Robust Wardrop Equilibrium and How to Price to Get It

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June 2007

Introduction

- Models of User Behavior:
  - Deterministic delay functions $\ell(x)$ map flows to delays

    Wardrop'52
  - **Actual delays** are subject to uncertainty
    (= nominal delay + prediction error)

Our model: agents use robust optimization to select routes

Introduction

- Models of User Behavior:
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Our model: agents use robust optimization to select routes

- As equilibria may be inefficient, we'd like to coordinate agents

  Dupuit 1849, Pigou'20

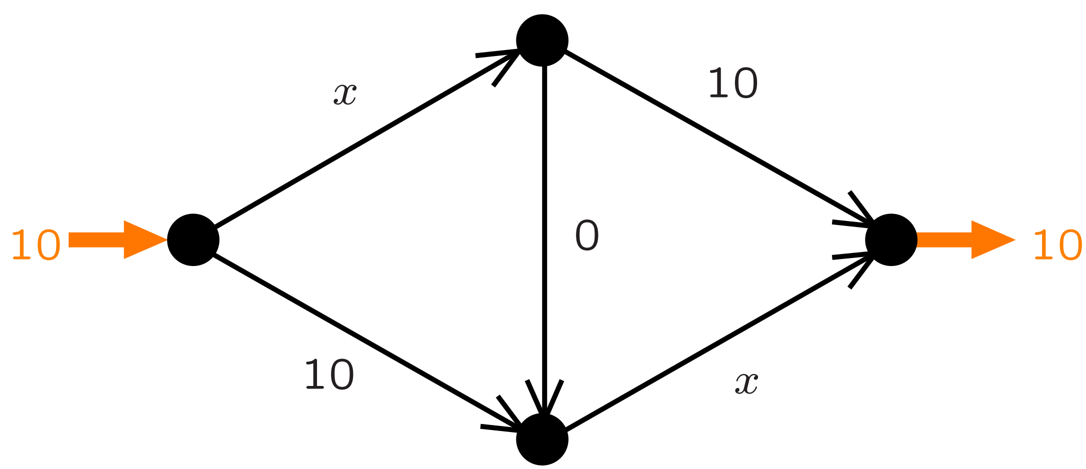
Our model: a network coordinator can do this by charging taxes

Outline

- Network Model
- Robust Wardrop Equilibria
- Computational Results
- Pricing Mechanisms

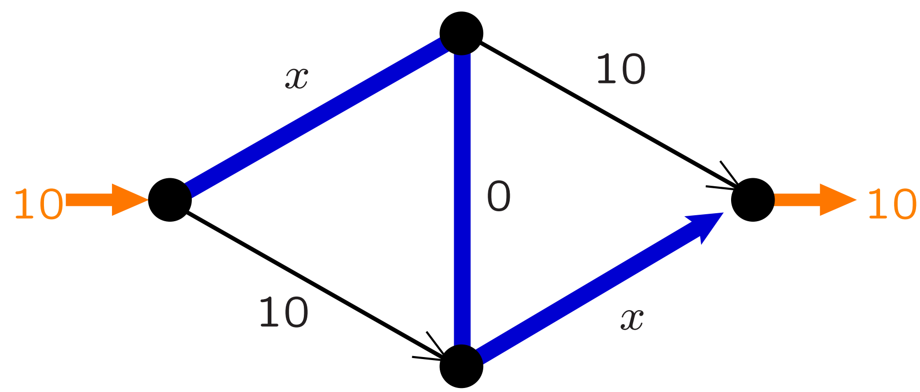
The Traditional Network Model

- Network $G = (N, A)$ with OD pairs of rate r_k, $k \in K$
- Nonatomic demand: infinitesimal agents
- $\mathcal{P}_k := \{$ paths connecting OD pair k $\}$, $\mathcal{P} := \cup \mathcal{P}_k$
- Nondecreasing & continuous delay functions $\ell_a : \mathbb{R}_+ \to \mathbb{R}_+$



(Deterministic) Wardrop Equilibrium

Definition: A flow x^{WE} is a **WE** of the network game if no agent can switch to a path with smaller delay Wardrop'52



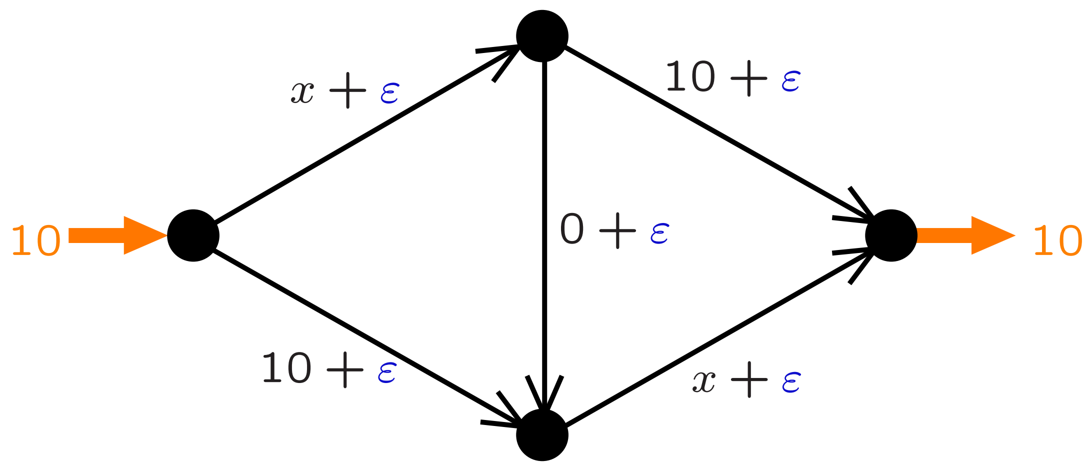
- **WE** characterized by a Variational Inequality: Smith'79
  $\langle \ell_a(x_a^{\text{WE}}), x_a - x_a^{\text{WE}} \rangle_{a \in A} \geq 0$ for all x Dafermos'80

- **WE** exists & is essentially unique Beckmann, McGuire & Winsten'56

Prediction Errors

We associate independent RVs ε (not necessarily identically distributed) to each arc to account for the uncertainty in delays

Agents don't know their distribution but have an idea of their variability (i.e., they know the support)



Robust Optimization

Ben-Tal & Nemirovski'98; El-Ghaoui, Oustry & Lebret'98

A method to address uncertainty in optimization that

- seeks a solution that optimizes the worst case
- distribution-less: uncertain parameters $\in$ **uncertainty region**

Attractive features of robust optimization:

- **Simple** uncertainty model
- Robust counterpart is **tractable**
  - often similar complexity as deterministic problem
- Solution is efficient under uncertainty: "solution is **immunized**"

Robust Shortest Path Problem

Bertsimas & Sim'03

- Our behavioral model is that agents choose paths with minimum delay. The uncertainty is present only in the objective function.

- If, for arc $a \in A$, the nominal delay is ℓ_a and the uncertainty is ε_a, agents solve the following **Robust Shortest Path Problem**:

$$\min_{P} \quad \max_{\varepsilon} \Big\{ \sum_{a \in P} (\ell_a + \varepsilon_a) : \varepsilon \in \mathcal{U} \Big\}$$

s.t. P is a feasible path for the agent,

where we still need to specify the uncertainty region $\mathcal{U}$

- This problem can be solved efficiently

Uncertainty Region

- Nominal delay $\ell_a(x_a)$ depends on flow x_a along a
  Error ε_a is random but doesn't depend on x_a. Support in $[0, \gamma_a]$

- Assume optimistic view:
  $\longrightarrow$ unlikely that all random variables take worst-case values

- The uncertainty region is the intersection of a norm-1 and norm-∞ balls:

  – We fix an uncertainty budget Γ, and constrain the deviation from the nominal delay on a path $P \in \mathcal{P}$ to be less than Γ

  – If path P is selected, the worst-case deviation is:

$$\gamma_P^\Gamma := \max \Big\{ \sum_{a \in P} z_a \gamma_a : \sum_{a \in \mathcal{P}} z_a \leq \Gamma, \ 0 \leq z_a \leq 1 \Big\}$$

Robust Wardrop Equilibria (RWE)

Definition: A flow x^{RWE} is a Robust Wardrop Equilibrium $\Leftrightarrow$

$$\ell_P(x^{\text{RWE}}) + \gamma_P^{\Gamma} \leq \ell_Q(x^{\text{RWE}}) + \gamma_Q^{\Gamma} \quad \text{for all } P, Q \in \mathcal{P}_k \text{ with } x_P^{\text{RWE}} > 0$$

One can extend the arc-by-arc VI formulation to one by paths:

$$\langle \ell_P(x^{\text{RWE}}) + \gamma_P^{\Gamma} \ , \ x_P - x_P^{\text{RWE}} \rangle_{P \in \mathcal{P}} \geq 0 \quad \text{for all feasible flows } x$$

This shows that a **RWE** exists   Hartmand and Stampacchia'66

Computation

RWE can be characterized using a non-linear complementarity problem (NCP) equivalent to the VI

Aashtiani and Magnanti'81

$$0 \leq f_P \perp \ell_P(f) + \gamma_P^{\Gamma} - u_k \geq 0 \quad \forall\ P \in \mathcal{P}$$

Column generation algorithm

Gabriel and Bernstein'97

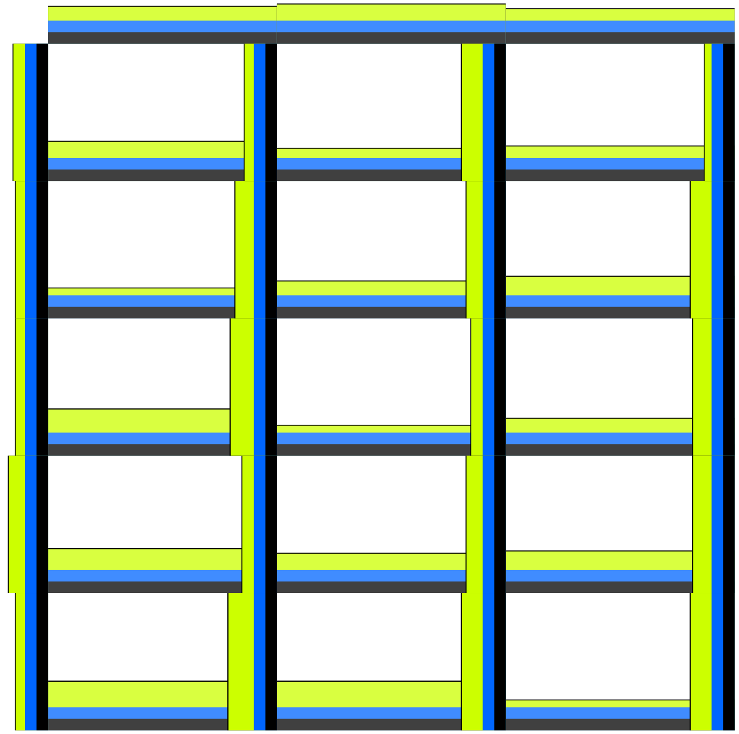
1. compute a solution f^* to the reduced NCP (with $\mathcal{P}' \subset \mathcal{P}$):

$$0 \leq f_P \perp \ell_P(f) + \gamma_P^{\Gamma} - u_k \geq 0 \quad \forall\ P \in \mathcal{P}'$$

2. find new paths to be added to reduced NCP by solving

$$\begin{aligned} v(f^*) = \ \min \ & \textstyle\sum_{P \in \mathcal{P}} \left(\ell_P(f^*) + \gamma_P^{\Gamma}\right) x_P & \\ \text{s.t.} \ & \textstyle\sum_{P \in \mathcal{P}_k} x_P = d_k & k \in K \\ & x_P \geq 0 & P \in \mathcal{P} \end{aligned}$$

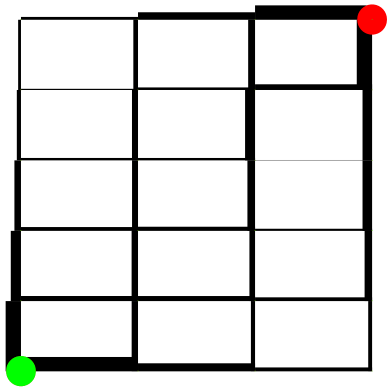
Example: Random Grid



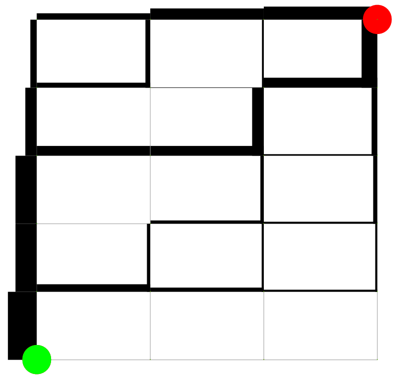
Widths and Colors represent Delays:

free-flow, **full capacity**, **latter plus uncertainty** γ_a

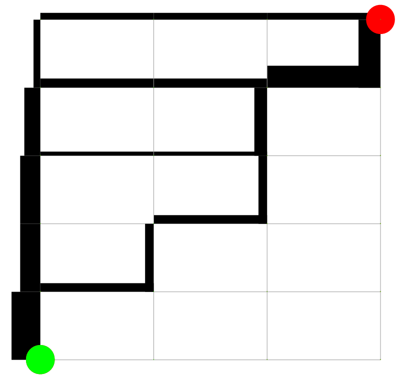
Robust Wardrop Equilibria in the Random Grid



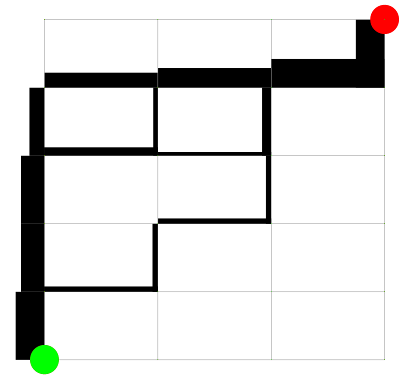
$\Gamma = 0$



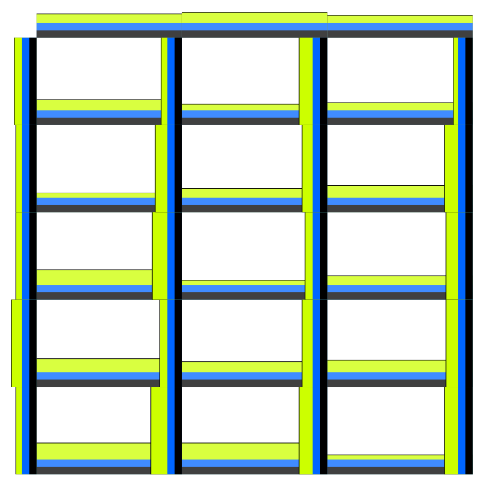
$\Gamma = 1$



$\Gamma = 3$



$\Gamma = 5$



Evaluation of Equilibria

Given ex-post delays $\tilde{\ell}$, we compare equilibria according to the following metrics: Jahn et al.'04

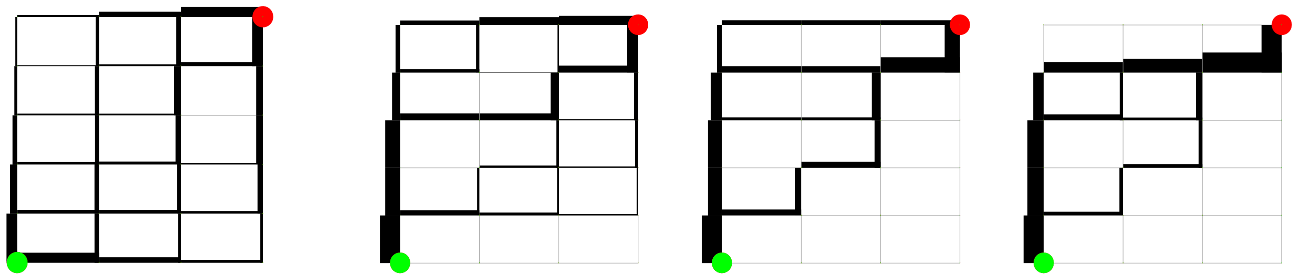
- Agent delay: $\tilde{\ell}_P(x^{\text{WE}})$ vs. $\tilde{\ell}_P(x^{\text{RWE}})$
- Unfairness: distance to a "real" equilibrium (ex-post)

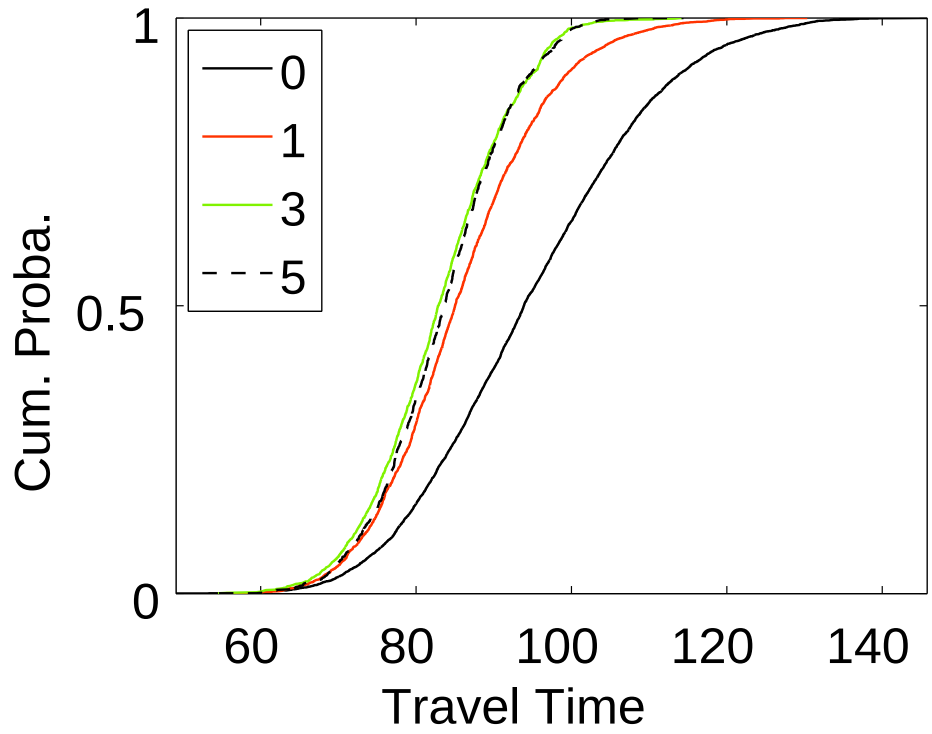
Definition. The unfairness of a flow f is defined by

$$\frac{\max_{P \in \mathcal{P}_k : f_P > 0} \ \tilde{\ell}_P(f)}{\min_{P \in \mathcal{P}_k : f_P > 0} \ \tilde{\ell}_P(f)}$$

Note that unfairness $\begin{cases} = 1 \text{ for an equilibrium} \\ > 1 \text{ if not at equilibrium} \end{cases}$

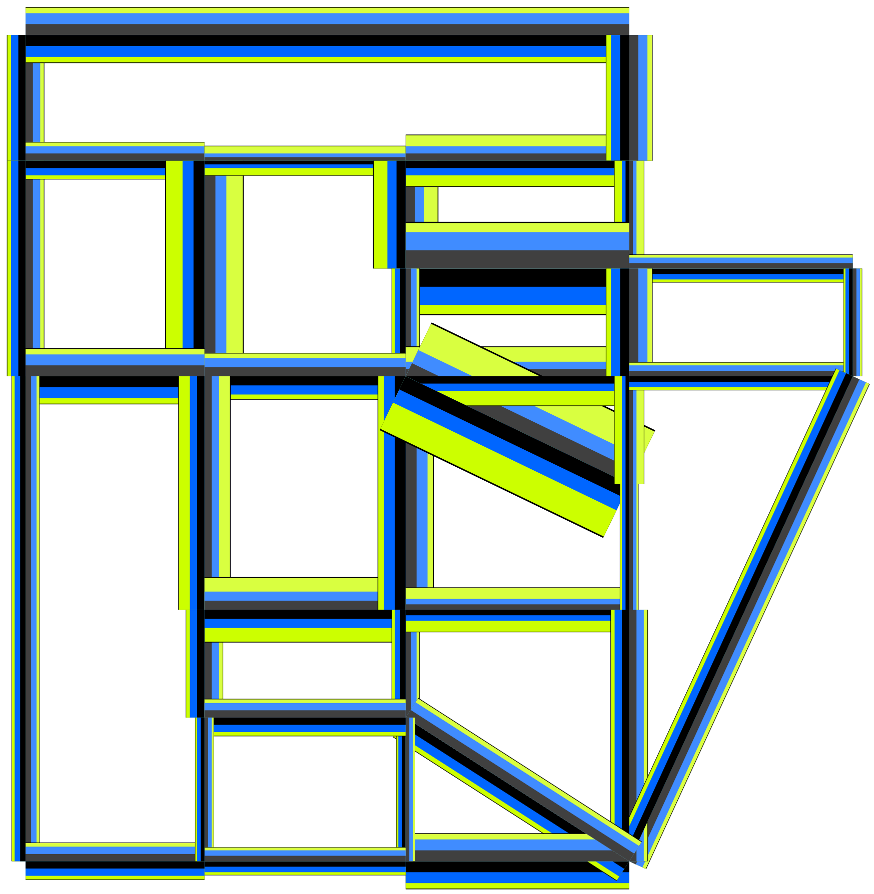
Distribution of Delays in the Random Grid





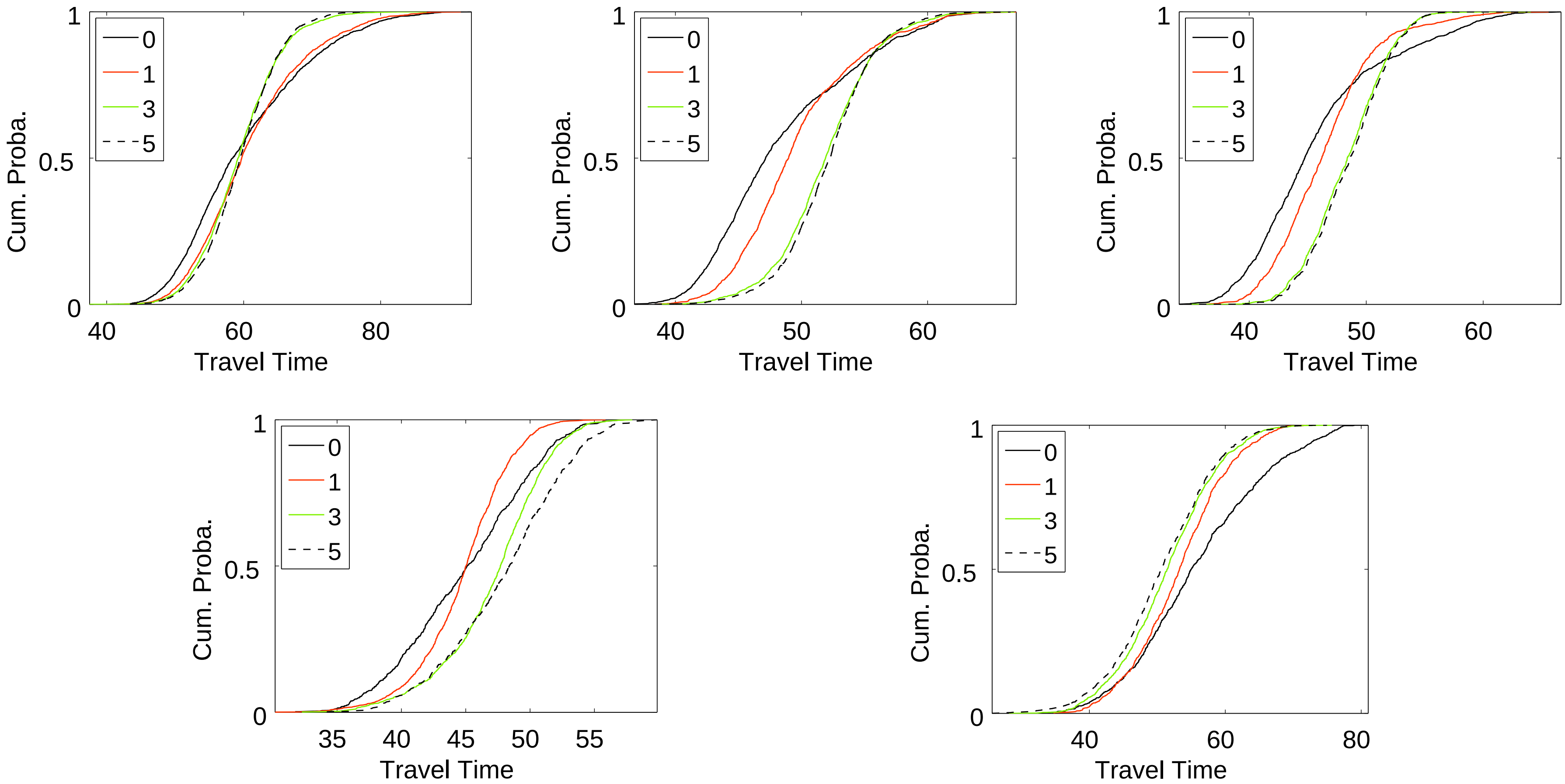
Example: Sioux Falls

http://www.bgu.ac.il/~bargera/tntp/

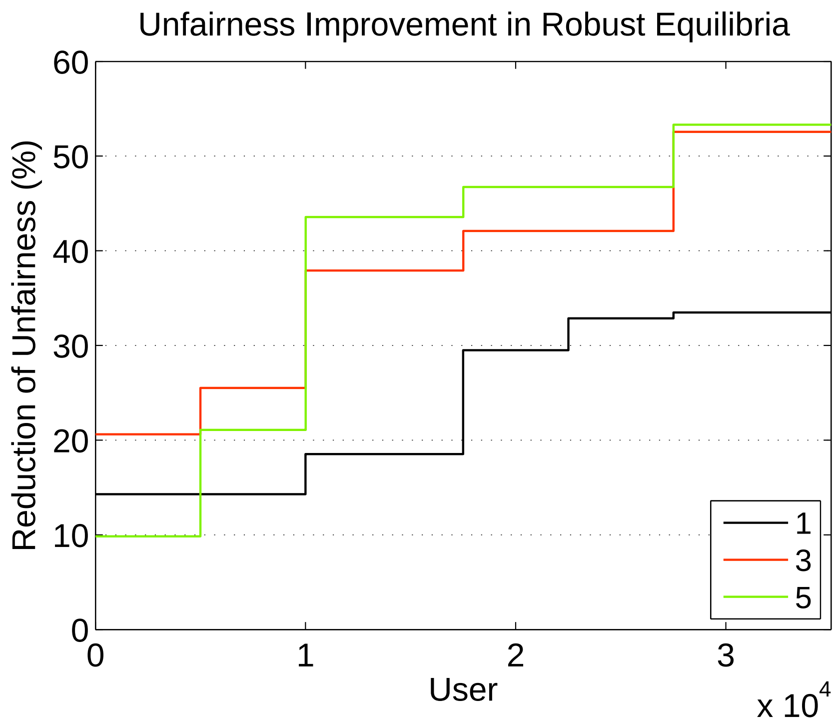


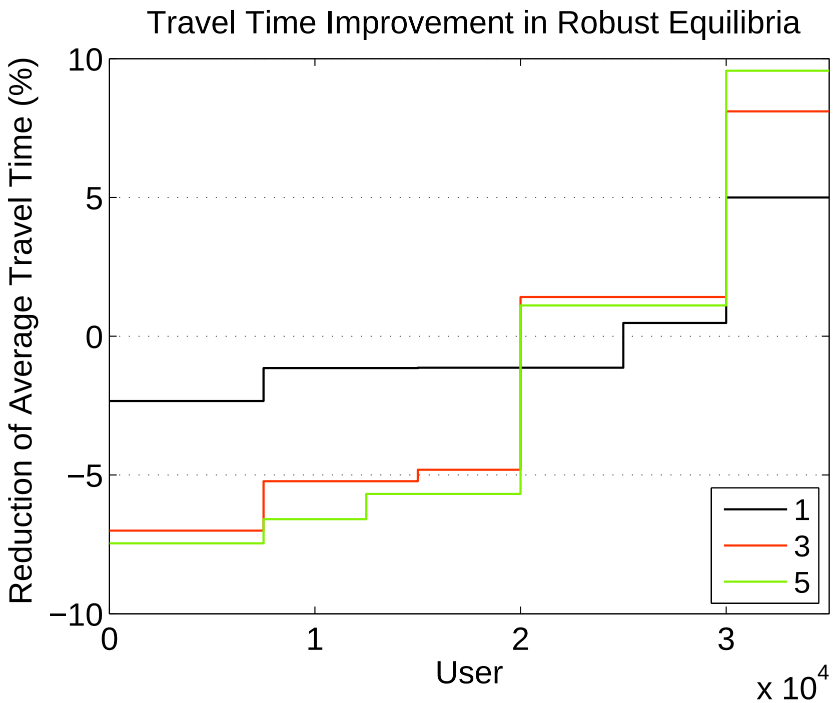
Delays: free-flow, full capacity, latter plus uncertainty γ_a

Distribution of Delays for 5 OD Pairs

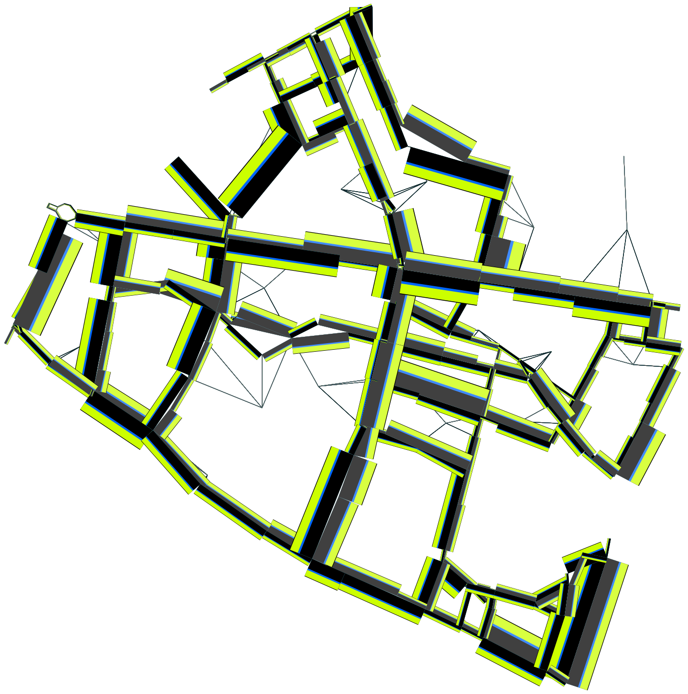


Change of Fairness and Delay wrt Nominal WE

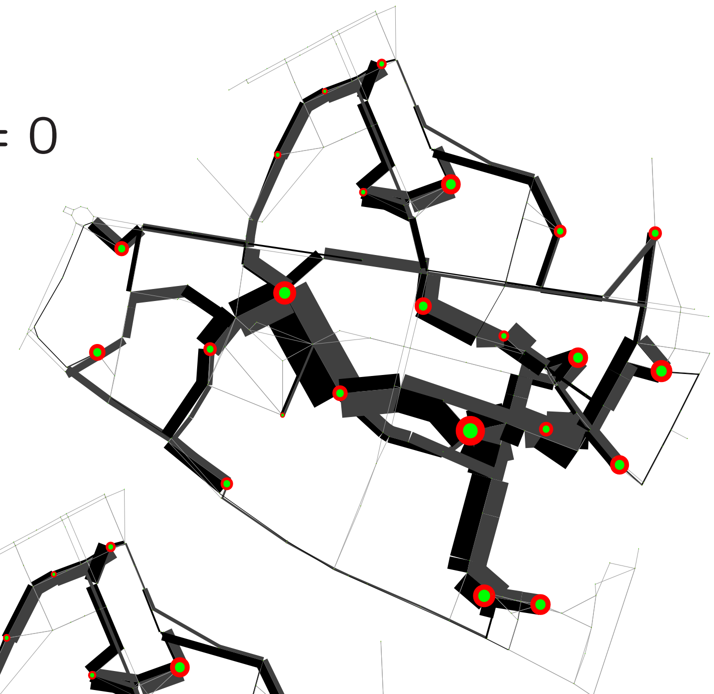




Example: Friedrichshain



$\Gamma = 0$



$\Gamma = 1$

Agents make selfish decisions

How efficient is the system?

centralized	**competition**
global perspective	selfish
social optimum =	**Nash equilibrium** =
min total cost s.t. solution is feasible	$\forall$ agent: min individual cost s.t. solution is feasible
efficient, unfair	not efficient, no regret

Pricing for (Deterministic) Wardrop Equilibria

Can one impose tolls/taxes/charges on arcs s.t. the resulting **WE** with tolls equals the **SO**?

Beckmann, McGuire & Winsten'56

Yes! On all arcs a, charge marginal prices
$x_a \ell_a'(x_a)$ to internalize the externalities:
a **SO** is a **WE** wrt costs $\ell_a(x_a) + x_a \ell_a'(x_a)$

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Can one find tolls to coordinate agents into an arbitrary flow f ?

Yes too! Solve the following LP

Fleischer, Jain & Mahdian'04

$$\begin{aligned} \min \quad & \sum x_a \ell_a(f_a) \\ \text{s.t.} \quad & x_a \leq f_a \qquad a \in A \\ & x \text{ is a feasible flow} \end{aligned}$$

and use the dual variables t_a as taxes

Pricing for Robust Wardrop Equilibria

- System planner wishes to minimize total delay
  - It has a model of uncertainty of the system
  - Also knows the uncertainty beliefs of agents

It sets f := social optimum or robust social optimum

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- Solve a version of the LP with path-variables:

$$\min \sum_{P \in \mathcal{P}} x_P(\ell_P(f) + \gamma_P^{\Gamma})$$

$$\text{s.t. } x_a \leq f_a \quad a \in A$$

$$x \text{ is a feasible flow}$$

and use the dual variables t_a as taxes

Pricing for Robust Wardrop Equilibria II

- Actually, we solve the dual problem:

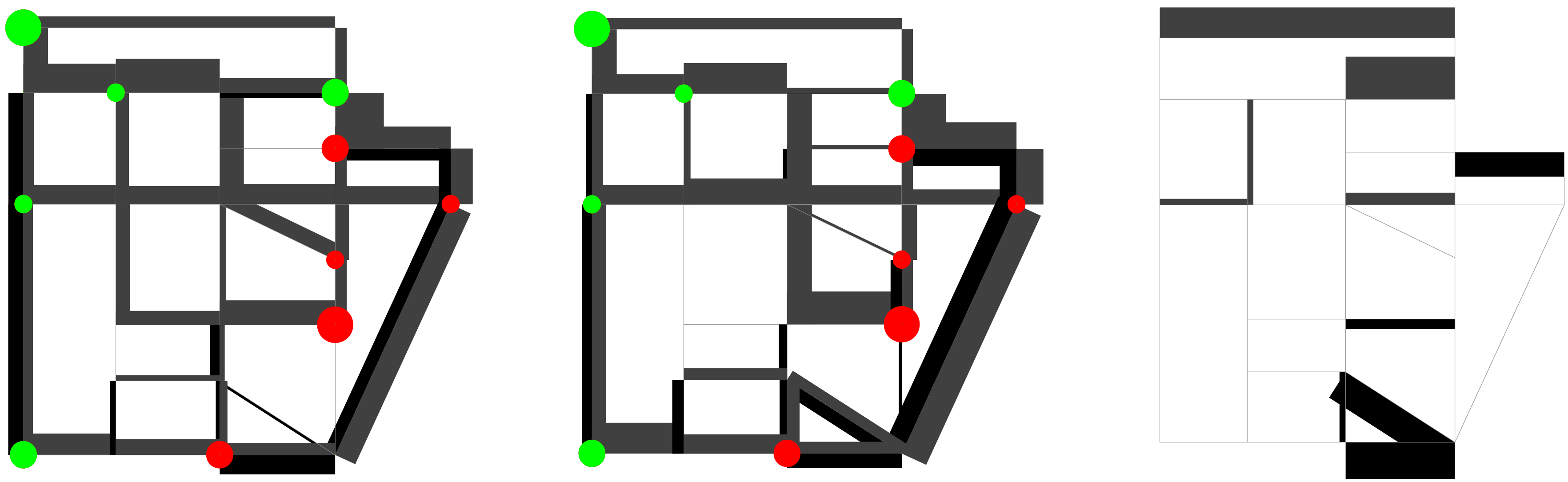
$$\begin{aligned} \max \quad & \sum_{k \in K} r_k z_k - \sum_{a \in A} f_a t_a \\ \text{s.t.} \quad & z_k \leq \sum_{a \in P} (\ell_a(f_a) + t_a) + \gamma_P^\Gamma \qquad P \in \mathcal{P}_k \\ & t_a \geq 0 \qquad a \in A \end{aligned}$$

  and get the taxes t_a explicitly

- The problem is that there are too many paths

- To get around it, we generate constraints by solving robust shortest path problems

Robust Equilibrium and System Optimum in Sioux Falls



SO RWE Taxes

Concluding Remarks

- Uncertainty should be modeled explicitly: Computations suggest that **RWE** are more (ex-post) fair than **WE**

- Can compute **RWE** efficiently

- Can compute optimal prices solving a series of LPs and robust shortest path problems
  (Can it be done in P-time?)

- The budgets of uncertainty can be player dependent and can be different for the planner and for the agents.
  $\longrightarrow$ can model that there is more uncertainty for
  OD pairs that are further away
  $\longrightarrow$ varying risk preferences

The End

	GridA								
Γ	0	1	2	3	4	5	6	7	8
Avg t.t.	95	86	84	83	83	84	84	83	83
Stdev t.t.	14	10	9	9	9	8	8	8	8
5th perc.	73	70	70	69	69	70	70	69	69
95th perc.	119	104	99	97	97	98	98	97	97
Unfairness	1.64	1.49	1.42	1.41	1.41	1.40	1.40	1.40	1.40

	GridB								
Γ	0	1	2	3	4	5	6	7	8
Avg t.t.	89	79	74	74	74	74	75	75	75
Stdev t.t.	14	10	9	8	8	8	7	7	7
5th perc.	67	63	60	61	61	61	62	62	62
95th perc.	113	97	90	87	86	87	87	87	87
Unfairness	1.67	1.54	1.49	1.43	1.41	1.41	1.40	1.40	1.40

	GridC								
Γ	0	1	2	3	4	5	6	7	8
Avg t.t.	65	60	57	56	57	57	57	57	57
Stdev t.t.	8	6	5	4	4	4	4	4	4
5th perc.	52	50	50	50	51	51	51	51	51
95th perc.	78	70	66	62	63	63	63	63	63
Unfairness	1.50	1.41	1.33	1.24	1.23	1.23	1.23	1.23	1.23

Column Generation Algorithm

Algorithm 1 Column Generation

1: Initialize: Add arbitrary paths to $\mathcal{P}'_1, \ldots, \mathcal{P}'_K$.

2: Set $f^* = 0$ and $v(f^*) = -\infty$.

3: **while** $v(f^*) < \sum_{P \in \mathcal{P}'} \left(\ell_P(f^*) + \gamma_P^\Gamma \right) f_P^*$ **do**

4: Compute a solution f^* to the reduced NCP.

5: Solve the robust shortest path problem with flow equal to f^*.
Let x^* be the optimal solution and $v(f^*)$ be its value.

6: Add paths used in x^* to $\mathcal{P}'$.

7: **STOP**. The flow f^* is a **RWE** .

Column Generation Algorithm

The termination condition

$$v(f^*) \;\; \geq \;\; \sum_{P \in \mathcal{P}'} \left(\ell_P(f^*) + \gamma_P^\Gamma \right) f_P^*$$

implies that f^* **is a RWE** as it solves the VI

Indeed, setting $f_P^* = 0$ for $P \notin \mathcal{P}'$:

$$\sum_{P \in \mathcal{P}} \left(\ell_P(f^*) + \gamma_P^\Gamma \right) x_P^* \geq \sum_{P \in \mathcal{P}'} \left(\ell_P(f^*) + \gamma_P^\Gamma \right) f_P^*$$

$$\sum_{P \in \mathcal{P}} \left(\ell_P(f^*) + \gamma_P^\Gamma \right) x_P - \sum_{P \in \mathcal{P}} \left(\ell_P(f^*) + \gamma_P^\Gamma \right) f_P^* \;\; \geq \;\; 0 \qquad \forall \; x$$

$$\langle \ell_P(f^*) + \gamma_P^\Gamma, x_P - f_P^* \rangle_{P \in \mathcal{P}} \;\; \geq \;\; 0 \qquad \forall \; x$$

Robust Shortest Path

Solving $v(f^*)$:

- $|K|$ robust shortest path problems
- unit demand per commodity
- non-additive term γ_P^Γ accounts for delay uncertainty

$$\gamma_P^\Gamma := \max \left\{ \sum_{a \in P} z_a \gamma_a : \sum_{a \in \mathcal{P}} z_a \leq \Gamma,\ 0 \leq z_a \leq 1 \right\}$$

$\Rightarrow$ for each solve $|A|$ shortest path problems Bertsimas and Sim'03

Simulation

We simulated the realizations of the delay errors to understand **Robust Wardrop Equilibria**, the dependance on Γ, and the differences with their deterministic counterpart

To do that, we:

- consider distributions for arcs with support in $[0, \gamma_a]$ (distributions are consistent with uncertainty model)
- perform Monte Carlo simulations to obtain sampled delays $\tilde{\ell}_P(\cdot)$
- compare x^{WE} and x^{RWE} for different values of Γ using these sampled delays

Change of Fairness and Delay wrt Nominal WE