Selfish routing in capacitated networks

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Outline

- The Network Model and the Price of Anarchy
- User Equilibria in Networks with Capacities
- A New Technique
- Performance Guarantees
- Concluding Remarks

The Goal

- Study performance of selfish traffic

- Price of Anarchy is the worst-case ratio of total travel time of selfish to coordinated. — Papadimitriou 2001

- For selfish traffic: use Game Theory — Wardrop 1952

  "users cannot do better by unilaterally changing their routes"

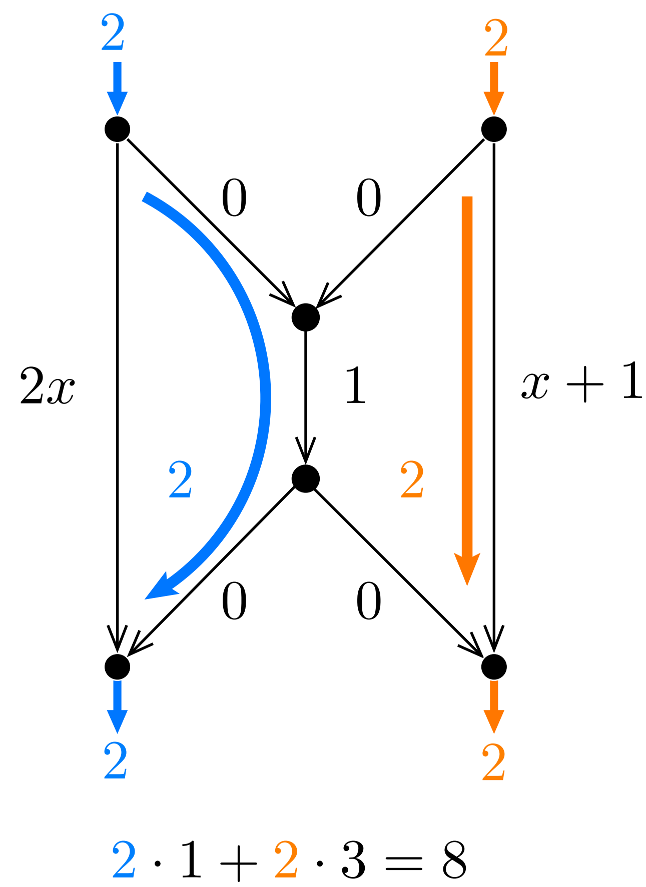
- For coordinated traffic: use Network Optimization

  "the total travel time is minimum"

The Traffic Model

- Directed graph $G = (V, A)$
- OD pairs (o_i, d_i) of rate r_i, $i = 1, \ldots, k$
- Flows on paths f_P. Can be non-integral.
- Traversal times: *latency functions* $t_a(\cdot)$
  - *continuous* and *nondecreasing*
  - belong to a given set $\mathcal{L}$ (e.g. polynomials)
- The total travel time of a flow f is:

$$C(f) := \sum_{a \in A} t_a(f_a) f_a$$



System Optimum

Nonlinear Multicommodity Min-Cost Flow Problem

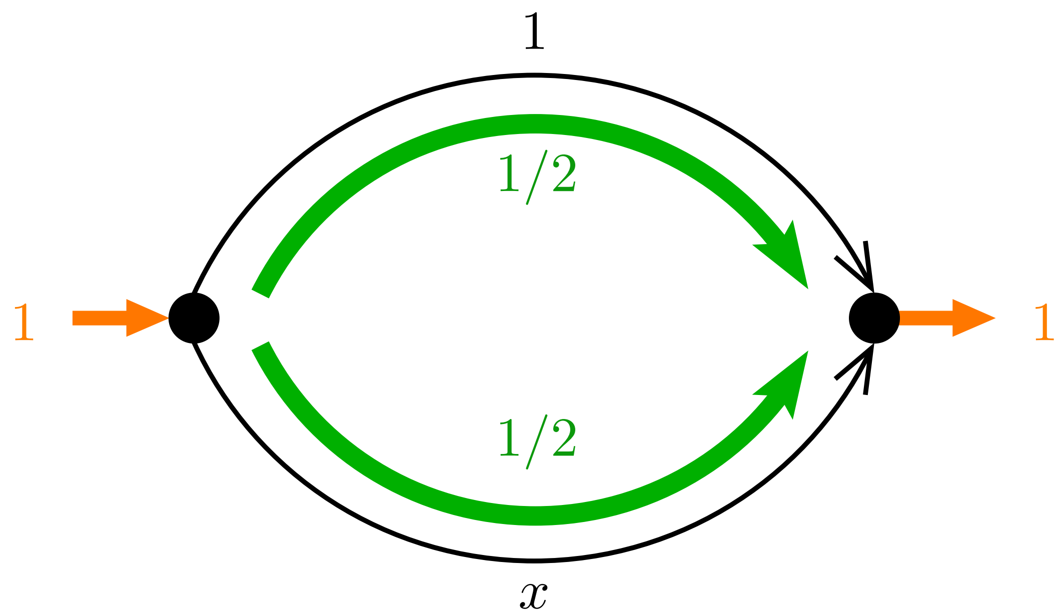
$$
\begin{aligned}
\min \quad & \sum_{a \in A} t_a(f_a) f_a & \\
\text{s.t.} \quad & \sum_{P \ni a} f_P = f_a & \text{for all } a \in A \\
& \sum_{P \in \mathcal{P}_i} f_P = r_i & \text{for all } i = 1, \ldots, k \\
& f_P \geq 0 & \text{for all } P \in \mathcal{P}
\end{aligned}
$$

where $\mathcal{P}_i :=$ set of paths from o_i to d_i

$\mathcal{P} := \cup \mathcal{P}_i$

Example of SO

Pigou 1920



$$\textbf{SO} = \min f_a + f_b^2 \qquad = \min f_b^2 + 1 - f_b = 3/4 \qquad \text{and} \qquad f_a = \tfrac{1}{2}$$

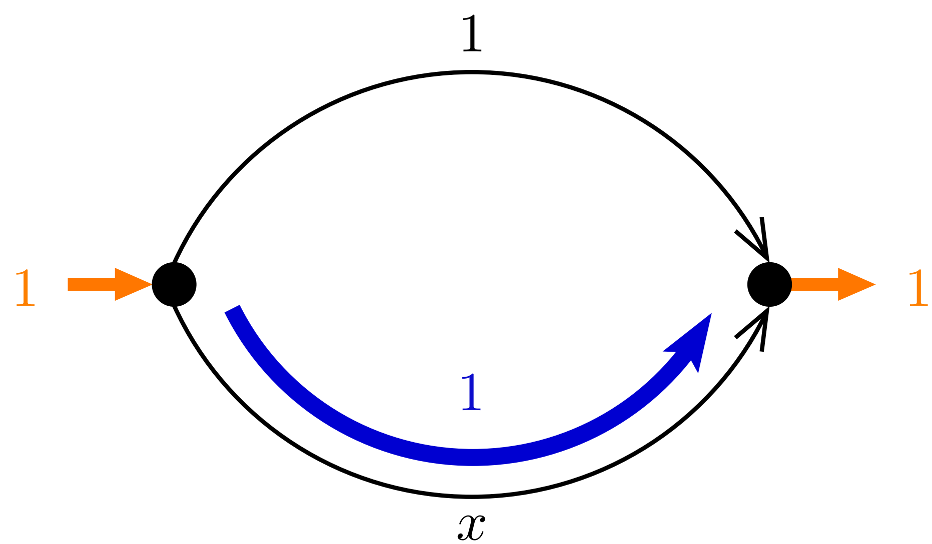
$$\text{s.t. } f_a + f_b = 1 \qquad \text{s.t. } 0 \le f_b \le 1 \qquad f_b = \tfrac{1}{2}$$

$$f_a, f_b \ge 0$$

User Equilibrium

Definition : A flow is a **UE** iff nobody can switch to a path with smaller travel time.

- Travel times of users between the same OD pair are equal
- **UE** always exists and is unique Beckmann et al. 1956



Price of Anarchy measures impact of lack of Central Coordination

$$\text{Price of Anarchy } \gamma := \max_{\text{inst.}} \frac{C(\mathbf{UE})}{C(\mathbf{SO})}$$

- In general, γ unbounded — Roughgarden & Tardos 2000
- If latencies are differentiable and in $\mathcal{L}$, and $C(f)$ is convex:
  $\gamma \leq \alpha(\mathcal{L})$, where $\alpha(\mathcal{L})$ depends only on $\mathcal{L}$ — Roughgarden 2002

  In particular, $\alpha(\{\text{linear latencies}\}) = 4/3$ — Roughgarden & Tardos 2000
- Pigou's example is worst possible

No Arc Capacities !?

- "not realistic in the sense that the resulting travel times are finite whenever the link flows are finite"

  "links are actually . . . able to carry arbitrarily large volumes"

  Larsson & Patriksson 1995

- "the predicted flow on some links will be far lower or far greater than the traffic engineer knows they should be" Hearn 1980

- In some of the first models, capacity constraints were used to model congestion effects Charnes & Cooper 1961, Jorgensen 1963, Tomlin 1966

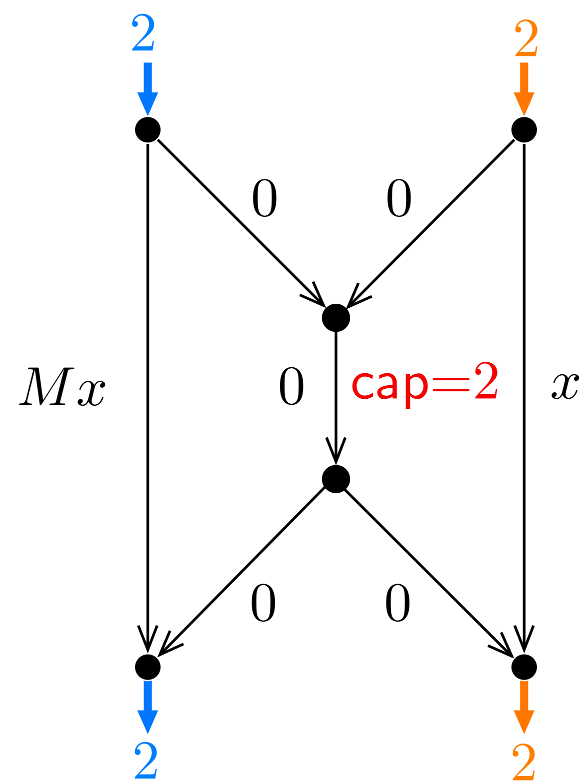
- Capacities can be used to derive tolls for the reduction of flows on overloaded links

Networks with Capacities

- Improve the quality of the model by introducing arc capacities

What is the impact of having explicit capacities on arcs?

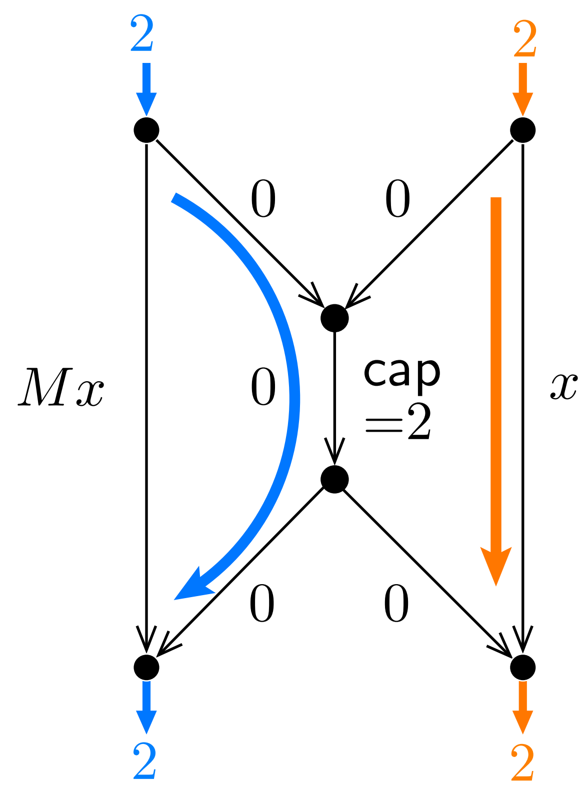
- Straightforward to define **SO**
- What is now a **UE**?



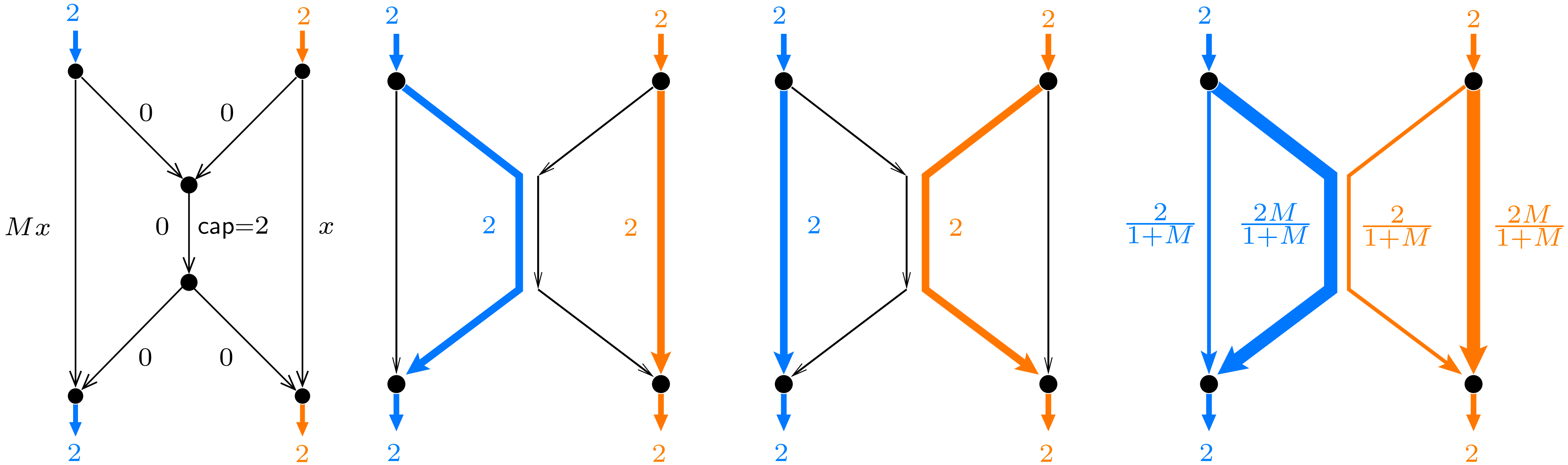
Capacitated Equilibria

Definition: A flow is a **capacitated UE** iff nobody can switch to a path with smaller travel time and residual capacity

- Travel times for users of same OD pair may differ (were constant w/o cap.)
- There may be multiple equilibria (**UE** was unique w/o cap.)
- How good is the best / worst eq.?



Multiple Capacitated Equilibria



with costs of: 4 $\quad 4\,M$ $\quad 4\,\frac{M}{1+M}$

Worst UE can be unbounded!

A good equilibrium: **Beckmann UE**

- Space of **UE** non-convex: Difficult to characterize **Best UE**

- Instead, **Beckmann UE (BUE)**:

$$\min \quad \sum_{a \in A} \int_0^{f_a} t_a(x)\,dx$$

subject to f feasible flow
capacity constraints

- Optimality Conditions: f is a **BUE** iff

for all flows x: $C^f(f) \leq C^f(x)$, where $C^f(x) = \sum_a x_a t_a(f_a)$

Linear Latencies: Coordination Ratio

Theorem. *For any instance of a network with capacities and linear latencies,*

$$C(\textbf{BUE}) \le 4/3\ C(\textbf{SO})$$

Proof: Assume $t_a(x_a) = q_a x_a + r_a : q_a, r_a \ge 0$, for all a and $f = \textbf{BUE}$

$$C^f(f) \le C^f(x)$$

$$= \sum_a x_a(q_a f_a + r_a)$$

$$\le \sum_a x_a(q_a x_a + r_a) + \frac{1}{4}\sum_a {f_a}^2 q_a \quad \text{because } (x_a - f_a/2)^2 \ge 0$$

$$\le C(x) + \frac{1}{4}C(f)$$

Networks without Capacities: Coordination Ratio
(Roughgarden 2002)

Theorem. *For any instance of a network without capacities and latencies in* $\mathcal{L}$

$$C(\textbf{UE}) \leq \alpha(\mathcal{L})\, C(\textbf{SO})$$

Assumptions on functions $t \in \mathcal{L}$*:*
continuous, nondecreasing, differentiable and $xt(x)$ *convex*

Here, $\alpha(\mathcal{L}) := \sup_{t\in\mathcal{L}} \;\alpha(t)$, where

$$\alpha(t) := \sup_{v>0:\, t(v)>0} \left[\lambda \frac{t(\lambda v)}{t(v)} + (1-\lambda)\right]^{-1}, \text{ and}$$

$\lambda \in [0,1]$ solves $(xt(x))'|_{x=\lambda v} = t_a(v)$

Networks with Capacities: Coordination Ratio

Theorem. *For any instance of a network* **with** *capacities and latencies in $\mathcal{L}$,*

$$C(\textbf{BUE}) \leq \alpha(\mathcal{L})\, C(\textbf{SO})$$

Assumptions on functions $t \in \mathcal{L}$:
continuous, nondecreasing, ~~differentiable and $xt(x)$ convex~~

Guarantee does not change with introduction of capacities.

Plus simpler proofs and wider classes of latency functions.

Networks with Capacities: Proof of Coord. Ratio

- For $t \in \mathcal{L}$ and $v \geq 0$, let $\nu(v,t) := \dfrac{1}{v\,t(v)} \max_{x \geq 0} \left\{ x\big(t(v) - t(x)\big) \right\}$

- Let $\nu(\mathcal{L}) := \sup_{t \in \mathcal{L}, v \geq 0} \nu(v,t)$, $f =$ **BUE** and $f^* =$ **SO**

- $$\begin{aligned} C(f) = C^f(f) &\leq C^f(x) \\ &\leq \textstyle\sum_a \nu(f_a, t_a)\, f_a t_a(f_a) + \sum_a x_a t_a(x_a) \\ &\leq \nu(\mathcal{L}) C(f) + C(x) \end{aligned}$$

Therefore, $C(f) \leq \dfrac{1}{1 - \nu(\mathcal{L})} C(f^*)$, where $\dfrac{1}{1-\nu(\mathcal{L})} \leq \alpha(\mathcal{L})$

Computing $\alpha(\mathcal{L})$

- If $\mathcal{L}$ contains only concave functions $\quad \longrightarrow \quad \alpha(\mathcal{L}) \leq 4/3$
  (Note this includes linear functions)

- If $\mathcal{L}$ contains polynomials of degree $n \quad \longrightarrow \quad \alpha(\mathcal{L}) \leq \frac{(n+1)^{1+1/n}}{(n+1)^{1+1/n}-n}$

$\longrightarrow$ $\alpha(\{$polynomials of degree $2\}) = 1.626$

$\longrightarrow$ $\alpha(\{$polynomials of degree $3\}) = 1.896$

$\longrightarrow$ $\alpha(\{$polynomials of degree $4\}) = 2.151$

$\cdots$

$\longrightarrow$ $\alpha(\{$polynomials of degree $n\}) = \Omega(n/\ln n)$

Example: Computing $\alpha(\{$polynomials of degree $n\})$

- Assume polynomials have positive coefficients

- Then, for $c \in [0,1]$: $\quad t(cx) \geq c^n t(x)$

- $$\begin{aligned} \nu(v,t) &= \max_{0 \le x \le v} \left\{ \frac{x}{v} \left(1 - \frac{t(x)}{t(v)} \right) \right\} \qquad \text{(rewriting } x \text{ as } v\,\frac{x}{v}\text{)} \\ &\leq \sup_{0 \le x \le v} \left\{ \frac{x}{v} \left(1 - \left(\frac{x}{v} \right)^n \right) \right\} \\ &= \sup_{0 \le x \le 1} \left\{ x \left(1 - x^n \right) \right\} \\ &= \frac{n}{(n+1)^{1+1/n}} \end{aligned}$$

Recap

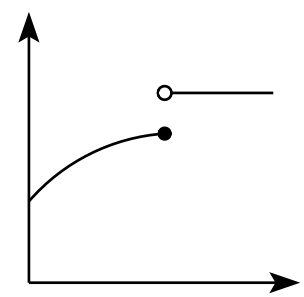
- Objective: quantify (in-)efficiency of "selfish" solutions (= Nash eq.)

- Considered networks with capacities and significantly broader classes of latency functions

- Main result: Selfish solutions are still close-to-optimal so long as Beckmann **UE** selected

- Simplified proofs

- In particular, lower semi-continuous functions OK



Recap II

No capacities	With capacities
UE unique	Set of **UE** in general non-convex
UE/**SO** $\geq \alpha(\mathcal{L})$	**UE**/**SO** unbounded
UE/**SO** $\leq \alpha(\mathcal{L})$	**BUE**/**SO** $\leq \alpha(\mathcal{L})$