Network Games with Atomic Players

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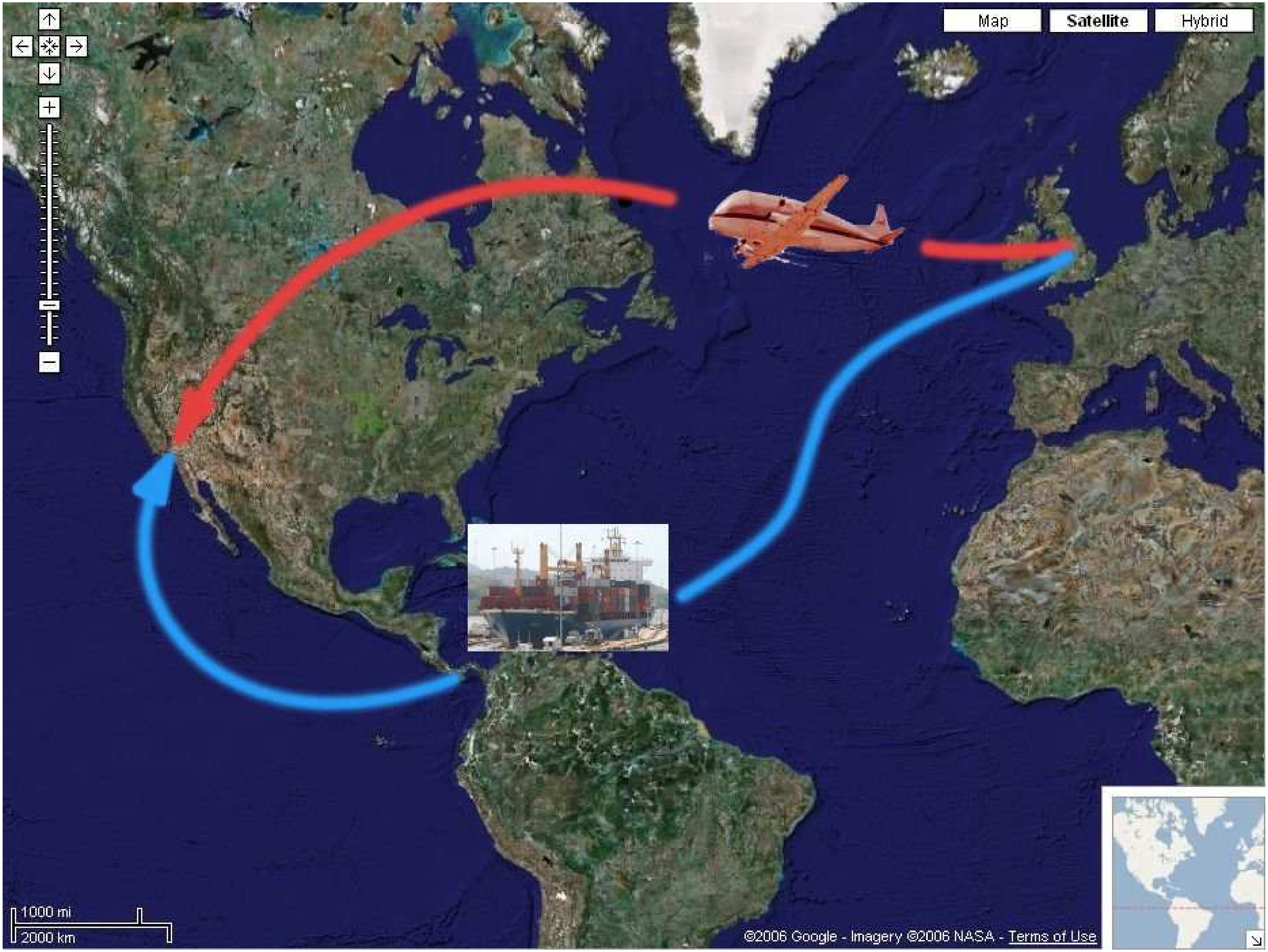
Bell Labs, Lucent Technologies, June 8, 2006



Map
Satellite
Hybrid
1000 mi
2000 km
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Competitors make selfish decisions

How efficient is the system?

regulated	**competitors**
global perspective	selfish
social optimum = min total cost s.t. solution is feasible	**Nash equilibrium** = $\forall$ user: min individual cost s.t. solution is feasible
efficient, unfair	not efficient, no regret

Pigou'20, Knight'24, Wardrop'52, . . .





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HSBC


Past Approaches

Centralized Optimization : leader dictates decisions for everybody

– Not realistic in many settings,
  but can give useful bounds on efficiency

Mechanism Design : create mechanisms s.t. equilibria are optimal

– **Toll Pricing**: charge users tolls equal to externalities.
  When users pay for externalities they create, system is optimal

Computational Studies : compare equilibria and optima for a fixed group of instances

– Provides good insight if instances are representative.
  No guarantee that same will happen for other instances

Price of Anarchy

- Ideas and techniques: in the interface of Economics, Computer Science and ORMS
- Use worst-case analysis to measure efficiency of equilibria
- Compare equilibria to good upper bounds on social surplus

$\longrightarrow$ Usually social optimum

POA=worst-case performance ratio of equilibrium to optimum

POA measures the "price" of not having central coordination in system

Price of Anarchy: Consequences

If **POA** small, designer may want to let users choose:
there is not much to gain from dictating what people should do!

If **POA** large, designer may want to re-design system or to give incentives to achieve more efficient results:

- Mechanism Design

  e.g., if certain design guidelines are used, then equil. $\approx$ opt.

- Pricing and Stackelberg games

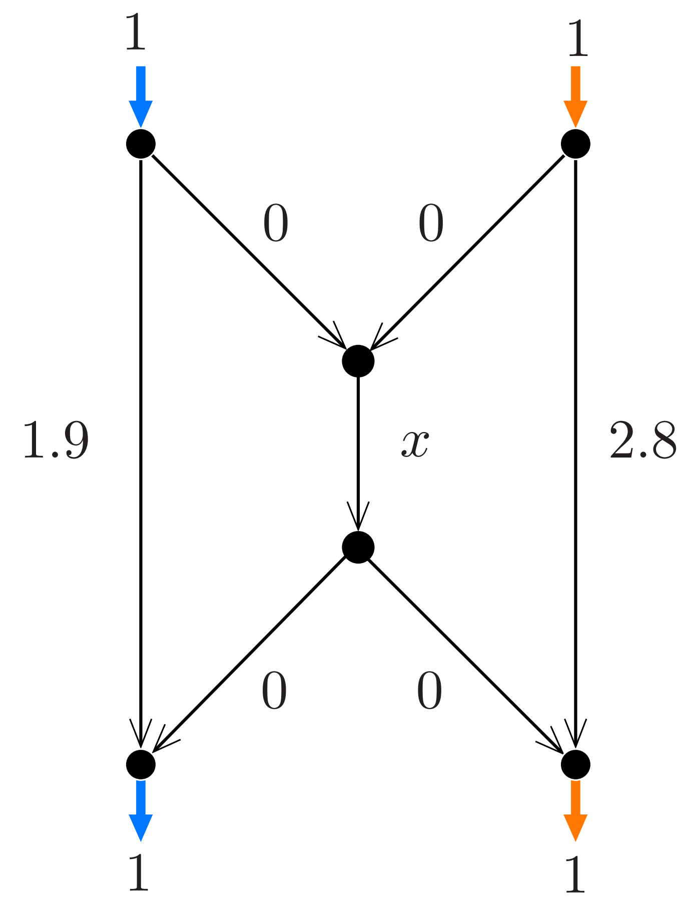
  e.g., designer charges tolls such that equil. $=$ opt.

Outline

- Warm-up: Nonatomic Games,
  Wardrop Equilibrium & System Optimum
- Atomic Games: Nash Equilibrium
- Bounds depending on Market Power
- Symmetric Games

Definition of General Network Games

- Network $G = (N, A)$
- Nondecreasing continuous (and usually convex) cost functions
$$c_a : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \text{ for } a \in A$$
- Set of origin-destination pairs, each with an associated demand



The Nonatomic Case: Wardrop Equilibrium

Definition: In a **WE**, every infinitesimal user minimizes its own cost Wardrop'52

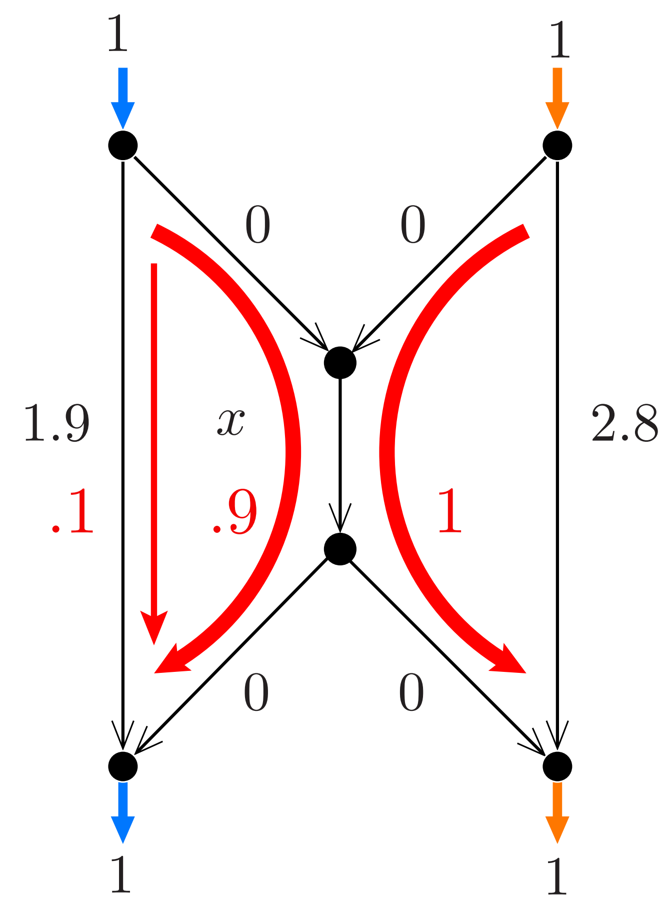
- **WE** minimizes $\sum_{a \in A} \int_0^{x_a} c_a(\tau) d\tau$

  $\Rightarrow$ it exists and is "unique"

  Beckmann et al.'56

- Equivalently, it solves the variational inequality Smith'79, Dafermos'80

  $\sum_{a \in A} c_a(x_a^{\text{WE}})(x_a - x_a^{\text{WE}}) \geq 0$ for all x

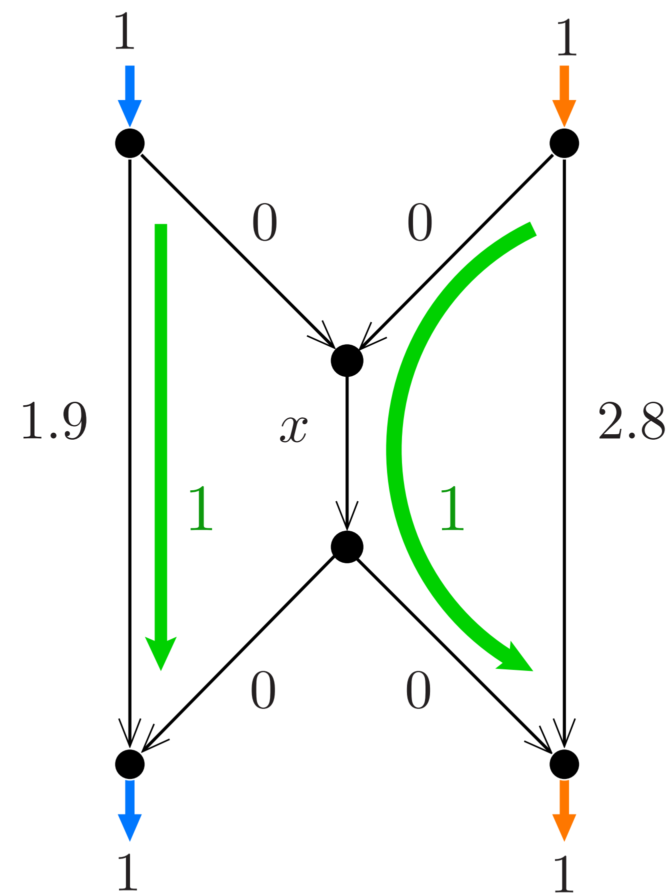


System Optimum

- Social welfare is given by total cost:

$$C(x) \;:=\; \sum_{a \in A} c_a(x_a)x_a$$

- A **SO** flow x^{SO} is a feasible flow that minimizes the total cost $C(x)$



Price of Anarchy

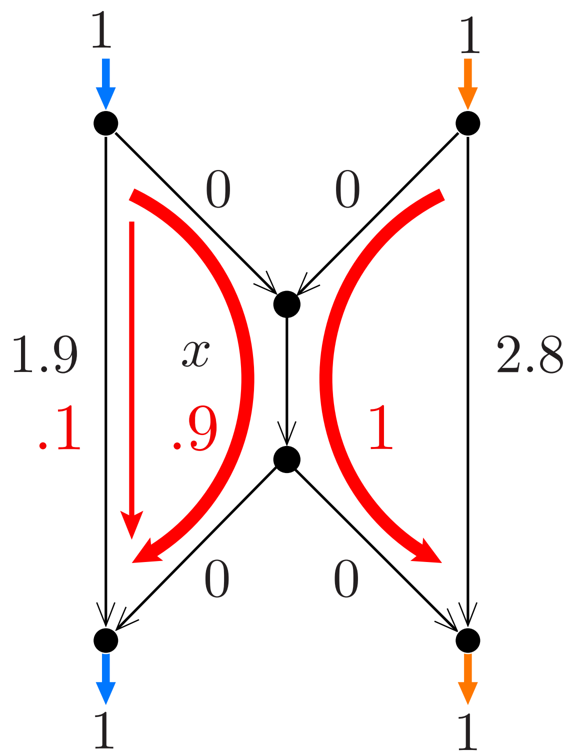
Price of Anarchy measures impact of lack of central coordination

Papadimitriou STOC'01

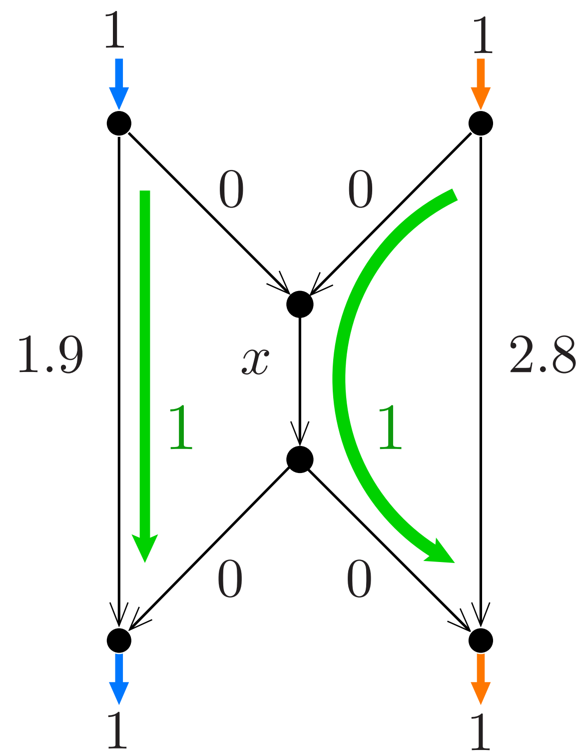
$$\textbf{POA} := \max_{\text{instances}} \frac{C(\textbf{WE})}{C(\textbf{SO})}$$

- For unrestricted cost functions, **POA** is unbounded
- We will assume a fixed set of cost functions $\mathcal{C}$
  e.g., affine

Inefficiency of WE for the Example



$C(\textbf{WE}) = 3.8$



$C(\textbf{SO}) = 2.9$

Therefore, **POA** is $3.8/2.9 = 1.31\ldots$

Price of Anarchy — Affine Costs

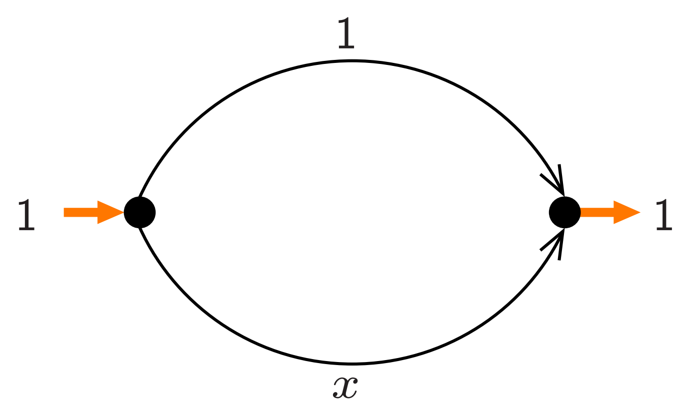
Theorem. Roughgarden & Tardos JACM'02

In networks with *affine* costs,

$$C(\textbf{WE}) \le \frac{4}{3} C(\textbf{SO})$$

Selfishness drives the system close to optimality

Remark. This bound is tight: $C(\textbf{WE}) = 1$ while $C(\textbf{SO}) = 3/4$

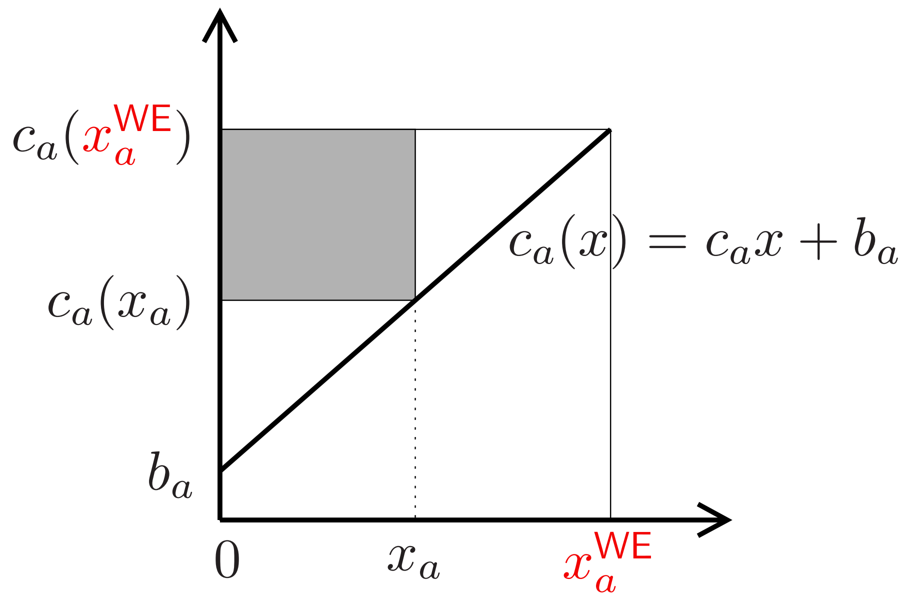


Proof of 4/3 Result

Correa, Schulz & S.M. IPCO'05

$$C(x^{\text{WE}}) = \sum c_a(x_a^{\text{WE}})x_a^{\text{WE}} \le \sum c_a(x_a^{\text{WE}})x_a \qquad (\leftarrow \textbf{VI})$$

$$= \sum c_a(x_a)x_a + \sum (c_a(x_a^{\text{WE}}) - c_a(x_a))x_a$$



$$\le C(x) + \tfrac{1}{4} \sum c_a(x_a^{\text{WE}})x_a^{\text{WE}} = C(x) + \tfrac{1}{4} C(x^{\text{WE}})$$

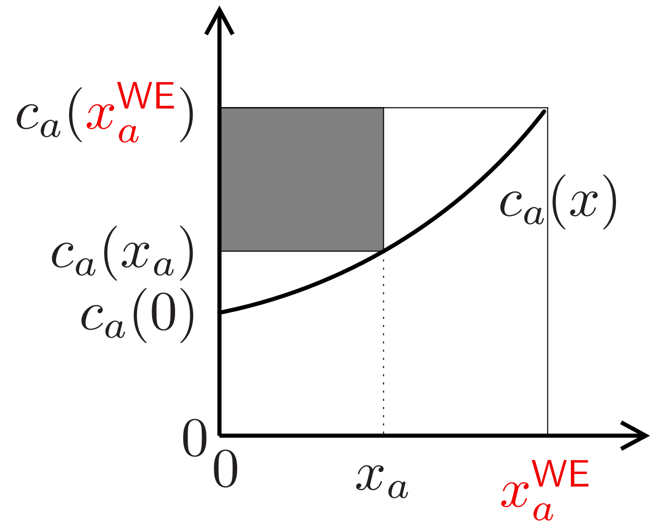
Price of Anarchy — General Costs

Roughgarden JCSS'03

Correa, Schulz & S.M. MOR'04

Let $\beta(\mathcal{C}) := \sup_{c \in \mathcal{C}} \left\{ \frac{\text{shaded area}}{\text{big rectangle}} \right\}$

$$\sup_{c \in \mathcal{C}} \sup_{0 \le y \le x} \frac{y(c(x) - c(y))}{x c(x)}$$



Theorem. If costs are drawn from $\mathcal{C}$, then

$$C(\textbf{WE}) \le (1 - \beta(\mathcal{C}))^{-1}\ C(\textbf{SO})$$

$\longrightarrow$ Gives: 4/3 for affine, 1.626 for quadratic and 1.9 for cubic fn.

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Atomic Network Games with Splittable Flows

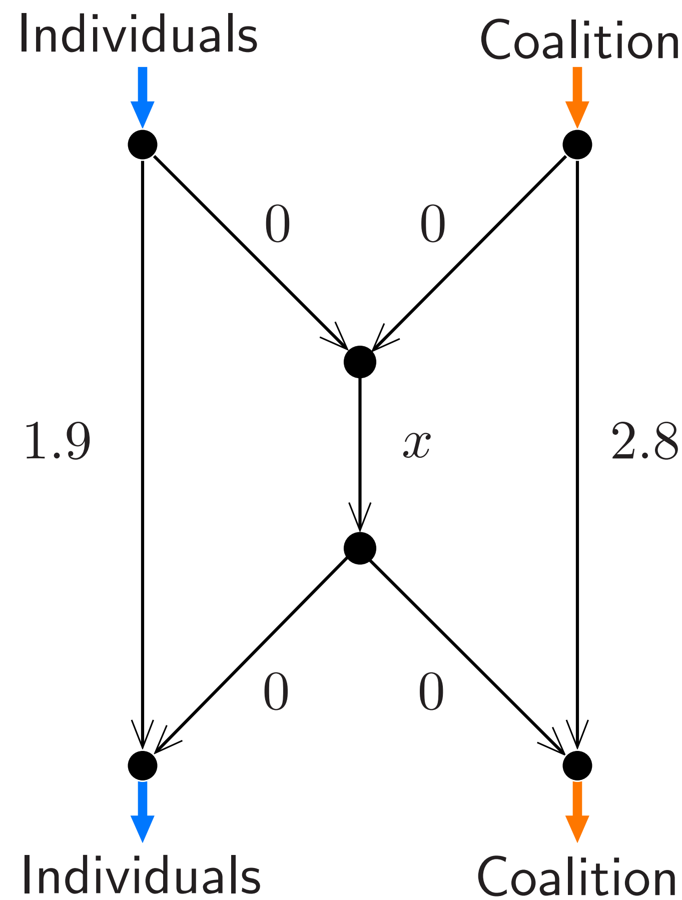
- What if coalitions are formed?

- User $k = 1, \ldots, K$ has to route d_k units of flow from s_k to t_k

- Users can divide flow along paths

- The cost that player k sees is:

$$C^k(x) \ := \ \sum_{a \in A} c_a(x_a) x_a^k$$



Some players cooperate and form a coalition

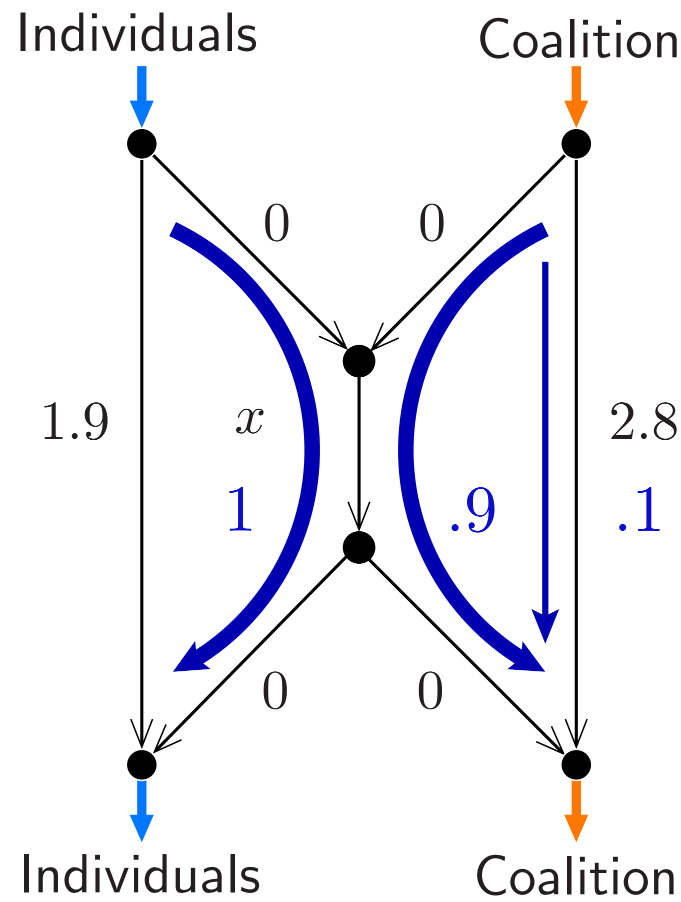
Nash Equilibrium

Definition: A flow $x^{\text{NE}} = (x^k)_{k \in [K]}$ is a **NE** of the network game if flows are optimal for each player when other flows are fixed:

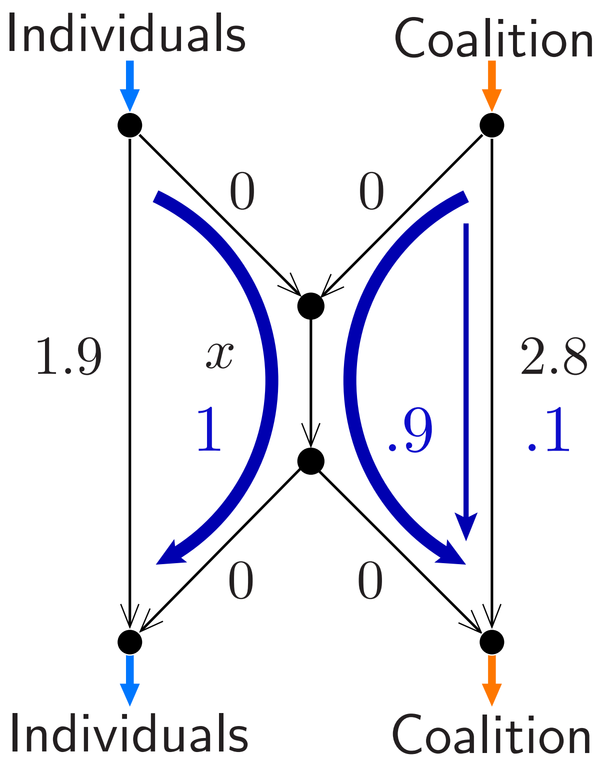
$$x^k \in \underset{\boldsymbol{x}}{\operatorname{argmin}}\, C^k(\boldsymbol{x} + x^{-k})\ \forall k \in [K]$$

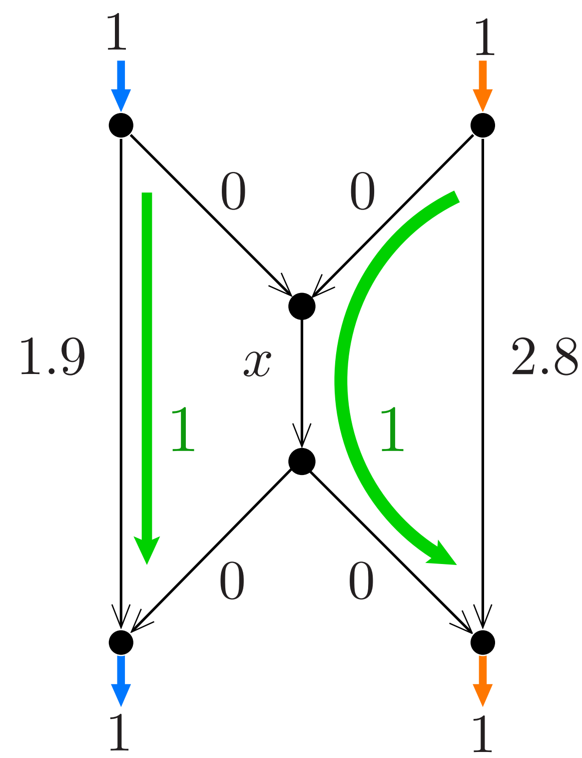
Nash'51

- **NE** exists — Rosen'65
- **NE** unique? — Orda et al.'93



Atomic Games can be Less Efficient





$C(\textbf{NE}) = 3.89$

$C(\textbf{SO}) = 2.9$

Therefore, **POA** is $1.341 > 4/3$

Price of Anarchy for Atomic Games

Theorem. If costs are drawn from $\mathcal{C}$, then

$$\boxed{C(\textbf{NE}) \leq (1 - \beta^\infty(\mathcal{C}))^{-1}\ C(\textbf{SO})}$$

$$\text{where } \beta^\infty(\mathcal{C}) := \sup_{c \in \mathcal{C}} \sup_{0 \leq y \leq x} \frac{y(c(x) - c(y)) + c'(x)y/4}{xc(x)}$$

- Gives: 3/2 for affine, 2.464 for quadratic and 7.826 for cubic fn.
- We don't know if tight but conjecture it's not
- Proof uses that **NE** solves the variational inequality $\sum_{a \in A} c_a^k(x_a^{\text{NE}})(y_a^k - x_a^k) \geq 0$ for any feasible flow y^k for player k

Proof Idea

Def. $\beta^K(c) := \sup_{\vec{x},\vec{y}\in\mathbb{R}_+^K} \frac{\sum_{k=1}^K \{(c^k(\vec{x}) - c(y))y^k + (c(x) - c^k(\vec{x}))x^k\}}{xc(x)}$,

and $\beta^K(\mathcal{C}) := \sup_{c\in\mathcal{C}} \beta^K(c)$

Let $\vec{x}^{\textbf{NE}}$ be a **NE**. Using the VI and the definition of β^K:

$$C(\textbf{NE}) = \sum_{a\in A}\sum_{k=1}^{K}\{(c_a(x_a^{\textbf{NE}}) - c_a^k(\vec{x}_a^{\textbf{NE}}))x_a^{\textbf{NE},k} + c_a^k(\vec{x}_a^{\textbf{NE}})x_a^{\textbf{NE},k}\}$$

$$\leq \sum_{a\in A}\sum_{k=1}^{K}\{(c_a(x_a^{\textbf{NE}}) - c_a^k(\vec{x}_a^{\textbf{NE}}))x_a^{\textbf{NE},k} + c_a^k(\vec{x}_a^{\textbf{NE}})\vec{x}_a^{\textbf{SO}}\}$$

$$\leq \beta^K(\mathcal{C})C(\textbf{NE}) + C(\textbf{SO})$$

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Single OD Pair — Variable Demands

We quantify market power variability by $\boldsymbol{M} := \sum_{k \in [K]} \left(\frac{d_k}{\sum_{j \in [K]} d_j} \right)^2$

$\boldsymbol{M}$, called the Herfindahl index, is $\begin{cases} 1 \text{ for a monopoly} \\ K^{-1} \text{ for symmetric players} \end{cases}$

Theorem. If we only consider instances with a single OD pair, the **POA** is at most

$$\left(1 - \sup_{c \in \mathcal{C}} \sup_{0 \le y \le x} \frac{y(c(x) - c(y)) + c'(x) y \boldsymbol{M}/4}{x c(x)} \right)^{-1}$$

Example: Oligopoly

Each player controls at most $\phi(D)$ (say $\ln(D)$) units of demand ($D = \sum d_k$), such that $\phi(D)/D \to 0$

Then, the **POA** approaches that of a nonatomic game

Indeed, $M = \sum_{k \in [K]} \left(\frac{d_k}{D}\right)^2 \leq \frac{D}{\phi(D)} \cdot \frac{\phi(D)^2}{D^2} \longrightarrow 0$

So, **POA** $\to \left(1 - \sup_{c \in \mathcal{C}} \sup_{0 \leq y \leq x} \frac{y(c(x) - c(y))}{xc(x)}\right)^{-1}$

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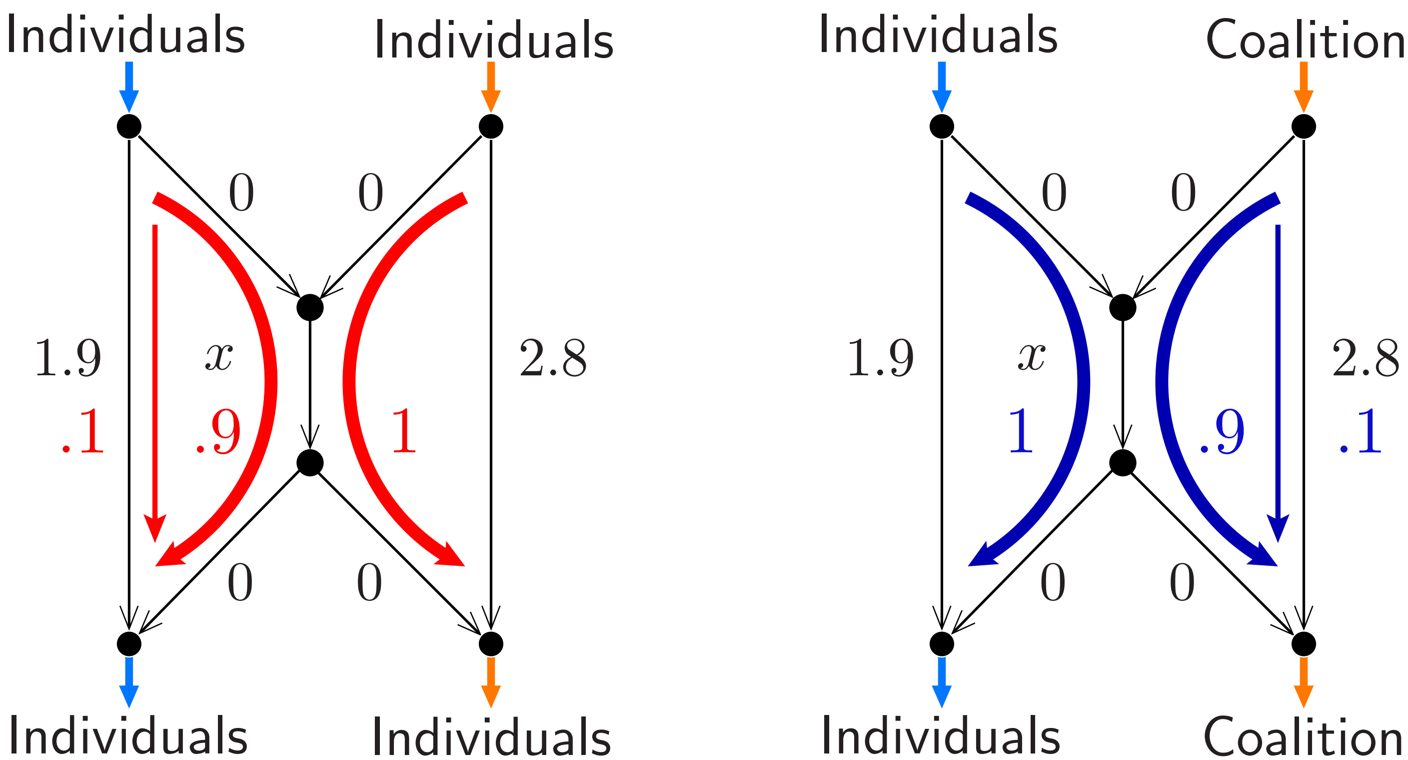
Paradox: More Concentration ⇒ Less Coordination

Catoni and Pallotino'91



$C^{\text{right}}(\textbf{WE}) = 1.9$ $\quad C^{\text{right}}(\textbf{NE}) = 1.99$

Coalition forms and does worse!

Single OD Pair — Symmetric Players

All players have to route $d_k = d$ from O to D

Let $\boldsymbol{x}$ be an optimal solution to the convex program:

$$\textbf{NE} = \min \quad \frac{1}{K}\overbrace{\sum_{a\in A} x_a c_a(x_a)}^{\text{SOCIAL COST}} + \frac{K-1}{K}\overbrace{\sum_{a\in A}\int_0^{x_a} c_a(\tau)d\tau}^{\text{WARDROP}}$$

s.t. flow x routes dK from O to D

Theorem. The **NE** is unique, and each player uses flow $\boldsymbol{x}/K$

Proof

KKT cond. say that $\boldsymbol{x}$ is optimal iff it is a feasible flow satisfying:

$$\begin{aligned} Kc_a(x_a) + x_a c_a'(x_a) =& \lambda_u - \lambda_v + \mu_a && \text{for all } a=(u,v)\in A \\ 0 =& \lambda_t - \lambda_s + \lambda_{(t,s)} && \\ \mu_a x_a =& 0,\ \mu_a \geq 0 && \text{for all } a \in A. \end{aligned}$$

Letting $x^k = x/K$, $\lambda^k = \lambda/K$ and $\mu^k = \mu/K$ and dividing all equations by K, we obtain that x^k is feasible for player k and it satisfies:

$$\begin{aligned} c_a(x_a) + x_a^k c_a'(x_a) =& \lambda_u^k - \lambda_v^k + \mu_a^k && \text{for all } a=(u,v)\in A \\ 0 =& \lambda_t^k - \lambda_s^k + \lambda_{(t,s)}^k && \\ \mu_a^k x_a^k =& 0,\ \mu^k \geq 0 && \text{for all } a \in A, \end{aligned}$$

which are the KKT conditions corresponding to $\min_{\boldsymbol{y}} C^k(\boldsymbol{y} + x^{-k})$.

Paradox Does Not Occur with Symmetric Players

Theorem. Let x^K be a **NE** in a game with K players who control d units of flow each; and let $x^{\tilde{K}}$ be a **NE** with $\tilde{K} < K$ players who control $dK/\tilde{K}$ units of flow each

$$\text{Then, } C(x^{\tilde{K}}) \leq C(x^K)$$

- Implies: $C(\textbf{NE}) \leq C(\textbf{WE})$
- **POA** of symmetric games $\leq$ **POA** of nonatomic games
- For affine fns. the bound can be improved to $\frac{4K^2}{(K+1)(3K-1)}$

Proof

The optimality of $x = x^K$ and $y = x^{\tilde{K}}$ in their problems says:

$$\sum_{a\in A} x_a c_a(x_a) + (K-1)\sum_{a\in A}\int_0^{x_a} c_a(\tau)d\tau \le \sum_{a\in A} y_a c_a(y_a) + (K-1)\sum_{a\in A}\int_0^{y_a} c_a(\tau)d\tau$$

$$\le \sum_{a\in A} x_a c_a(x_a) + (\tilde{K}-1)\sum_{a\in A}\int_0^{x_a} c_a(\tau)d\tau + (K-\tilde{K})\sum_{a\in A}\int_0^{y_a} c_a(\tau)d\tau$$

Thus, $\sum_{a\in A}\int_0^{x_a} c_a(\tau)d\tau \le \sum_{a\in A}\int_0^{y_a} c_a(\tau)d\tau$, which implies that

$$\sum_{a\in A} y_a c_a(y_a) + (\tilde{K}-1)\sum_{a\in A}\int_0^{y_a} c_a(\tau)d\tau \le \sum_{a\in A} x_a c_a(x_a) + (\tilde{K}-1)\sum_{a\in A}\int_0^{x_a} c_a(\tau)d\tau$$

$$< \sum_{a\in A} x_a c_a(x_a) + (\tilde{K}-1)\sum_{a\in A}\int_0^{y_a} c_a(\tau)d\tau$$

Final Remarks and Some Open Questions

- Atomic games are less efficient than nonatomic ones
  – There is a gap for the **POA**: between 1.343 and 3/2 (affine fn.)

- We can improve the bounds when market power has low variability
  – Can we have bounds parametrized by market power that work well in situations close to a monopoly?

- Symmetric games are “well-behaved” and as efficient as nonatomic
  – Uniqueness of NE? is it a potential game?

- All results valid for **Atomic Congestion Games**

- For a network of parallel links (single OD), the **POA** is at most that of the nonatomic game Hayrapetyan, Wexler, Tardos STOC'06

The End

Nonseparable Latency Functions

Costs usually depend on flow on other arcs: $\boxed{c_a(f) : \mathbb{R}^A \to \mathbb{R}}$

All works with $\beta(c, v) := \max_{x \in \mathbb{R}^A, x \geq 0} \frac{\langle c(v) - c(x), x \rangle}{\langle c(v), v \rangle}$

Side Constraints

- To capture real-world phenomena we can add side constraints to traffic network

  Charnes & Cooper '61
  Jorgensen '63, Hearn '80
  Larsson & Patriksson '95
  . . . , C., Schulz & Stier Moses MOR'04

- It is easy to generalize **SO** (add constraints to math. prog.)

- There are multiple equilibria and **POA is** ∞ even with linear fn.

- The **best NE** is hard to characterize but a good one can be found with same **VI**

- Then: $C($**best NE**$) \leq (1 - \beta(\mathcal{C}))^{-1} C($**SO**$)$ **Price of Stability**