathsf{SO}}$ and polynomials of maximum degree p, the cost is bounded by $C(\mathbf{SO})$ times:

degree $\longrightarrow$	0	1	2	3	4	$\ldots$	p
$C^x(\mathbf{SE})$	1	1	1.185	1.688	2.621	$\ldots$	$\Omega(2^p)$
$C(\mathbf{SE})$	1	4/3	1.626	1.896	2.151	$\ldots$	$\Omega(p/\ln p)$

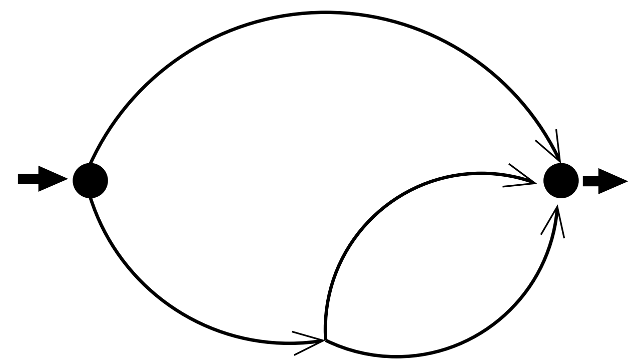
"Reasonable" strategies guarantee that
leader and system objectives are partially aligned

Price of Anarchy parametrized by d

Previous slide independent of d. What about explicit dependence?

Theorem. In series-parallel networks,
if leader uses strategy **LP**,

$$C(\textbf{SE}) \le (1 + \tfrac{1}{d})\, C(\textbf{SO})$$



– Swamy'07 proved this result independently

– Improves on $1/d$ for parallel-link networks by Roughgarden'04
(a similar argument also proves the $1/d$ result in a simpler way)

Final Remarks

Leader can exploit its position in market

Look for additional insights for:

- Is the leader better off by having market power?
- More general market structures
- Other definitions of social cost